LEBESGUE MEASURE THEORETIC DYNAMICS OF RATIONAL MAPS

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Abstract. We survey the role of one and two dimensional Lebesgue measure in complex dynamics. Before computers, rational maps with Julia set the whole sphere, a circle, or an arc were the only accessible maps so we begin with these classical examples. We then discuss some more recently studied families of rational maps that preserve finite or infinite measures equivalent to one and two dimensional Lebesgue measure. We end with a brief look at the idea behind Julia sets of quadratic polynomials with positive two-dimensional Lebesgue measure.

1. Introduction

Ever since the advent of computers, interest in the measurable dynamics of rational maps whose Julia sets are either the sphere or a smooth arc has been overshadowed by the multitude of beautifully illustrated studies of measures supported on fractal Julia sets. The fractal measures considered are a natural generalization of Lebesgue measure to noninteger dimension. However Lebesgue measurable dynamics still play an important and interesting role in the field of complex dynamics. One of the deepest results in the field in the past decade was the construction and proof of a Julia set which has positive area, but is not the entire sphere [12].

In this short survey article we start with the classical examples and move to more unusual families of rational maps of the Riemann sphere that exhibit interesting Lebesgue measurable dynamics. Many of the results mentioned in this paper have appeared elsewhere but we bring them together here under this common theme of Lebesgue measurable dynamics of rational maps. One of the main points of this survey is to illustrate that in the midst of fractal Julia sets lie many parametrized families of maps with smooth Julia sets and chaotic Lebesgue measure theoretic behavior.

In Section 2 we lay some groundwork with basic definitions and the classical examples. We continue the discussion in this section with two

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families of maps with chaotic one dimensional behavior; the measure we use is the arc length measure on the sphere, which we denote by m_1. One family of maps has the property that it is a nonpolynomial family (except for one map) of S-unimodal maps all of which are ergodic, exact, and admit an equivalent invariant probably measure.

The next family is parametrized by one real parameter, and each map preserves Lebesgue measure ℓ on $\mathbb{R}$, the Julia set of each map. We connect these maps to inner functions by showing they are square roots of them. Many of the results in this section were obtained by two of the author's Ph.D. students [6, 17].

We then turn to m_2, normalized surface area measure in Section 4; we begin with a quick view of the Lattès examples dating back to 1918 and we then move forward to the 1980s and look at postcritically finite rational maps more generally. Theorem 4.1 is a new result. In Section 4.4 we give a brief idea of how one can show a Julia set for a polynomial could have positive m_2 measure.

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2. Preliminary definitions and the first examples

We start with the first two classical examples of rational maps that one studies; they have smooth Julia sets. They are $p_0(z) = z^2$ and the polynomial occurring at the tip of the Mandelbrot set: $p_{-2}(z) = z^2 - 2$. The second map is more often presented to us as the degree two Chebychev polynomial $\tau_2(z) = 2z^2 - 1$. The maps p_{-2} and τ_2 are related via the map $\phi(z) = 2z$, which maps $[-1, 1]$ linearly onto $[-2, 2]$, and satisfies $\phi \circ \tau_2(z) = p_{-2} \circ \phi(z)$.

Throughout this paper we let $\mathbb{C}_\infty$ denote the Riemann sphere. The maps of interest are rational maps of the form: $R(z) = \dfrac{p(z)}{q(z)}$, with p, q polynomials over $\mathbb{C}$ with no common factor, and such that the maximum degree of p and q is at least 2. Rational maps characterize the analytic maps of the Riemann sphere; by R^n we denote the n-fold composition of R with itself. Two rational maps $R, S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ are *conformally conjugate* if there exists a linear fractional transformation ϕ on $\mathbb{C}_\infty$ such that

$$\phi \circ R = S \circ \phi. \tag{1}$$

We showed above that p_{-2} and τ_2 are conformally conjugate maps.

The notion of conjugacy has a few meanings in this setting. It often occurs that we have two rational maps R, S related by a map ϕ on $\mathbb{C}_\infty$ such that $\phi \circ R = S \circ \phi$, but ϕ is only continuous, in which case we say R and S are *topologically conjugate*.

We consider the circle $S^1 = \{z : |z| = 1\}$; by m_1 we denote arc length measure normalized so that $m_1(S^1) = 1$. In most introductory dynamics courses, for the map $p_0(z) = z^2$ we see the following:

- $p_0(S^1) = S^1 = p_0^{-1}(S^1)$;
- $\lim_{n\to\infty} p_0^n(z) = 0$ for any z with $|z| < 1$;
- $\lim_{n\to\infty} p_0^n(z) = \infty$ for any z with $|z| > 1$.

The dynamics of p_0 restricted to S^1 are interesting and will be discussed further. Using the Euler identity: $e^{i\theta} = \cos\theta + i\sin\theta$, if we set $\psi(z) = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$, restrict our attention to the circle, and write $z = e^{i\theta}$, then we see that $\psi(e^{i\theta}) = \cos(\theta)$, so the unit circle is mapped by ψ in a 2-to-one manner, except at the critical points ± 1, onto $[-1, 1]$. It is easy to show that for all $z \in \mathbb{C}$,

$$\psi(p_0(z)) = \tau_2(\psi(z)) = \cos(2\theta), \tag{2}$$

and in fact replacing 2 by any integer $d \geq 2$, we use the following to define the d^{th} Chebychev polynomial τ_d:

$$\tau_d(\cos(z)) = \cos(dz), \tag{3}$$

One other useful and noteworthy property of the maps $\mathfrak{X}_d(z) = z^d$, $d \geq 2$ and τ_d, is that they are measure theoretically isomorphic to one-sided Bernoulli shifts with respect to their invariant probability measures $\sim m_1$. We turn to a brief review of some needed measure theoretic definition, but first we recall the definition of a one-sided Bernoulli shift.

Definition 2.1. We fix an integer $d \geq 2$ and consider $\mathcal{A} = \{1, \ldots, d\}$, a finite state space with the discrete topology. A vector $p = \{p_1, \ldots, p_d\}$ such that $p_k > 0$ and $\sum p_k = 1$ determines a measure on $\mathcal{A}$, namely $p(\{k\}) = p_k$. Let $\Omega = \prod_{i=0}^{\infty} \mathcal{A}$ be the product space endowed with the product topology and product measure ρ determined by $\mathcal{A}$ and p. The map σ is the one-sided shift to the left, $(\sigma x)_i = x_{i+1}$. We say σ is a *one-sided Bernoulli shift* and denote it by $(\Omega, \mathcal{D}, \rho; \sigma)$, where $\mathcal{D}$ denotes the Borel σ-algebra generated by the cylinder sets, completed with respect to ρ.

2.1. **Measure theoretic preliminaries.** We assume throughout that every space $(X, \mathcal{B}, \mu)$ under consideration is a Lebesgue probability space though sometimes we specify that μ is a σ-finite infinite measure. In our setting, $X \subset \mathbb{C}_\infty$ is always a closed set and $\mathcal{B}$ denotes the σ-algebra of Borel measurable sets. We assume that the measure space is complete with respect to μ (every subset of every null set for μ is measurable), and that T is a surjective *nonsingular endomorphism*; i.e., $T : X \to X$ satisfies: $\mu(A) = 0 \iff \mu(T^{-1}A) = 0$ for every $A \in \mathcal{B}$, and $\mu(T(X)\triangle X) = 0$. If ν is a σ-finite measure such that $\nu \sim \mu$, and $\nu(T^{-1}A) = \nu(A)$ for all $A \in \mathcal{B}$, we say that T is *measure-preserving*, or equivalently T *preserves* ν. Without loss of generality we can assume that T is forward measurable and forward nonsingular; i.e., for all measurable sets A, $T(A) \in \mathcal{B}$ and $\mu(A) = 0 \iff \mu(TA) = 0$. When we say that a property holds on X (μ mod 0) or μ a.e., we mean that there is a set $N \in \mathcal{B}$ with $\mu(N) = 0$, (N is possibly the empty set), such that the property holds for all $x \in X \setminus N$.

Definition 2.2. Let $T_1 : (X_1, \mathcal{B}_1, \mu_1) \circlearrowleft$ and $T_2 : (X_2, \mathcal{B}_2, \mu_2) \circlearrowleft$ be two measure-preserving endomorphisms.

A measurable map $\phi : X_1 \to X_2$ is a *homomorphism* if there exists a set $Y_1 \in \mathcal{B}_1$ of full measure and a set $Y_2 \in \mathcal{B}_2$ of full measure in X_2 such that ϕ maps Y_1 onto Y_2.

If there exists a homomorphism ϕ such that $T_1(Y_1) = Y_1$, $T_2(Y_2) = Y_2$, $\phi \circ T_1 = T_2 \circ \phi$ on Y_1, and $\mu_2(A) = \mu_1(\phi^{-1}(A))$ for all $A \in \mathcal{B}_1$, then T_2 is called a *factor* of T_1 (w.r.t. the measures μ_1 and μ_2), *with factor map* ϕ.

If in addition ϕ is injective on Y_1 we say it is an *isomorphism*. If T_2 is a factor of T_1 and ϕ is an isomorphism, then we say that the endomorphisms T_1 and T_2 are *isomorphic* endomorphisms.

For any two sets $A, B \in \mathcal{B}$ we define $A\triangle B = (A \setminus B) \cup (B \setminus A)$. The map T is *ergodic* if T has a trivial field of invariant sets, or equivalently, if any measurable set T with the property that $\mu(B\triangle T^{-1}B) = 0$ has either zero or full measure.

A map is *exact* if it has a trivial tail field $\cap_{n\geq 0} T^{-n}\mathcal{B} \subset \mathcal{B}$, or equivalently, if any set B with the property $\mu(T^{-n} \circ T^n(B) \triangle B) = 0$ for all n has either zero or full measure. It is clear that every exact map is also ergodic.

Assume $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ preserves μ. We recall a condition which is strictly weaker than one-sided Bernoulli for endomorphisms [16], but equivalent to Bernoulli in the invertible case [15]. We refer the reader to [11, 23, 24] if more detail is needed.

We consider partitions $\zeta = \{P_1, P_2, \cdots\}$ and $\eta = \{Q_1, Q_2, \cdots\}$ of X; each set is measurable and $\cup_{i\geq 1} P_i = \cup_{j\geq 1} Q_j = X$ (μ mod 0), with each union disjoint (μ mod 0). The notation $\zeta \vee \eta$ denotes the partition such that each set is of the form $P_i \cap Q_j$ for some i, j. The partition ζ *is independent of* η if

$$|\mu(P_i \cap Q_j) - \mu(P_i)\mu(Q_j)| = 0 \text{ for all } i, j.$$

The partition ζ is defined to be $\varepsilon-$*independent of* η if

$$\sum_i \sum_j |\mu(P_i \cap Q_j) - \mu(P_i)\mu(Q_j)| \leq \varepsilon.$$

Definition 2.3. For an ergodic measure-preserving endomorphism T on $(X, \mathcal{B}, \mu)$, (invertible or noninvertible) a partition ζ is *weak Bernoulli* if given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m \geq 1$,

$$\bigvee_0^m T^{-i}\zeta \quad \text{is} \quad \varepsilon - \text{independent of} \bigvee_N^{N+m} T^{-i}\zeta.$$

Definition 2.4. (1) We say that a noninvertible endomorphism T on $(X, \mathcal{B}, \mu)$ has the *weak Bernoulli property* or that T is *weak Bernoulli* if there exists a weak Bernoulli partition ζ for T such that $\bigvee_{i=0}^{\infty} T^{-i}(\zeta) = \mathcal{B}\,(\mu \bmod 0)$.

(2) An automorphism $\tilde{T}$ is *the natural extension* of the (noninvertible endomorphism) T if T is a measurable factor of $\tilde{T}$ and any other automorphism S which has T as a factor also has $\tilde{T}$ as a factor.

The first example of a noninvertible endomorphism with a weak Bernoulli generator, that is not one-sided Bernoulli is due to Furstenberg [16]. It was proved by Friedman and Ornstein in [15] that for an invertible transformation T, if there exists a weak Bernoulli partition ζ such that:

$$\bigvee_{i=-\infty}^{\infty} T^{-i}(\zeta) = \mathcal{B}\,(\mu \bmod 0),$$

(ζ is a two-sided generator), then T is isomorphic to an (invertible) Bernoulli shift. It is clear that a measure-preserving endomorphism T is one-sided Bernoulli if and only if there exists an independent generating partition. The construction of the natural extension leads to a straightforward proof that the a weakly Bernoulli endomorphism has a weakly Bernoulli, hence Bernoulli natural extension (see e.g. [23]).

Weakly Bernoulli endomorphisms exhibit many highly mixing properties, and conditions under which piecewise smooth bounded-to-one interval maps are weakly Bernoulli were given by Ledrappier in [23].

When we endow the Riemann sphere $\mathbb{C}_\infty$ with the σ-algebra of Borel sets, we consider rational maps R and nonsingular Borel measures μ for R, supported on the Julia set $J(R)$ (see Defn 3.1) , such that R is a nonsingular d-to-1 endomorphism of $\mathbb{C}_\infty$, where d is the degree of the map. We only consider measures with respect to which R is d-to-one in the following measurable sense.

Definition 2.5. A nonsingular map T on $(X, \mathcal{B})$ is d-to-one with respect to a measure μ if there exists a partition $\zeta = \{A_1, A_2, A_3, \dots\}$ of X into d disjoint atoms of positive measure, called a *Rohlin partition*, and satisfying:

(1) the restriction of T to each A_i, which we will write as T_i, is one-to-one (μ mod 0);
(2) each A_i is of maximal measure in $X \setminus \cup_{j<i} A_j$ with respect to property 2;
(3) T_1 is one-to-one and onto X (μ mod 0) by numbering the atoms so that
$$\mu(TA_i) \geq \mu(TA_{i+1})$$
for $i \in \mathbb{N}$.

3. A family of ergodic and exact nonpolynomial maps with an invariant probability measure equivalent to ℓ

The stereographic projection map $\mathfrak{S}$ takes $\mathbb{C}_\infty \setminus \{\infty\}$ injectively onto $\mathbb{C}$, and circles containing $\{\infty\}$ and their arcs are mapped to lines and line segments under $\mathfrak{S}$. Moreover on these sets the measure $\mathfrak{S}_* m_1 \sim \ell$, therefore when we consider rational maps in $\mathbb{C}$ we use Lebesgue measure ℓ on $\mathbb{R}$ if the Julia set is a line or line segment. We know that the map p_0 preserves m_1, the map τ_2 preserves a probability measure $\mu \sim \ell$, and that both are one-sided $\{1/2, 1/2\}$ Bernoulli with respect to their invariant measures. Hence p_0 and τ_2 are measure theoretically isomorphic. We now give some examples of maps that are topologically conjugate to τ_2, but neither measure theoretically nor conformally conjugate to it. These examples were studied by the author and Barnes in [5] and their Bernoulli properties proved by the author and Bruin in [11].

We consider the family of maps on $([-2, 2], \mathcal{B}, \ell)$:

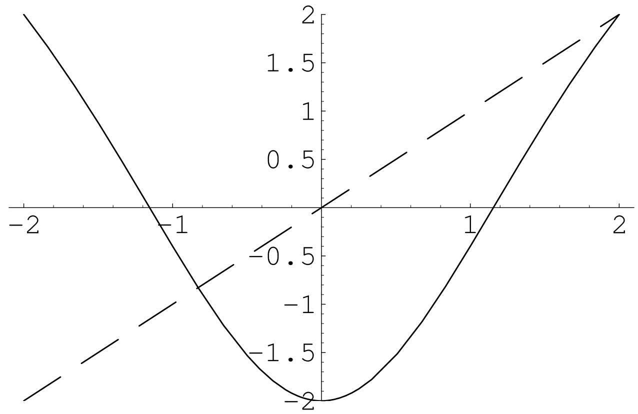

FIGURE 1. The graph of a nonBernoulli S-unimodal map I_a with the dashed line $y = x$

$$I_a(x) = \frac{-8 + (2 + 8a)x^2}{4 + (4a - 1)x^2},\ a \in (0, 1). \tag{4}$$

Proposition 3.1 below from [5], establishes that each map I_a, $a \in (0, 1)$ is a unimodal map in the sense that it is piecewise monotone with one turning point (the critical point at 0), and is has a finite postcritical orbit. When $a = \frac{1}{4}$, the map I_a is exactly p_{-2}, so isomorphic to τ_2. From [5] we list some properties, and show the graph of a typical map in Figure 1.

Proposition 3.1. For each $a \in (0, 1)$, and I_a given in (4),

- $I_a(-2) = I_a(2) = 2$, $I_a(0) = -2$, and I_a has one critical point at $x = 0$, is strictly decreasing on $[-2, 0)$ and strictly increasing on $(0, 2]$.
- $I_a''(0) = 8a \neq 0$.
- $I_a[-2, 2] = I_a^{-1}[-2, 2] = [-2, 2]$.
- $I_a'(-2) = -1/a$ and $I_a'(2) = 1/a$, so $x = 2$ is a repelling fixed point.
- Each I_a is finite postcritical (has a finite forward critical orbit).
- There is one other fixed point in $[-2, 2]$, namely $p = \frac{-2}{1+2\sqrt{a}} \in (-2, -2/3)$, with derivative $I'(p) = -(1 + 2\sqrt{a})$. That is, p is always repelling.

These properties imply the existence of an invariant probability measure $\mu_a \sim \ell$ [36], and by [23] these maps are weakly Bernoulli, hence ergodic and exact with respect to ℓ (see [5, 11] for further details.)

It is also the case that the Schwarzian derivative of I_a is negative; namely,

$$S(I_a) := \frac{R_a'''}{R_a'} - \frac{3}{2}(\frac{R_a''}{R_a'})^3 = \frac{-3}{2z^2} < 0,$$

but using ([36], Theorem D) the equivalent invariant probability measure follows from the fact that it is postcritically finite so this is not needed to prove the existence of an equivalent invariant measure.

In [11] it was proved that except for the $\{1/2, 1/2\}$ Bernoulli Chebyshev case, I_a is not one-sided Bernoulli. This is proved by showing that if I_a were $(\beta, 1-\beta)$ one-sided Bernoulli, then all periodic points of period k would need to have derivatives which are products of the form $\beta^i(1-\beta)^j$, with $i+j=k$, which cannot occur unless $\beta = 1/2$ and $a = 1/4$, the Chebychev case.

Proposition 3.2. The map I_a is not one-sided Bernoulli except w.r.t. m if $a = 1/4$, the Chebyshev polynomial. For all $a \in (0,1)$, I_a is ergodic, exact, and weak Bernoulli with respect to an invariant probability measure $\mu_a \sim \ell$.

Moreover, while the topological entropy of each map I_a, $a \in (0,1)$ is $\log 2$, only the map $I_{\frac{1}{4}}$ has $h_\ell(I_{\frac{1}{4}}) = \log 2$, and for $a \in (0,1) \setminus \frac{1}{4}$, preserving $\mu_a \sim \ell$, $h_{\mu_a} < \log 2$; this is discussed in [5]. The strict inequality on entropy follows from a result of Zdunik on rational maps ([40], Thm 1) in which it is shown that (when the Julia set is an interval) the unique measure of maximal entropy is singular with respect to ℓ except in the case of a Chebyshev polynomial.

Corollary 3.1. *The map $I_{\frac{1}{4}}$ is not measure theoretically isomorphic to I_a for any $a \in (0,1) \setminus \{\frac{1}{4}\}$.*

I_a as a map of $\mathbb{C}_\infty$. Up to now we have not treated the maps I_a as maps of the sphere; a natural question is to ask is: what happens to points $z \in \mathbb{C}_\infty \setminus [-2,2]$? It is not difficult to see that for $a \in (0,1)$, there is an attracting fixed point outside the interval $[-2,2]$; in particular, $q = \frac{2}{(2\sqrt{a}-1)}$ is fixed, and has derivative $I_a'(q) = -1 + 2\sqrt{a}$. The derivative $I_a'(q)$ is an increasing function of a and is in $(-1,0)$, so is attracting. We can view the fixed point q as a decreasing function of a, with a pole at $a = \frac{1}{4}$. It follows then that if $a \in (0, 1/4]$, $q \in [-\infty, -2)$ and if $a \in (1/4, 1)$, $q \in (2, \infty)$ From this and the general theory of complex dynamics (see e.g. [29]), we can deduce:

$$\forall z \in \mathbb{C}_\infty \setminus [-2,2], \quad \lim_{n\to\infty} I_a^n(z) = q$$

It is also natural to ask what happens if a is a complex number not in $(0,1)$. For this we recall the definition of a Julia set and Fatou set.

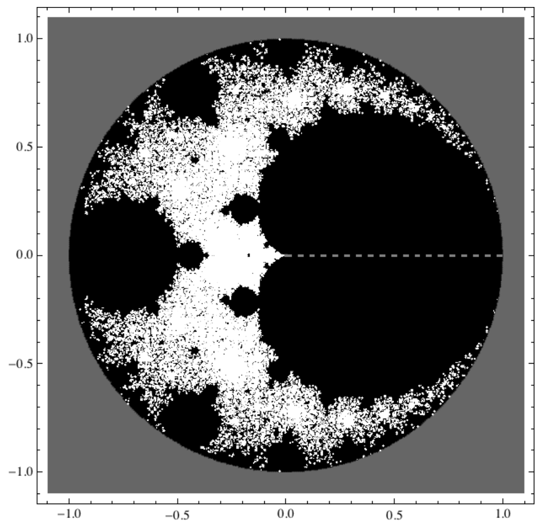

FIGURE 2. Parameter space for the maps I_a; the black parameters correspond to maps with attracting periodic orbits in $\mathbb{C}_\infty$ and outside the unit disk $J(I_a)$ is a Cantor set. The dotted line corresponds to the parameters

Definition 3.1. Let R be rational function of degree $d \geq 2$. The *Fatou set* of R is the open subset of $\mathbb{C}_\infty$ defined by:

$$\{z \in \mathbb{C}_\infty : \{R^n\} \text{ is equicontinuous at } z\}.$$

The *Julia set* of R is its complement in $\mathbb{C}_\infty$.

The terminology Julia set has been used for a long time - the first appearance in the literature seems to be around 1965 [33] - but the use of Fatou set for the set of normality only dates back to 1984 [8]. Earlier the Fatou set was usually referred to in the literature as the stable set and the set of normality. We write $F(R)$ and $J(R)$ for the Fatou and Julia sets respectively.

Remark 3.1. There are other versions of the maps I_a discussed in [5]; we mention two here.

(1) The maps $f_a(z) = a(z + 1/z + 2)$, $a \in (0, 1)$ have $J(f_a) = [-\infty, 0]$. Each f_a is conformally conjugate to I_a via the linear fractional transformation $M(z) = \frac{2(z+1)}{z-1}$.
(2) The so-called modified Boole maps are defined by:

$$b_a(x) = \sqrt{a}(x - 1/x),$$

taking the positive square root of $a \in (0,1)$. Each f_a is a factor of the modified Boole map b_a, via the two-fold branched covering map $\phi(x) = -x^2$.

In Figure 2 we see that the parameters $a \in (0,1)$ form the axis of a cardioid; this parameter space appeared in ([28], Fig. 11) and has been studied by the author in [18]. In particular the black areas of the parameter space represent open sets of parameters where an attracting periodic orbit exists. There are other areas of Fig. 2 that give maps of great interest in the study of Lebesgue dynamics. For example, we study a map corresponding to a non-real parameter, whose Julia set is the entire sphere, in Section 4. We now turn to the parameter $a = -1$, which generalizes to another parametrized family of quadratic maps, one whose only intersection with the space shown in Fig. 2 is that one point.

Lemma 3.2. *We define the map $\mathcal{I}(z) \equiv I_{-1}(z)$. We have $\mathcal{I} : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \setminus (-2, \frac{6}{5})$, and $\mathcal{I}$ has the following properties as a rational map on $\mathbb{C}_\infty$:*

(1) *$\mathcal{I}(z) = \frac{-(8+6z^2)}{4-5z^2}$ has fixed points at 2 and $-\frac{2}{5} \pm \frac{4}{5}i$.*
(2) *$\mathcal{I}'(2) = -1$ and the other two fixed points are repelling.*
(3) *There are no period 2 cycles for $\mathcal{I}$; i.e., the fixed points of $\mathcal{I}^2$ are $2, 2, 2, -\frac{2}{5} + \frac{4}{5}i$ and $-\frac{2}{5} - \frac{4}{5}i$.*

Proof. One can check the fixed points and their derivatives by using Prop 3.1. Solving for period 2 points of $\mathcal{I}$ is equivalent to finding the roots of the polynomial $j(z) = 5z^5 - 26z^4 + 40z^3 - 16z^2 + 16z - 32$; using long division one can easily determine that $j(z) = (z-2)^3(5z^2+4z+4)$, from which the result follows. □

It is also stated in ([28], caption of Figure 6) that a quadratic rational map has a fixed point with derivative -1 if and only if it has no period two orbits. This is proved in [17] along with a complete classification of which rational maps of degree d lack periodic orbits of period k; most of these maps result in nonsmooth Julia sets. However the quadratic family contains some examples of interest. The study of rational maps missing periodic orbits of certain periods goes back to Baker [3] and examples are given in [7]. Hagihara gave a complete list of such maps in her Ph.D. thesis [17].

Corollary 3.3. [17] *The map $\mathcal{I}$ is conformally conjugate to a map of the form:*

$$S_\alpha(z) = \frac{z^2 - z}{1 + \alpha z} \tag{5}$$

for some $\alpha \in \mathbb{C} \setminus \{1\}$.

Proof. It is proved in [17] that every quadratic map with no period 2 cycles is conformally conjugate to a map of the form (5). □

While $J(\mathcal{I})$ is fractal and does not have nice Lebesgue measurable properties (meaning with respect to m_2, m_1 or ℓ), the family of maps S_α, which includes a conformally conjugate copy of $\mathcal{I}$, leads us to our discussion about infinite measure preserving maps.

3.1. **A family of ergodic and exact rational maps with an infinite σ-finite invariant measure equivalent to ℓ.** In this section we turn to infinite Lebesgue measure ℓ on $\mathbb{R}$. We have so far looked at families of finite measure preserving examples. There are maps with smooth one-dimensional Julia sets with the property that they preserve an infinite ergodic and exact σ-finite measure equivalent to m. Therefore they cannot preserve an equivalent ergodic finite measure so are fundamentally different from the family given in Eqn (4). We now turn to the family of maps from Eqn (5): $S_\alpha(z) = \frac{z^2-z}{1+\alpha z}$, and we are interested only in $\alpha < -1$ for now.

The next modification we make is to conjugate S_α by the involution $\phi(z) = 1/z$, which gives us a map of the form:

$$U_{1+\alpha}(z) = -z - \frac{1+\alpha}{z-1} - (1+\alpha). \tag{6}$$

If $\alpha < -1$, then $U_{1+\alpha}$ is called a negative R-function; it can also be thought of as the square root of an inner function in the following sense. Recall that an inner function is a map f of the upper half plane satisfying: $\lim_{y\to 0} f(x+iy) = f(x)$, for ℓ a.e. $x \in \mathbb{R}$. The dynamics of inner functions have been studied extensively by Aaronson [1], along with the criteria supplied by Letac [25] on when they (and rational negative R-functions) preserve the infinite measure ℓ on $\mathbb{R}$. It is also directly verifiable that U_α preserves Lebesgue measure on $\mathbb{R}$. Rational maps that are negative R-functions flip the upper and lower half plane, but every even iteration is an inner function, so $U^2_{1+\alpha}$ is an inner function. To simplify notation, we write $b = 1+\alpha$, and we restrict to $\alpha \in [-3,-1)$ (see [7, 17]); for any $\alpha < -3$, S_α is conformally conjugate to $S_{\frac{3-\alpha}{1+\alpha}}$; then $b \in [-2, 0)$. For the next two theorems we need to recall the following concepts from infinite measure dynamics.

Remark 3.2. (1) Since U_b preserves ℓ on $\mathbb{R}$, (for $b < 0$), and ℓ is infinite, for any fixed $c > 0$, U_b also preserves the scaled measure given by $(c\,\ell)(A) \equiv c \cdot \ell(A)$ for all measurable A. We say the maps U_{b_1} and U_{b_2} are *c-isomorphic* if for some $c > 0$, if there exists a measurable isomorphism $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi_* \ell \equiv \ell \circ \phi^{-1} = (c\,\ell)$, and such that $\phi \circ U_{b_1} = U_{b_2} \circ \phi$ ℓ - a.e.

(2) Assume $A \subset \mathbb{R}$ satisfies $0 < \ell(A) < \infty$ and $\ell(\mathbb{R} \setminus \cup_{i=0}^{\infty} U_b^{-i}(A)) = 0$. Then replacing U_b by U for simplicity of notation, by U_A we mean the map on A given by: $U_A(x) = U^{p(x)}(x)$ where $p(x) = \min\{i : U^i(x) \in A\}$.

(3) Since U preserves ℓ, (2) defines a measure preserving endomorphism $U_A : (A, \mathcal{B} \cap A, \ell|_A) \to (A, \mathcal{B} \cap A, \ell|_A)$ where $\ell|_A$ denotes the (non normalized) restriction of ℓ. We call U_A the *induced map on A*.

We have the following results from the Ph. D. thesis of Bayless [6]; (1) was shown in [17].

Theorem 3.4. [6] *For $b \in [-2, 0)$, for the map $U_b(z) = -z - b + \frac{b}{1-z}$, the following hold with respect to ℓ:*

(1) $J(U_b) = \mathbb{R} \cup \{\infty\}$;
(2) *U_b is conservative, exact, and ergodic;*
(3) *For each U_b, there exists a set $A \in \mathcal{B}$ of finite measure such that for m-a.e. $x \in \mathbb{R}$ there exists a $p(x) \in \mathbb{N}$ such that $U_b^{p(x)}(x) \in A$, and such that the first return time partition (to A) of $\mathbb{R}$ has finite entropy (i.e., U_b is said to be quasi-finite).*

3.1.1. *Entropy on infinite spaces.* We assume the reader has familiarity with the notion of measure theoretic entropy on probability spaces, and if not, a book by Walters ([38], Chap 4) is a good source. When dealing with infinite invariant measures, there is an issue of scaling by a constant, which can change the value of invariants; here we use ℓ on $\mathbb{R}$ so $\ell([0, 1]) = 1$. The earliest definition of entropy of an infinite measure preserving transformation is due to Krengel [**?**] and is given as follows:

Definition 3.2. If $T : (X, \mathcal{B}, m) \to (X, \mathcal{B}, m)$ is a conservative measure-preserving system, then we define

$$h_{\mathrm{kr}}(T) = \sup h(T_A, m_A),$$

where the supremum is take over all $A \in \mathcal{B}$ satisfying $0 < m(A) < \infty$. We write Krengel entropy of U_b with respect to Lebesgue measure on $\mathbb{R}$, m, as $h_{\mathrm{kr}}(U_b)$.

From Theorem 3.4 the Krengel entropy of each U_b is defined, and can be calculated for the maps U_b.

Theorem 3.5. [6] *For each* $b \in [-2, 0)$,

$$h_{\mathrm{kr}}(U_b) = \int_{\mathbb{R}} \log |U_b'(t)| d\ell(t) = 2\pi\sqrt{-b}.$$

Moreover Bayless showed that every map of the form: $f(z) = -z - \beta - \dfrac{p}{t-z}$, with $\beta, t \in \mathbb{R}$, and $p > 0$ (i.e., any quadratic rational negative R-function), is c-isomorphic to exactly one map in the family U_b, and U_b and $U_{b'}$ are in turn mutually non-c-isomorphic for $b, b' \in [-2, 0)$.

We finish this section by mentioning that this family of mappings is embedded in a much more complicated parameter space studied by Hagihara in [17]. The parameter space resembles "the usual" ubiquitous Mandelbrot set (missing the period 2 limb due to Corollary 3.3), and is difficult to produce due to the neutral fixed point. In Figure 3 the dark unit disk centered at the origin shows parameters corresponding to maps S_α with an attracting fixed point at ∞, while the white dotted line shows parameters corresponding to maps S_α with Julia set the real line. There is a hole at $\alpha = -1$ where S_α does not have degree 2. We refer to a parameter space as *reduced* when no two points in parameter space represent conformally equivalent maps. Figure 4 shows the reduced Mandelbrot set, arising for quadratic polynomials of the form $q_\kappa = \kappa z^2 + z$ (each of which is conformally conjugate to exactly one map p_c).

4. Two dimensional Lebesgue measure

We now turn to maps whose Julia set is two dimensional, and write m_2 for the normalized surface area measure on $\mathbb{C}_\infty$. We are interested in rational maps whose Julia set is all of $\mathbb{C}_\infty$ so that we can analyze their m_2 dynamical properties. Examples of this type are very old and it is natural they would predate computers. In fact there are many one-sided Bernoulli maps on the sphere that date back to Lattès in 1918. Milnor has written a clear and comprehensive exposition of Lattès' work along with updated examples, so we do not repeat that here, but refer the reader to the article [31]. Instead we illustrate the basic construction of these examples by giving a parametrized family of such maps (2 complex dimensions) that does not appear explicitly in [31] (though it can certainly be found there implicitly). Nonpolynomial Lattès maps are characterized by the property that the unique invariant probability measure of maximal entropy is equivalent to m_2 [40].

The idea behind a Lattès map in measure theoretic terms is to start a uniformly expanding map of the torus, viewed as $\mathbb{C}/\Lambda$ with $\Lambda \subset \mathbb{C}$ a lattice; usually the map is as simple as $z \mapsto nz$, with $n \geq 2$ an integer.

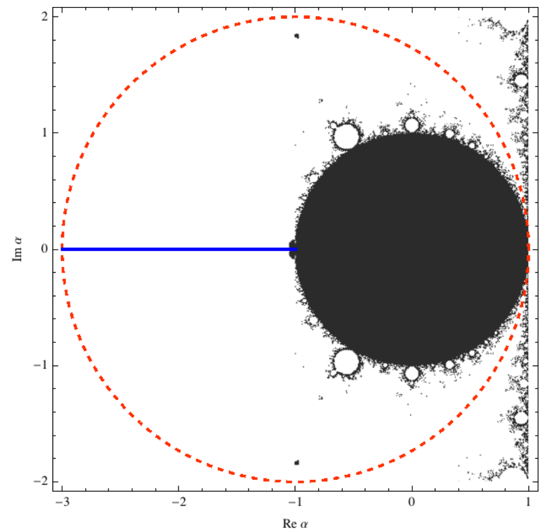

FIGURE 3. Parameter space for the maps S_α, equivalently U_b, with ℓ- preserving maps from Thm 3.4 marked by a solid (blue) line. The dashed (red) circle encloses the reduced parameter space.

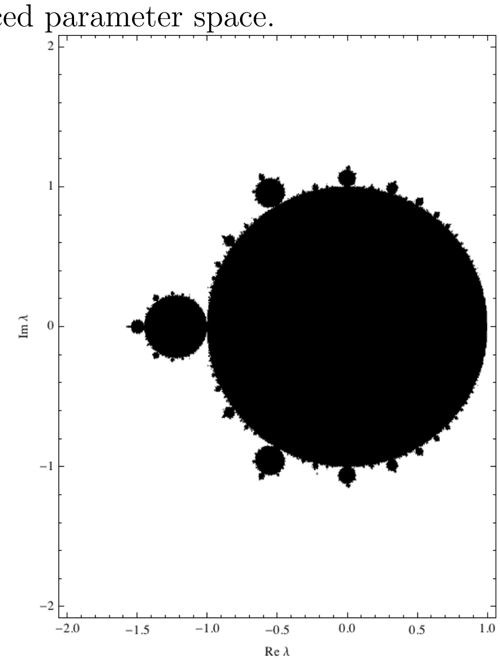

FIGURE 4. Reduced parameter space for the maps $q_\kappa(z) = \kappa z + z^2$, known as the Mandelbrot set

Such a map is easily seen to be ergodic, preserves ℓ^2, two dimensional Lebesgue measure on $\mathbb{C}/\Lambda$, and the generating partition to make it one-sided Bernoulli (i.e., isomorphic to the one-sided $\{\frac{1}{n^2}, \frac{1}{n^2}, \ldots, \frac{1}{n^2}\}$ shift) is obvious. We then find a suitable factor map from the torus to the sphere, and look at the map generated on $\mathbb{C}_\infty$. If the factor map is finite-to-one, the pullback measure will inherit the Bernoulli measurable properties.

We also discuss postcritically finite maps that are not Lattès maps, and give the idea behind the result that such maps admit an ergodic finite invariant measure equivalent to m_2. There are many open questions about their measure theoretic entropy as well as the Hausdorff dimension of the maximal entropy measure whose support is necessarily the entire sphere.

4.1. **A family of Lattès examples of degree** 4. We begin with a complex torus: Let $\lambda_1, \lambda_2 \in \mathbb{C}\backslash\{0\}$ with $\lambda_2/\lambda_1 \notin \mathbb{R}$. A lattice is defined by $\Lambda = [\lambda_1, \lambda_2]\colon = \{m\lambda_1 + n\lambda_2 \colon m, n \in \mathbb{Z}\}$; Λ is a discrete additive subgroup of $\mathbb{C}$ of rank 2, so $\mathbb{C}/\Lambda$ is a torus. Different sets of vectors can generate the same lattice Λ, but if $\Lambda = [\lambda_1, \lambda_2]$, and $\Lambda = [\lambda_3, \lambda_4]$, the generators are related by

$$\begin{pmatrix} \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = \pm 1$. Therefore given any lattice $\Lambda = [\lambda_1, \lambda_2]$, we interchange the order if needed, so that

$$\tau = \lambda_2/\lambda_1 \in \mathbb{H} = \{z : z = x + iy, \, y > 0\}.$$

It is useful to define an equally proportioned lattice $\Omega = [1, \tau]$, and set

$$\Lambda = k\Omega = k[1, \tau], \quad \tau \in \mathbb{H}, \quad k \neq 0. \tag{7}$$

A simple map such as $L(z) = \alpha z$, with the property that $\alpha\Lambda \subset \Lambda$ and $|\alpha| > 1$ defines a uniformly expanding map on $\mathbb{C}/\Lambda$ which gives rise via a meromorphic factor maps to a rational map of $\mathbb{C}_\infty$ of degree α^2. We use $n = 2$ in the family of examples in Theorem 4.1, and integers $n \geq 2$ are the easiest to work with; however there are infinitely many noninteger examples, and in Figure 2, at the parameter $a = -\frac{1}{4}$ we find a Lattès example obtained by using $\Lambda = [1, i]$ and $\alpha = \sqrt{2}$.

The factor map we use for these examples is an elliptic function, which by definition is a meromorphic function in $\mathbb{C}$ which is periodic with respect to a lattice Λ. In our examples we use the Weierstrass elliptic $\wp$ function, defined by

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{w\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right), z \in \mathbb{C}.$$

Replacing every z by $-z$ in the definition we see that $\wp_\Lambda$ is an even function, $\wp_\Lambda$ is meromorphic, periodic with respect to Λ, and each pole, occurring exactly at lattice points, has order 2. The function $\wp$ exhibits some characteristics of trigonometric functions; it is periodic (with respect to a rank 2 subgroup of $\mathbb{C}$ rather than rank 1), it is related to its own derivative via a simple differential equation (see Eqn (8) below), and we have an "angle doubling" formula (see Eqn (12) below).

4.1.1. **ODEs for $\wp_\Lambda$ and its derivatives:** We have

$$\wp_\Lambda'(z)^2 = 4\wp_\Lambda(z)^3 - g_2\wp_\Lambda(z) - g_3, \tag{8}$$

where

$$g_2(\Lambda) = 60 \sum_{w\in\Lambda\setminus\{0\}} \frac{1}{w^4} \quad \text{and} \quad g_3(\Lambda) = 140 \sum_{w\in\Lambda\setminus\{0\}} \frac{1}{w^6}. \tag{9}$$

Remark 4.1. We call (g_2, g_3) *invariants of* Λ ([13], Chap 2.22). In particular $g_2(\Lambda)$ and $g_3(\Lambda)$ are complete invariants of the lattice Λ since for any g_2 and g_3 such that $g_2^3 - 27g_3^2 \neq 0$ there exists a unique lattice having g_2 and g_3 as its sums in Eqn(9), ([13], Chap 2.11, and [21], Cor. 6.5.8), or equivalently, the pair (g_2, g_3) determines completely the value of $\wp_\Lambda(z)$ for all $z \in \mathbb{C}$. Moreover the invariants g_2, g_3 depend analytically on Λ in the sense that they vary analytically in $\tau \in \mathbb{H}$ when represented as in Eqn (7) (see [21], Thm 6.4.1), and from Eqn (9) it follows that

$$g_2(k\Omega) = k^{-4}g_2(\Omega), \text{ and } g_3(k\Omega) = k^{-6}g_3(\Omega). \tag{10}$$

Analogous to sine and cosine functions, we also have a second order ODE connecting the second derivative to the original function:

$$\wp_\Lambda''(z) = 6\wp_\Lambda^2(z) - \frac{g_2}{2}. \tag{11}$$

4.1.2. **Angle Doubling Formula for $\wp_\Lambda$:**

$$\wp_\Lambda(2z) = \frac{\wp_\Lambda''(z)^2}{4\wp_\Lambda'(z)^2} - 2\wp_\Lambda(z) \tag{12}$$

These classical identities and many others are worked out in detail in [13] or [21]. Putting together Eqns. (8) - (12), via the commuting diagram below, we obtain a rational map of the sphere of the form:

$$R(z) = \frac{z^4 + (g_2/2)z^2 + 2g_3 z + g_2^2/16}{4z^3 - g_2 z - g_3} \tag{13}$$

depending on $\Lambda(g_2, g_3)$.

$$\begin{array}{ccc} (\mathbb{C}/\Lambda, \mathcal{B}, \ell^2) & \xrightarrow{2z} & (\mathbb{C}/\Lambda, \mathcal{B}, \ell^2) \\ \downarrow \wp_\Lambda & & \downarrow \wp_\Lambda \\ \mathbb{C}_\infty & \xrightarrow{R} & \mathbb{C}_\infty \end{array}$$

Since $\wp_\Lambda$ is meromorphic, the pullback measure on $\mathbb{C}_\infty$, $(\wp_\Lambda)_*\ell^2$, is equivalent to m_2. We are now ready to state our theorem, and give a brief proof.

Theorem 4.1. *Given any* 3 *distinct complex numbers,* a, b, c *such that* $a + b + c = 0$*, there exists a rational map of degree* 4 *of the form:*

$$R(z) = \frac{z^4 + (\alpha/2)z^2 + 2\beta z + \alpha^2/16}{4z^3 - \alpha z - \beta} \tag{14}$$

with $\alpha, \beta \neq 0$*, such that:*

1. $J(R) = \mathbb{C}_\infty$
2. *The critical values of* R *are exactly* a, b*, and* c*, each of multiplicity* 2*.*
3. *The critical values get mapped to a fixed point at* ∞*, which is repelling.*
4. *There are 6 distinct critical points for* R*, and they sum to* 0*.*
5. *There exists a lattice* Λ *such that* $\wp_\Lambda$ *induces the map* R *from a toral endomorphism on* $\mathbb{C}/\Lambda$*;*
6. *The map* R *is isomorphic to the one-sided* $\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\}$ *Bernoulli shift with respect to the invariant probability measure* $\mu = (\wp_\Lambda)_*\ell^2 \sim m_2$*.*

Proof. Given a, b, c we reorder them if necessary so that $a \neq 0, b \neq 0$; recall that at most one can be 0, so we assign that value to c if it occurs. We next set $p(z) = (z-a)(z-b)(z-c)$ and write it as:

$$p(z) = (z-a)(z-b)(z-(-a-b)) = z^3 - (a^2+ab+b^2)z + (a^2b+ab^2).$$

Using the angle doubling identities we get: $\alpha = 4(a^2+ab+b^2), \quad \beta = -4(a^2b + ab^2)$ to obtain

$$R(z) = \frac{z^4 + (\alpha/2)z^2 + 2\beta z + \alpha^2/16}{4z^3 - \alpha z - \beta}$$

While the statements about the critical values follow from some identities, in this case they can also be verified by hand. In particular, the 6 critical points for the map R are: $a \pm \sqrt{2a^2 - ab - b^2}$, $b \pm \sqrt{-a^2 - ab + 2b^2}$, and $c \pm \sqrt{2a^2 + 5ab + 2b^2}$, which sum to 0. The discriminant given for each critical point does not vanish because a, b, and c are distinct. Using $g_2 = \alpha$ and $g_3 = \beta$, we find a unique lattice $\Lambda(g_2, g_3)$ which we use for the covering torus. $\square$

It is not hard to vary the theorem a little in order to specify that $c_1, c_2 = \pm q \in \mathbb{C}$ be (distinct) critical points, instead of specifying the critical values. Choosing $g_2 = 4(\frac{q}{1+\sqrt{2}})^2$ and $g_3 = 0$ in (13) will work.

4.2. **An example illustrating Theorem 4.1.** In Theorem 4.1, we choose any nonzero $a \in \mathbb{C}, b = -a$, and $c = 0$. Following the proof, we see that $\alpha = 4(a^2+ab+b^2) = 4a^2$ (which is g_2), and $\beta = -4(a^2b+ab^2) = 0$ (which is g_3), yielding the rational map:

$$F(z) = \frac{z^4 + 2a^2z^2 + a^4}{4z(z^2 - a^2)} \tag{15}$$

It is a calculation to show that the 6 critical points for F are: $\pm ai, a(1 \pm \sqrt{2}), a(-1 \pm \sqrt{2})$. The critical values are as follows: $F(ai) = F(-ai) = 0$. One can substitute the other critical points and see that

$$F(a+a\sqrt{2}) = F(a-a\sqrt{2}) = a \text{ and } F(-a+a\sqrt{2}) = F(-a-a\sqrt{2}) = -a;$$

so we recover the three distinct critical values. It is then easy to see that each of the critical values has multiplicity 2 and is a pole, so is mapped under f to $\infty \in \mathbb{C}_\infty$.

The four fixed points for F that are in $\mathbb{C}$, are:

$$\pm ai\sqrt{-1 + \frac{2}{\sqrt{3}}}, \quad \pm a\sqrt{1 + \frac{2}{\sqrt{3}}},$$

and $F'(x) = -2$ for each of the fixed points. We note that ∞ is also a fixed point, and we calculate its multiplier, the analog of a derivative at ∞ when $\mathbb{C}_\infty$ is viewed as a compact surface. This is the derivative of: $z \mapsto 1/f(\frac{1}{z})$ at the origin, which can be shown to be 4.

Moreover every periodic point of period k, has derivative $\pm 2^m$ for some positive integer m, and in this example $m = k$ or $m = 2k$ for each k. This is a characteristic of Lattès examples, that their periodic points have derivatives of a particular form, but not verifiable unless you know how you arrived at the example (in our case, by multiplication by 2 on the torus, and then used $\wp$ as the factor map). This is discussed in ([31], Cor. 3.9).

Using a result from ([21], Chapter 6) we see that when $b = -a$, the underlying lattice Λ coming from the Theorem 4.1 is of the form $\Lambda = [\lambda, \lambda i]$, $\lambda \neq 0$ satisfying

$$\lambda = \frac{1}{\sqrt{a}} \cdot \gamma,$$

where $\gamma > 0$ is the lemniscate constant which arises as the lattice side length for invariants $g_2 = 4, g_3 = 0$. Many details behind the constant γ and the invariants g_2, g_3 are discussed in [19].

4.3. **Postcritically finite rational maps.** The Lattès examples are special cases of postcritically finite rational maps. By definition, a rational map is *postcritically finite* if every critical point is preperiodic but not periodic. This excludes maps like $\mathfrak{X}_d(z) = z^d$, where 0 and ∞ are both fixed critical points on $\mathbb{C}_\infty$, but includes maps like the one in Example 4.2 and more general maps. We define the postcritical set of R by:

$$P(R) = \overline{\bigcup_{c \in C(R),\, n \in \mathbb{N}} R^n(c)}, \tag{16}$$

where $C(R)$ is the set of critical points of R; by definition $P(R)$ is closed and may or may not contain points from $C(R)$.

The reason a postcritically finite map R is of interest in Lebesgue ergodic theory is that $J(R) = \mathbb{C}_\infty$; a proof of this can be found in many places (see [7], Thm 9.4.4), but the rough idea is that any time there is a Fatou component, it maps eventually to a periodic component which must have a critical point associated to it (either in the Fatou set or with an infinite forward orbit, or both). This means that it is worth considering the measure theoretic structure of a postcritically finite map R with respect to the 2-dimensional probability measure m_2, as each such map is always topologically transitive (see eg. [7], Thm 4.2.5). The measurable dynamics of postcritically finite maps have been extensively written about (eg., [4, 14, 18, 27, 34]) and generalized to other settings (meromorphic postcritically finite, entire postcritically finite, and rational maps not postcritically finite but "close" to one). Here we review only a few of the basic measure theoretic results.

Remark 4.2. (1) Assume that $P(R) = \{a_1, a_2, \ldots, a_k\}$ with each $a_j \notin C(R)$; it follows that $|P(R)| \geq 3$ ([27], Thm 3.6). To each a_j we assign the positive integer ν_j which is the least common multiple of the local degrees $\deg(R^k, y)$, for all $k > 0$ and y such that $y \in R^{-k}(x)$. We then have a set of ramification indices: $\mathcal{N} = \{\nu_1, \nu_2, \ldots, \nu_k\}$ with each $\nu_k \geq 2$. This leads to a metric which has a finite set of singularities for an otherwise smooth Riemannian metric $\rho = \gamma(z)|dz|$ on $\mathbb{C}_\infty$, and the singularities are of the type $\rho = \dfrac{|dz|}{|z|^{1-1/\nu_j}}$ in local coordinates near each a_j.

(2) The pair $(\mathbb{C}_\infty \setminus P(R), \mathcal{N})$, with a complex atlas structure on $\mathbb{C}_\infty \setminus P(R)$, is called the *orbifold of* R and the metric ρ is its associated *orbifold metric*.

An application of the Schwarz lemma to the covering space of the orbifold $\mathbb{C}_\infty \setminus P(R)$, which is the disk with the hyperbolic metric since $|P(R)| \geq 3$, gives the following result. Details can be found in any one of these sources ([14, 27, 29])

Theorem 4.2. [14] *A postcritically finite map R is expanding with respect to its orbifold metric and $||R'(z)|| > \lambda > 1$ for all $z \in \mathbb{C}_\infty \setminus P(R)$.*

The next result is a compendium and a simplification of results by many authors. We cite a few places where the statements can be found, more or less as stated here; most of these results were shown by Rees [34].

Theorem 4.3. *If R is a postcritically finite rational map then the following hold.*

(1) *R is ergodic, exact, and conservative with respect to m_2* [4, 27, 34]*;*
(2) *R admits an invariant probability measure $\nu \sim m_2$* [34]*.*
(3) *Let $\mathfrak{M}$ be a connected 1 complex dimensional manifold, and suppose $\{R_a, a \in \mathfrak{M}\}$ is a parametrized family of rational mappings that vary holomorphically in a; assume there exists some $a_0 \in \mathfrak{M}$ such that R_{a_0} is postcritically finite. If R_a satisfies a nondegeneracy condition (given in Remark 4.3 below), then there exists a set of parameters of positive measure in $\mathfrak{M}$ such that (1) and (2) hold* [34]*.*

Remark 4.3. (1) Polynomials never satisfy the hypotheses of Theorem 4.3 since the point at ∞ is a fixed critical point for every polynomial. The examples and results from Sec. 3.1, S_α and U_b, can never be post critically finite either since a parabolic fixed point forces a map to have an infinite forward orbit (see [7], Thm 9.3.2).

(2) The nondegeneracy condition in Theorem 4.3 is described as follows. Fix the postcritically finite parameter a_0. By considering a higher iterate of R_a if necessary, assume each critical orbit terminates in a (necessarily repelling) fixed point. Assume the map

R_{a_0} has critical points $c_1(a_0), \ldots, c_m(a_0)$, $m \leq 2d-2$. Write the fixed point at the end of the orbit of $c_i(a_0)$ as: $R^{s_i}(c_i(a_0)) = y_i(a_0) = R_{a_0}(y_i(a_0)) = z_i(a_0)$ for each $i = 1, \ldots, m$, where $s_i = \min\{k : R^k(c_i(a_0)) \text{ is fixed}\}$.

Each of these points moves holomorphically in $a \in \mathfrak{M}$. The nondegeneracy condition is that the function:

$$F_i(a) = R^{s_i+1}(c_i(a)) - R^{s_i}(c_i(a))$$

should satisfy:

$$\frac{DF_i}{Da}(a_0) \neq 0, \tag{17}$$

or equivalently,

$$\frac{Dz_i}{Da}(a_0) - \frac{Dy_i}{Da}(a_0) \neq 0.$$

(3) We give an example of a family of maps satisfying the nondegeneracy condition, from [34]. If $R_a(z) = a\dfrac{(z-2)^2}{z^2}$, with $a \in \mathbb{C} \setminus \{0\}$ then the critical points are $c_1 = 0$ and $c_2 = 2$, and the critical orbits look like:

$$2 \mapsto 0 \mapsto \infty \mapsto a \mapsto \frac{(a-2)^2}{a} \cdots$$

We set $R_a(a) = a$, and note that $R_a'(a) = \frac{4(a-2)}{a^2}$. Therefore choosing $a_0 = 1$, yields both critical orbits terminating at a repelling fixed point at 1. We can check the condition in Eqn (17) by brute force to see that both resulting derivatives yield $-4 \neq 0$.

(4) The family of maps shown in Figure 2, which we write as: $f_a(z) = a(z + 1/z + 2)$, has a fixed critical orbit at $c_1 = -1$. Namely $-1 \mapsto 0 \mapsto \infty$ for all nonzero a. The multiplier of the fixed point at ∞ is $1/a$, so as long as a stays in the open unit disk D of parameters, this is a repelling fixed point. Therefore we only look at the second critical point, $c_2 = 1$ to check the nondegeneracy condition given by (17); the purpose of Condn (17) in [34] is to estimate the measure of the set of parameters on which both critical orbits "stay far enough away from" critical points, which is unaffected by only checking for c_2. If we set $a_0 = \frac{-3+\sqrt{7}i}{8}$, we see that under f_{a_0},

$$1 \mapsto \frac{-3+\sqrt{7}i}{2} \mapsto \frac{-3-\sqrt{7}i}{8},$$

and $z_0 = \frac{-3-\sqrt{7}i}{8}$ is fixed. Moreover $|f'_{a_0}(z_0)| = |1/a_0| = 2$.

Using the notation in (1),

$$F_2(a) = f_a^2(1) - f_a(1) = \frac{(1+4a)^2}{4} - 4a = \frac{(1-4a)^2}{4},$$

and $\frac{DF_2}{Da}(a_0) = -5 + \sqrt{7} \neq 0$ when $a_0 = \frac{-3+\sqrt{7}i}{8}$. This implies that the family of map I_a, defined by Eqn (4), when parametrized over $a \in \mathbb{C}$ contains a set of parameters of positive Lebesgue measure for which $J(I_a) = \mathbb{C}_\infty$ and for which Thm 4.3 holds. This is mentioned in [18].

(5) Apart from the Lattès examples we do not look for parametrized families of rational maps whose Julia set is the whole sphere; they are very unstable mappings, and in a neighborhood of a parameter of a postcritically finite map one usually sees other similar maps as well as tiny Mandelbrot sets signaling the presence of maps with attracting orbits (see [18, 27] for example).

4.4. **Recent results: fractal Julia sets of positive m_2 measure.** In this section we review some recent results about the existence of quadratic polynomials whose Julia sets have positive m_2 measure [12]. We have a limited goal here of giving the reader an idea of how a positive Lebesgue measure Julia set could possibly arise. This result, Theorem 4.4 below, is quite remarkable and represents a creative and technical leap forward in the field. Like many good ideas, the basic construction can be explained in simple terms. And also like many significant breakthroughs the process of arriving at the construction involved many people, difficult technical tools (e.g., [20]), and a lot of groundwork before the key idea fell into place. The result appears in a paper by Buff and Cheritat [12] but their introduction mentions many contributions by others. Background material for this section can be found in [29].

We start by recalling the familiar definition of a filled Julia set of a polynomial map $p(z)$: namely,

$$K(p) = \{z \in \mathbb{C} : \{p^n(z)\}_{n\in\mathbb{N}} \text{ is bounded}\} \tag{18}$$

We consider the family of quadratic polynomials parametrized as:

$$q_\kappa(z) = \kappa z + z^2, \quad \kappa \in \mathbb{C}. \tag{19}$$

In this form we see immediately that 0 is a fixed point with multiplier κ. For the construction we are only interested in $|\kappa| = 1$, and more precisely in $\kappa = e^{2\pi i\xi}$ with ξ an irrational number, so we replace the notation $q_{e^{2\pi i\xi}}$ with q_ξ. We refer to 0 as an irrationally indifferent fixed point, and these points are of two types: either 0 is a *Siegel point* if there exists a local holomorphic change of coordinate $z = \phi(w)$ such that:

$$q_\xi \circ \phi(w) = \phi \circ e^{2\pi i \xi} w$$

for all w near the origin, or it is a *Cremer point* if no such local map exists. While irrational numbers giving Siegel points for q_ξ have full Lebesgue measure in $[0,1]$, Cremer point values of ξ form a dense G_δ set. The rotation numbers that occur for Siegel disks are relatively poorly approximable by rational numbers while the Cremer numbers are well approximable. Siegel points give rise to maps that have a Fatou component containing 0, so that q_ξ is locally conjugate to rotation through $2\pi\xi$ on a disk. In Figure 5 we show a Siegel disk. By contrast, Cremer points are always in the Julia set of q_ξ.

In general, if $J(p)$ is connected, $\partial K(p) = J(p) \subsetneq K(p)$, but it can happen, as is shown in Figure 6 that $K(p)$ has empty interior and is $J(p)$. If one starts with irrational numbers that have Siegel disks, and then takes a sequence ξ_n that has been carefully chosen so that the Lebesgue measure of $K(q_{\xi_{n+1}})$ stays close to the measure of $K(q_{\xi_n})$, then a limiting value ζ might appear which is not a Siegel number, but $m_2(K(q_\zeta)) > 0$ and $K(q_\zeta) = J(q_\zeta)$. This is roughly how the construction goes, after showing that the map $\xi \mapsto m_2(K(q_\xi))$ is upper semicontinuous. In other words, the authors are able to construct $\xi_n \to \zeta$ with these (and more) properties.

Theorem 4.4. [12]

- *There exist quadratic polynomials that have a Cremer fixed point and a Julia set of positive m_2 measure.*
- *There exist quadratic polynomials which have a Siegel disk and a Julia set of positive m_2 measure.*

Some topological properties of the Julia sets resulting from this work have been published (see [9] for example.) Since there exists a unique invariant probability measure μ supported on the Julia set such that $h_\mu(p) = \log 2$, and it must be singular with respect to m_2 [40], the Lebesgue measure properties of these maps are still largely unknown.

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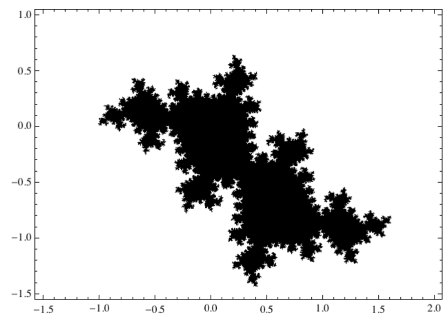

FIGURE 5. The map $q_{\sqrt{7}}(z) = e^{-2\pi i\sqrt{7}}z + z^2$ has a filled Julia set that is a Siegel disk centered at the fixed point at $z = 0$.

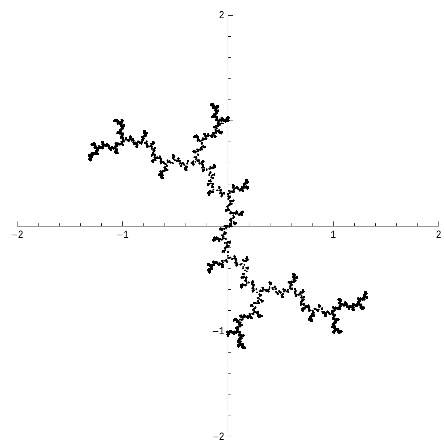

FIGURE 6. For the map $p_{-1}(z) = z^2 + i$, $K(p_i) = J(p_i)$

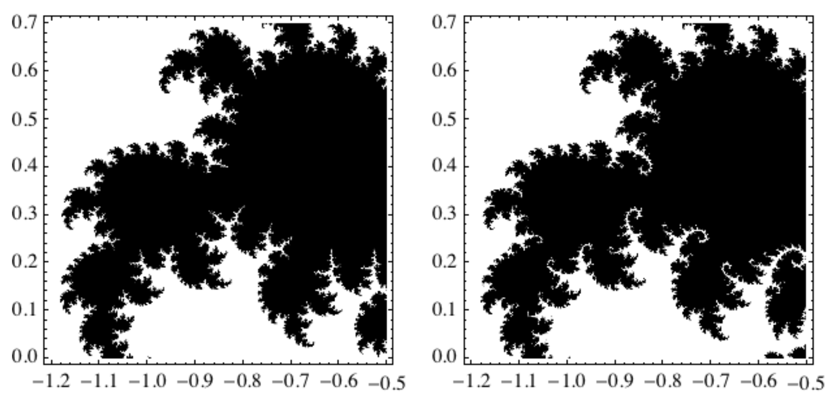

FIGURE 7. For the filled Julia set of the map $q_\xi(z) = e^{2\pi i\xi}z+z^2$, a small perturbation in ξ can cut deeper fjords in $K(q)$; on the left $\xi_1 = .131578\ldots$ and on the right, $\xi_2 = \xi_1 + \varepsilon$ with $\varepsilon < .00125$

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