

THE MAXIMAL ENTROPY MEASURE OF FATOU BOUNDARIES

JANE HAWKINS * AND MICHAEL TAYLOR

Mathematics Dept. CB #3250
University of North Carolina at Chapel Hill
Chapel Hill, NC 27599, USA

(Communicated by Mark F. Demers)

ABSTRACT. We look at the maximal entropy measure (MME) of the boundaries of connected components of the Fatou set of a rational map of degree ≥ 2 . We show that if there are infinitely many Fatou components, and if either the Julia set is disconnected or the map is hyperbolic, then there can be at most one Fatou component whose boundary has positive MME measure. We also replace hyperbolicity by the more general hypothesis of geometric finiteness.

1. Introduction. Let $R : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be a rational map of degree $d \geq 2$, defined on the Riemann sphere $\widehat{\mathbb{C}}$, \mathcal{F} its Fatou set, \mathcal{J} its Julia set, and λ the unique maximal entropy measure (MME) on \mathcal{J} (i.e., $h_\lambda(R) = \log d$). Recall that \mathcal{F} and \mathcal{J} are invariant under R ,

$$R \text{ preserves } \lambda \text{ and acts ergodically on } (\mathcal{J}, \mathcal{B}, \lambda), \quad (1.1)$$

where \mathcal{B} consists of the Borel subsets of \mathcal{J} , and

$$\text{supp } \lambda = \mathcal{J}. \quad (1.2)$$

See [6], [4], or §5.4 of [10] for (1.1)–(1.2). We want to study $\lambda(\partial\mathcal{O})$, when \mathcal{O} is a connected component of \mathcal{F} , and in particular to understand when we can say $\lambda(\partial\mathcal{O}) = 0$.

The results in this paper hold for any Borel probability measure μ satisfying (1.1) and (1.2). It is well known that for many rational maps there are measures in addition to the MME measure λ that have these properties. We focus on λ to complement an earlier result by the authors on the construction of the MME. In the jointly written appendix of [5], the authors showed that the backward random iteration method works for drawing the Julia set of any rational map. They proved the result by showing the associated delta mass measures, averaged along almost any randomly chosen backward path, converge to λ . It is observed that certain details of Julia sets have a fuzzy appearance for many maps when this algorithm is used, and the results of this paper provide an explanation of this observation.

2010 *Mathematics Subject Classification.* Primary: 37F10, 30D05; Secondary: 37A05.

Key words and phrases. Complex dynamics, Julia sets, Fatou sets, measure.

The second author was partially supported by NSF grant DMS-1500817.

* Corresponding author.

To get started, given a rational map R and a component $\mathcal{O} \subset \mathcal{F}$, Sullivan's Non-wandering Theorem implies there exist $m \in \mathbb{N}$ and a collection $\mathcal{O}_1, \dots, \mathcal{O}_k$ of mutually disjoint connected components of \mathcal{F} such that

$$R : \mathcal{O}_j \longrightarrow \mathcal{O}_{j+1}, \quad j, j+1 \in \mathbb{Z}/(k), \quad (1.3)$$

and

$$R^m(\mathcal{O}) \subset \mathcal{O}_1, \quad \text{hence } R^m(\partial\mathcal{O}) \subset \partial\mathcal{O}_1. \quad (1.4)$$

Our first goal is to establish the following.

Theorem 1.1. *Let $\mathcal{O}_1, \dots, \mathcal{O}_k$ be mutually disjoint connected components of \mathcal{F} such that (1.3) holds. Then either*

$$\lambda(\partial\mathcal{O}_j) = 0, \quad \forall j \in \mathbb{Z}/(k), \quad (1.5)$$

or

$$\partial\mathcal{O}_1 = \dots = \partial\mathcal{O}_k = \mathcal{J}. \quad (1.6)$$

If (1.5) holds, then $\lambda(\partial\mathcal{O}) = 0$ whenever \mathcal{O} is a component of \mathcal{F} such that $\mathcal{O} \subset R^{-M}(\mathcal{O}_j)$, $M \in \mathbb{N}$, and any j .

The following general result will be useful in the proof of Theorem 1.1.

Lemma 1.2. *Let $(\mathcal{J}, \mathcal{B}, \lambda)$ be a probability space, and assume $R : \mathcal{J} \rightarrow \mathcal{J}$ is \mathcal{B} -measurable and λ is ergodic and invariant under R . Then, for $E \in \mathcal{B}$,*

$$R^{-1}(E) \supset E \implies \lambda(E) = 0 \quad \text{or} \quad 1. \quad (1.7)$$

Proof. Let $F_j = R^{-j}(E)$. Then $E = F_0 \subset F_j \nearrow F \in \mathcal{B}$. The hypotheses also imply $\lambda(F) = \lambda(E)$ and $R^{-1}(F) = F$. Ergodicity implies $\lambda(F) = 0$ or 1 , so we have (1.7). \square

Proof of Theorem 1.1. Lemma 1.2 applies to $E = \partial\mathcal{O}_1 \cup \dots \cup \partial\mathcal{O}_k$, and we have (1.7). Also $R^{-1}(\partial\mathcal{O}_{j+1}) \supset \partial\mathcal{O}_j$, so

$$\lambda(\partial\mathcal{O}_{j+1}) \geq \lambda(\partial\mathcal{O}_j), \quad \forall j \in \mathbb{Z}/(k). \quad (1.8)$$

This implies

$$\lambda(\partial\mathcal{O}_1) = \dots = \lambda(\partial\mathcal{O}_k). \quad (1.9)$$

Applying (1.7) leads to two cases:

$$\lambda(\partial\mathcal{O}_1 \cup \dots \cup \partial\mathcal{O}_k) = 0, \quad \text{or} \quad \lambda(\partial\mathcal{O}_1 \cup \dots \cup \partial\mathcal{O}_k) = 1. \quad (1.10)$$

In the first case, (1.5) holds. Furthermore, $\lambda(\partial\mathcal{O}) = 0$ for each component $\mathcal{O} \subset R^{-M}(\mathcal{O}_j)$, $M \in \mathbb{N}$, since

$$\begin{aligned} R^M(\mathcal{O}) = \mathcal{O}_j &\implies R^M(\overline{\mathcal{O}}) = \overline{\mathcal{O}_j} \text{ and } R^M(\partial\mathcal{O}) \subset \partial\mathcal{O}_j \\ &\implies \partial\mathcal{O} \subset R^{-M}(\partial\mathcal{O}_j). \end{aligned} \quad (1.11)$$

In the second case, since $\partial\mathcal{O}_1 \cup \dots \cup \partial\mathcal{O}_k$ is compact and (1.2) holds, we have

$$\partial\mathcal{O}_1 \cup \dots \cup \partial\mathcal{O}_k = \mathcal{J}. \quad (1.12)$$

Going further, let us note that R and R^k have the same Julia set and the same maximal entropy measure. Given (1.3), we have $R^k : \mathcal{O}_j \rightarrow \mathcal{O}_j$ for each $j \in \mathbb{Z}/(k)$, hence $R^k : \partial\mathcal{O}_j \rightarrow \partial\mathcal{O}_j$. Hence, by Lemma 1.2 (applied to R^k), for each such j ,

$$\lambda(\partial\mathcal{O}_j) = 0 \text{ or } 1. \quad (1.13)$$

Then

$$\lambda(\partial\mathcal{O}_j) = 1 \implies \partial\mathcal{O}_j = \mathcal{J}. \quad (1.14)$$

We still have (1.9), so

$$\lambda(\partial\mathcal{O}_j) = 1 \implies \partial\mathcal{O}_1 = \dots = \partial\mathcal{O}_k = \mathcal{J}. \tag{1.15}$$

This proves Theorem 1.1. □

The following result of [3] is a significant consequence of Theorem 1.1.

Theorem 1.3. *Either*

$$\lambda(\partial\mathcal{O}) = 0 \text{ for each connected component } \mathcal{O} \text{ of } \mathcal{F}, \tag{1.16}$$

or there is a connected component } \mathcal{O}_1 \text{ of } \mathcal{F} \text{ such that

$$\partial\mathcal{O}_1 = \mathcal{J}. \tag{1.17}$$

More precisely, } \mathcal{O}_1 \text{ can be taken as in (1.3)–(1.4), and then (1.6) holds.}

We can rephrase Theorem 1.3 using the notion of *residual Julia set*, defined by

$$\mathcal{J}_0 = \mathcal{J} \setminus \bigcup_j \partial\mathcal{O}_j, \tag{1.18}$$

where $\{\mathcal{O}_j\}$ consists of all the connected components of \mathcal{F} . Note that

$$(1.16) \implies \mathcal{J}_0 \neq \emptyset, \quad \text{and} \quad (1.17) \implies \mathcal{J}_0 = \emptyset. \tag{1.19}$$

As noted in [3], it follows from Theorem 1.3 that

$$\mathcal{J}_0 \neq \emptyset \implies \lambda(\mathcal{J}_0) = 1, \tag{1.20}$$

and we have the following basic result of [9].

Corollary 1.4. *Either } \mathcal{J}_0 \neq \emptyset \text{ or there is a Fatou component } \mathcal{O}_1 \text{ such that } \partial\mathcal{O}_1 = \mathcal{J}. \text{ If } \mathcal{J}_0 \neq \emptyset, \text{ then } \mathcal{J}_0 \text{ is a dense, } \mathcal{G}_\delta \text{ subset of } \mathcal{J}.*

To establish the results in the last sentence of Corollary 1.4, we note that denseness of \mathcal{J}_0 in \mathcal{J} follows from (1.20) and (1.2). The fact that \mathcal{J}_0 is obtained from \mathcal{J} by successively removing $\partial\mathcal{O}_j, j \in \mathbb{N}$, implies \mathcal{J}_0 is a \mathcal{G}_δ subset of \mathcal{J} .

Our goal is the study of $\lambda(\partial\mathcal{O})$ for various Fatou components in cases where this measure is not identically zero on these boundaries. An important class of rational maps with empty residual Julia set is the class of polynomials of degree $d \geq 2$, which we tackle in §2. In this case, the Fatou component \mathcal{O}^∞ containing ∞ satisfies

$$\partial\mathcal{O}^\infty = \mathcal{J}. \tag{1.21}$$

We give conditions under which we can show that, if \mathcal{O} is another component of \mathcal{F} , i.e., a bounded component of \mathcal{F} , then

$$\lambda(\partial\mathcal{O}) = 0. \tag{1.22}$$

We demonstrate this for a class of polynomials whose Fatou sets have an infinite number of components. One important property of \mathcal{O}^∞ used in the analysis is its complete invariance: $R(\mathcal{O}^\infty) \subset \mathcal{O}^\infty$ and $R^{-1}(\mathcal{O}^\infty) \subset \mathcal{O}^\infty$.

In §3 we extend the scope of this work to include other rational maps R for which there is a completely invariant Fatou component $\mathcal{O}^\#$, so

$$R(\mathcal{O}^\#) \subset \mathcal{O}^\# \quad \text{and} \quad R^{-1}(\mathcal{O}^\#) \subset \mathcal{O}^\#. \tag{1.23}$$

We extend the basic results of §2 to cover this more general situation. In §4 we consider hyperbolic maps and geometrically finite maps. Results of §§3–4 together yield the following.

Theorem 1.5. *Let R be a rational map of degree ≥ 2 , and assume \mathcal{F} has infinitely many connected components. Assume there is a Fatou component \mathcal{O}_1 such that $\lambda(\partial\mathcal{O}_1) \neq 0$. Then $\lambda(\partial\mathcal{O}) = 0$ for each Fatou component $\mathcal{O} \neq \mathcal{O}_1$, under either of the following conditions:*

$$\mathcal{J} \text{ is disconnected,} \tag{A}$$

or

$$\mathcal{J} \text{ is connected and } R \text{ is hyperbolic.} \tag{B}$$

Moreover, if (A) or (B) hold, then $\partial\mathcal{O}_1 = \mathcal{J}$, and \mathcal{O}_1 is completely invariant.

Theorem 1.5 follows from Propositions 3.3 and 4.1. We mention that Proposition 3.2 has the same conclusion, with hypothesis (B) replaced by

$$\begin{aligned} \mathcal{J} &\text{ is connected and locally connected, and} \\ \mathcal{F} &\text{ has a completely invariant component.} \end{aligned} \tag{B1}$$

The special case when R is a polynomial and \mathcal{J} is connected and locally connected is Proposition 2.3. As we show in §4, using (B1), we can also replace (B) by

$$\mathcal{J} \text{ is connected and } R \text{ is geometrically finite.} \tag{B2}$$

We end with an appendix, giving examples to illustrate our results.

2. Polynomials. Here we explore consequences of Theorem 1.1 for the class of polynomials of degree $d \geq 2$, e.g.,

$$R(z) = z^d + a_{d-1}z^{d-1} + \cdots + a_0, \quad d \geq 2. \tag{2.1}$$

(If the leading term were $a_d z^d$, we could conjugate by $z \mapsto cz$ with $c^{d-1} = a_d$, to obtain the form (2.1).) In such a case, \mathcal{F} has a connected component \mathcal{O}^∞ , containing ∞ , and (cf. [1], Theorem 5.2.1)

$$\mathcal{J} = \partial\mathcal{O}^\infty. \tag{2.2}$$

It clearly follows from (2.2) that for polynomials,

$$\mathcal{J}_0 = \emptyset \quad \text{and} \quad \lambda(\partial\mathcal{O}^\infty) = 1.$$

The set $\mathcal{K} = \widehat{\mathbb{C}} \setminus \mathcal{O}^\infty$ is called the *filled Julia set* of R , and one has $\mathcal{J} = \partial\mathcal{K}$. In some cases, $\mathcal{F} = \mathcal{O}^\infty$. If \mathcal{F} has exactly 2 connected components, say \mathcal{O}^∞ and \mathcal{O}_1 , then it is also the case that $\partial\mathcal{O}_1 = \mathcal{J}$. Otherwise, \mathcal{F} must have an infinite number of connected components (cf. [1], Theorem 5.6.2). We call the connected components of \mathcal{F} other than \mathcal{O}^∞ , i.e., those contained in \mathcal{K} , *bounded Fatou components*. The component \mathcal{O}^∞ is completely invariant. Hence, if \mathcal{O} is a bounded Fatou component, so is $R(\mathcal{O})$ and so is each connected component of $R^{-1}(\mathcal{O})$. We now look into when a bounded Fatou component \mathcal{O} can be shown to satisfy $\lambda(\partial\mathcal{O}) = 0$.

Proposition 2.1. *Let $R(z)$ have the form (2.1), and filled Julia set \mathcal{K} . Assume there is a point $z_0 \in \mathcal{K}$ such that $\mathcal{K} \setminus \{z_0\}$ is not connected. Then $\lambda(\partial\mathcal{O}) = 0$ for each bounded Fatou component \mathcal{O} of R .*

Proof. Take a bounded Fatou component $\mathcal{O} \subset \mathcal{K}$. The non-wandering theorem gives $m \in \mathbb{N}$ and Fatou components $\mathcal{O}_1, \dots, \mathcal{O}_k$ (necessarily bounded Fatou components) such that (1.3)–(1.4) hold. Say \mathcal{K}_1 and \mathcal{K}_2 are disjoint connected components of $\mathcal{K} \setminus \{z_0\}$. We can assume $\mathcal{O}_1 \subset \mathcal{K}_1$. Then $\overline{\mathcal{O}_1}$ does not contain $\partial\mathcal{K}_2$, so (1.6) fails. By Theorem 1.1, this forces $\lambda(\partial\mathcal{O}_1) = 0$, and hence $\lambda(\partial\mathcal{O}) = 0$. \square

Of course, Proposition 2.1 applies if \mathcal{K} is not connected. We will now assume \mathcal{K} is connected. Hence $\widehat{\mathbb{C}} \setminus \mathcal{K}$ is simply connected. By the Riemann mapping theorem, there is a unique biholomorphic map

$$\varphi : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \longrightarrow \widehat{\mathbb{C}} \setminus \mathcal{K}, \tag{2.3}$$

such that

$$\varphi(\infty) = \infty, \quad D\varphi(\infty) = \alpha > 0. \tag{2.4}$$

Here $\overline{\mathbb{D}}$ is the disk $\{z \in \mathbb{C} : |z| \leq 1\}$. We use this to recall from Chapter 6 of [7] conditions under which the hypothesis of Proposition 2.1 holds. It involves the notion of external rays,

$$\psi_\theta(r) = \varphi(re^{i\theta}), \quad r \in (1, \infty), \quad \theta \in \mathbb{T}^1 = \mathbb{R}/(2\pi\mathbb{Z}). \tag{2.5}$$

Given $\theta \in \mathbb{T}^1$, we say ψ_θ lands at a point $z \in \partial\mathcal{K}$ provided

$$\lim_{r \searrow 1} \psi_\theta(r) = z. \tag{2.6}$$

We mention two classical results, given as Theorems 6.1 and 6.2 of [7]. The first is that the limit in (2.6) exists for almost all $\theta \in \mathbb{T}^1$. The second is the following:

If $E \subset \mathbb{T}^1$ has positive measure, then there exist $\theta_0, \theta_1 \in E$ such that

$$\lim_{r \searrow 1} \psi_{\theta_0}(r) = z_0, \quad \lim_{r \searrow 1} \psi_{\theta_1}(r) = z_1, \quad \text{and} \quad z_0 \neq z_1. \tag{2.7}$$

These results lead to the following, which is part of Corollary 6.7 of [7].

Proposition 2.2. *Let R have the form (2.1) and assume \mathcal{K} is connected. Take φ and ψ_θ as in (2.3)–(2.5). Assume there exist $\xi_0 \neq \xi_1 \in \mathbb{T}^1$ such that*

$$\lim_{r \searrow 1} \psi_{\xi_0}(r) = \lim_{r \searrow 1} \psi_{\xi_1}(r) = z_0. \tag{2.8}$$

Then $\mathcal{K} \setminus \{z_0\}$ is not connected.

Proof. The union γ of the two rays $\{\psi_{\xi_0}(r) : 1 < r < \infty\}$ and $\{\psi_{\xi_1}(r) : 1 < r < \infty\}$, together with their endpoints z_0 and ∞ , forms a simple closed curve in $\widehat{\mathbb{C}}$. The Jordan curve theorem implies that $\widehat{\mathbb{C}} \setminus \gamma$ has exactly two connected components. Our hypotheses imply

$$\mathcal{K} \cap \gamma = \{z_0\}. \tag{2.9}$$

It remains to note that each connected component of $\widehat{\mathbb{C}} \setminus \gamma$ contains a point of \mathcal{K} , and this follows from (2.7), first taking E to be the open arc from ξ_0 to ξ_1 in \mathbb{T}^1 , then taking E to be the complementary open arc in \mathbb{T}^1 . \square

Regarding the question of when Proposition 2.2 applies, we note that a definite answer can be given under the additional hypothesis that

$$\mathcal{J} \text{ is connected and locally connected,} \tag{2.10}$$

or equivalently, that \mathcal{K} is connected and locally connected. In that case, a classical result of Caratheodory (cf. [8], Theorem 17.14) implies that φ in (2.3), mapping $\mathbb{D}^\infty = \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ to $\mathcal{O}^\infty = \widehat{\mathbb{C}} \setminus \mathcal{K}$, has a unique continuous extension to

$$\varphi : \overline{\mathbb{D}^\infty} \longrightarrow \overline{\mathcal{O}^\infty}. \tag{2.11}$$

The image of φ in (2.11) is both compact and dense in $\overline{\mathcal{O}^\infty}$, so φ in (2.11) is surjective. It restricts to a continuous map

$$\varphi : \partial\mathbb{D} \longrightarrow \mathcal{J}, \tag{2.12}$$

also surjective. In view of (2.5), we have

$$\varphi(e^{i\theta}) = \lim_{r \searrow 1} \psi_\theta(r). \quad (2.13)$$

Now if φ in (2.12) is also one-to-one, this makes $\mathcal{J} \subset \widehat{\mathbb{C}}$ a simple closed curve, so $\mathcal{F} = \widehat{\mathbb{C}} \setminus \mathcal{J}$ would have just two connected components. We have the following conclusion.

Proposition 2.3. *Let R have the form (2.1) and assume \mathcal{J} is connected and locally connected. If \mathcal{F} has infinitely many connected components, then*

$$\lambda(\partial\mathcal{O}) = 0 \text{ for each bounded component } \mathcal{O} \text{ of } \mathcal{F}. \quad (2.14)$$

Proof. As we have just seen, the hypotheses imply that φ in (2.12) is not one-to-one. Hence there exist $\xi_0 \neq \xi_1 \in \mathbb{T}^1$ such that $\varphi(e^{i\xi_0}) = \varphi(e^{i\xi_1})$ in (2.13). Thus, with $z_0 = \varphi(e^{i\xi_0}) = \varphi(e^{i\xi_1})$, Proposition 3.2 implies that $\mathcal{K} \setminus \{z_0\}$ is not connected, so (2.14) follows from Proposition 2.1. \square

3. Other maps with a completely invariant Fatou component. Extending our scope a bit, let us now assume that R (of degree $d \geq 2$) has the property that \mathcal{F} has a completely invariant connected component $\mathcal{O}^\#$, i.e.,

$$R(\mathcal{O}^\#) \subset \mathcal{O}^\# \text{ and } R^{-1}(\mathcal{O}^\#) \subset \mathcal{O}^\#. \quad (3.1)$$

If there is a fixed point $p \in \mathcal{O}^\#$ of R , then, conjugating by a linear fractional transformation, we can take $p = \infty$. If $R^{-1}(p) = p$ as well, then R is a polynomial (which guarantees that p must be a superattracting fixed point). There are many examples of rational maps with completely invariant Fatou domains and attracting fixed points that are not polynomials. We discuss some in Appendix A.

Generally when (3.1) holds, we have

$$\partial\mathcal{O}^\# = \mathcal{J}. \quad (3.2)$$

This follows from Theorem 5.2.1 of [1], which also contains the results that $\mathcal{O}^\#$ is either simply connected or infinitely connected; all the other components of \mathcal{F} are simply connected; and $\mathcal{O}^\#$ is simply connected if and only if \mathcal{J} is connected. Using these facts, we can set things up to obtain results parallel to Propositions 2.1–2.3. To formulate these results, let us arrange that $\infty \in \mathcal{O}^\#$. Then $\mathcal{O}^\#$ is the unbounded component of \mathcal{F} , and we call the other components of \mathcal{F} bounded Fatou components. As in the case of polynomials, we see that if \mathcal{O} is a bounded Fatou component, so is $R(\mathcal{O})$ and so is each connected component of $R^{-1}(\mathcal{O})$. We set $\mathcal{K} = \widehat{\mathbb{C}} \setminus \mathcal{O}^\#$, and call this the filled Julia set of R . The following result has the same proof as Proposition 2.1.

Proposition 3.1. *Let R be a rational map of degree ≥ 2 for which a Fatou component $\mathcal{O}^\#$ satisfies (3.1), and let $\mathcal{K} = \widehat{\mathbb{C}} \setminus \mathcal{O}^\#$ be its filled Julia set. Assume there exists $z_0 \in \mathcal{K}$ such that $\mathcal{K} \setminus \{z_0\}$ is not connected. Then $\lambda(\partial\mathcal{O}) = 0$ for each Fatou component $\mathcal{O} \neq \mathcal{O}^\#$.*

As before, Proposition 3.1 certainly applies if \mathcal{K} is not connected. If \mathcal{K} is connected, then $\mathcal{O}^\#$ is simply connected, and we again have the set-up (2.3)–(2.7), and Proposition 2.2 extends to this setting, as does the analysis leading to the following extension of Proposition 2.3.

Proposition 3.2. *Let R be a rational map of degree ≥ 2 for which there is a Fatou component $\mathcal{O}^\#$ satisfying (3.1), and assume \mathcal{J} is connected and locally connected. If \mathcal{F} has infinitely many connected components, then $\lambda(\partial\mathcal{O}) = 0$ for each Fatou component $\mathcal{O} \neq \mathcal{O}^\#$.*

In counterpoint, we have the following result when \mathcal{J} is not connected.

Proposition 3.3. *Let R be a rational map of degree ≥ 2 , and assume \mathcal{J} is disconnected. If there is a Fatou component \mathcal{O}_1 such that $\lambda(\partial\mathcal{O}_1) \neq 0$, then $\lambda(\partial\mathcal{O}) = 0$ for each Fatou component $\mathcal{O} \neq \mathcal{O}_1$.*

Proof. The hypotheses imply that the residual set \mathcal{J}_0 is empty. Hence, given \mathcal{J} disconnected, by Theorem 4.4.19 of [10], \mathcal{F} has a completely invariant component $\mathcal{O}^\#$. By Proposition 3.1, $\lambda(\partial\mathcal{O}) = 0$ for each Fatou component $\mathcal{O} \neq \mathcal{O}^\#$. This proves the proposition (and forces $\mathcal{O}_1 = \mathcal{O}^\#$). \square

Remark. As seen in §1, the hypothesis that $\lambda(\partial\mathcal{O}_1) \neq 0$ for some Fatou component \mathcal{O}_1 is equivalent to the hypothesis that $\mathcal{J}_0 = \emptyset$. The *Makienko conjecture* can be stated as saying that, if $\mathcal{J}_0 = \emptyset$, then there is a Fatou component that is completely invariant under R^2 . A discussion of conditions under which Makienko’s conjecture has been proved can be found in [2]. Of particular use here is the following result of [11]:

$$\text{The Makienko conjecture holds provided } \mathcal{J} \text{ is locally connected.} \tag{3.3}$$

We make use of this in the following section.

4. Hyperbolic maps and geometrically finite maps. A rational map $R : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$, of degree $d \geq 2$, with Julia set \mathcal{J} , is said to be *hyperbolic* provided

$$\mathcal{J} \cap \overline{C^+(R)} = \emptyset, \quad \text{where } C^+(R) = \bigcup_{k \geq 1} R^k(C_R), \tag{4.1}$$

where $C_R = \{z \in \widehat{\mathbb{C}} : DR(z) = 0\}$ is the set of critical points of R . An equivalent condition for hyperbolicity (cf. [10], Theorem 4.4.2) is that each critical point of R is in \mathcal{F} and each forward orbit of a critical point converges to an attracting cycle. Hyperbolic rational maps are relatively “tame,” from a topological point of view. For example (cf. [10], Theorem 4.4.5), if R is hyperbolic,

$$\mathcal{J} \text{ connected} \implies \mathcal{J} \text{ locally connected.} \tag{4.2}$$

There is also a partial converse to the implication (3.1) \implies (3.2) for hyperbolic maps (cf. [10], Theorem 4.4.16):

$$\text{If } R \text{ is hyperbolic and } \mathcal{J} \text{ is connected, and } \mathcal{O} \text{ is a component of } \mathcal{F}, \text{ then} \tag{4.3}$$

$$R(\mathcal{O}) \subset \mathcal{O}, \quad \partial\mathcal{O} = \mathcal{J} \implies \mathcal{O} \text{ is completely invariant.}$$

We therefore have the following.

Proposition 4.1. *Assume that R is hyperbolic and \mathcal{J} is connected, that \mathcal{F} has infinitely many connected components, and that there is a Fatou component \mathcal{O}_1 such that $\lambda(\partial\mathcal{O}_1) \neq 0$. Then $\lambda(\partial\mathcal{O}) = 0$ for each Fatou component $\mathcal{O} \neq \mathcal{O}_1$.*

Proof. By Theorem 1.3, there must be a component $\mathcal{O}^\#$ of \mathcal{F} such that $\partial\mathcal{O}^\# = \mathcal{J}$. Furthermore, $\mathcal{O}^\#$ is invariant under R^k , which is also hyperbolic. By (4.3), such $\mathcal{O}^\#$ must be completely invariant (under R^k). Thanks to (4.2), Proposition 3.2 applies (to R^k), and it implies that $\lambda(\partial\mathcal{O}) = 0$ for each component $\mathcal{O} \neq \mathcal{O}^\#$.

Incidentally, this forces $\mathcal{O}_1 = \mathcal{O}^\#$. Furthermore, taking into account (1.6), we see that k must be 1. Hence \mathcal{O}_1 must be completely invariant under R . \square

For further results, we bring in the following two generalizations of hyperbolicity, taking definitions from [10], p. 153.

Definition. A rational map $R : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ of degree ≥ 2 is said to be *subhyperbolic* provided the following two conditions hold:

$$\begin{aligned} &\text{the forward orbit of each critical point in } \mathcal{F} \\ &\text{converges to an attracting cycle,} \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} &\text{the forward orbit of each critical point in } \mathcal{J} \\ &\text{is eventually periodic.} \end{aligned} \tag{4.5}$$

If we merely assume (4.5) holds, we say R is *geometrically finite*.

Clearly R hyperbolic $\Rightarrow R$ subhyperbolic $\Rightarrow R$ geometrically finite. The usefulness of geometric finiteness for the work here stems from the following generalization of (4.2), established in [12], Theorem A:

$$\begin{aligned} &\text{If } R \text{ is geometrically finite and } \mathcal{J} \text{ is connected,} \\ &\text{then } \mathcal{J} \text{ is locally connected.} \end{aligned} \tag{4.6}$$

In concert with (3.3), this yields the following:

Lemma 4.2. *Assume that R is geometrically finite and \mathcal{J} is connected. Then either $\mathcal{J}_0 \neq \emptyset$ or \mathcal{F} has a completely invariant component for R^2 .*

We now establish the following extension of Proposition 4.1.

Proposition 4.3. *Assume R is geometrically finite, \mathcal{J} is connected, and \mathcal{F} has infinitely many connected components. Assume further that there is a Fatou component \mathcal{O}_1 such that $\lambda(\partial\mathcal{O}_1) \neq 0$. Then $\lambda(\partial\mathcal{O}) = 0$ for each Fatou component $\mathcal{O} \neq \mathcal{O}_1$.*

Proof. The hypotheses imply $\mathcal{J}_0 = \emptyset$, by (1.20). Hence, by Lemma 4.2, there is a component $\mathcal{O}^\#$ of \mathcal{F} that is completely invariant under R^2 . As in (3.2), this implies $\partial\mathcal{O}^\# = \mathcal{J}$. We are assuming \mathcal{J} is connected, and by (4.6) \mathcal{J} is also locally connected. Thus Proposition 3.2 applies (to R^2), giving $\lambda(\partial\mathcal{O}) = 0$ for each Fatou component $\mathcal{O} \neq \mathcal{O}^\#$.

As before, this forces $\mathcal{O}_1 = \mathcal{O}^\#$. It also implies that $\mathcal{O}^\#$ is completely invariant under R . \square

We turn to a proof of Theorem 1.5.

Proof of Theorem 1.5. If (A) holds, the conclusion that $\lambda(\partial\mathcal{O}) = 0$ for each Fatou component $\mathcal{O} \neq \mathcal{O}_1$ is a direct consequence of Proposition 3.3. The proof of Proposition 3.3 also ends with the comment that \mathcal{O}_1 must coincide with a completely invariant component $\mathcal{O}^\#$. Similarly, if (B) holds, the conclusion of the theorem follows from Proposition 4.1, the complete invariance of \mathcal{O}_1 again established in the course of the proof of Proposition 4.1. Validity of Theorem 1.5 with (B) replaced by (B1) follows from Proposition 3.2, and its validity with (B) replaced by (B2) follows from Proposition 4.3. \square

Appendix A. Examples. We include a few basic examples illustrating the results of §§1–4. In all these examples, $\mathcal{J}_0 = \emptyset$. Before getting to them, we make a few useful general comments whose proofs are found in the literature in the bibliography. If R is a rational map of degree $d \geq 2$ with a k -periodic point z_0 whose multiplier satisfies: $DR^k(z_0) = \exp(2\pi i/m)$, $m \in \mathbb{N}$, then the k -cycle is called *parabolic*; k always refers to the minimum period.

Remarks A.

1. If a rational map R has an attracting k -periodic point in \mathcal{F} , then there are at least k Fatou components in \mathcal{F} .
2. If R has a parabolic k -periodic point z_0 with multiplier $DR^k(z_0) = \exp(2\pi i/m)$, then there are at least mk Fatou components in \mathcal{F} .
3. These lead to the following lemma.

Lemma A.1. *Let $R : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be a rational map of degree ≥ 2 . Assume its Fatou set \mathcal{F} has a completely invariant component and that R has an attracting or parabolic k -periodic point with $k \geq 2$. Then \mathcal{F} has infinitely many components.*

Proof. The completely invariant component cannot be one of the components in the n -cycle, $n \geq 2$, of components in \mathcal{F} induced by the non-repelling k -cycle for R so \mathcal{F} has more than two components. □

4. Whenever $R : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a rational map of degree 2, the Julia set \mathcal{J} is either connected or totally disconnected.
5. If every critical point in \mathcal{J} has a finite forward orbit (i.e., R is geometrically finite), then all non-repelling periodic points of R are attracting or parabolic. If R has degree $d \geq 2$ and there are $2d - 2$ critical points in \mathcal{F} , then R is geometrically finite.

Example 1. As an example of a non-polynomial map with a completely invariant Fatou component, neither hyperbolic nor subhyperbolic, but geometrically finite, we set:

$$R(z) = z + \frac{1}{z} + \frac{3}{2}. \tag{A.1}$$

R has critical points $c_1 = 1$ and $c_2 = -1$.

For these maps ∞ is a fixed point with multiplier 1 so the map is parabolic; this implies that $\infty, 0 \in \mathcal{J}$ and there is an attracting petal in \mathcal{F} which sits in a component $\mathcal{O}^\# \subset \mathcal{F}$ containing a critical point. We have:

$$-1 \mapsto -1/2 \mapsto -1$$

so there is a superattracting 2-cycle of components in \mathcal{F} , \mathcal{O}_1 and $\mathcal{O}_2 = R(\mathcal{O}_1)$. Therefore $c_1 \in \mathcal{O}^\#$. Since there are no other critical points, R is geometrically finite by Remark A.5.

There are infinitely many components $\mathcal{O} \subset \mathcal{F}$; in fact all $\mathcal{O} \neq \mathcal{O}^\#$ map to $\mathcal{O}_1 \cup \mathcal{O}_2$; the first statement follows from Lemma A.1, and the second since we have exhausted the critical points so no other Fatou components are possible. Every component of \mathcal{F} is simply connected by Remark A.4.

Since $c_1 \in \mathcal{O}^\#$ and $R(\mathcal{O}^\#) = \mathcal{O}^\#$, $\mathcal{O}^\#$ is a degree 2 branched cover of itself, so $\mathcal{O}^\#$ is completely invariant. The map R is geometrically finite, and \mathcal{J} is connected so it is locally connected using (4.6). Then $\lambda(\partial\mathcal{O}^\#) = 1$ and $\lambda(\partial\mathcal{O}_1) = \lambda(\partial\mathcal{O}_2) = 0$ by Proposition 3.2.

Because of the parabolic point at ∞ , R is neither hyperbolic nor subhyperbolic, since $\lim_{n \rightarrow \infty} R^n(c_1) \in \mathcal{J}$ causing (4.1) and (4.4) to fail.

Example 2. Here is another example of a geometrically finite but non hyperbolic rational map.

$$R(z) = \frac{z^2 - z}{2z + 1}. \quad (\text{A.2})$$

We list a few easily verifiable properties of R :

- There are 3 fixed points at $-2, 0$, and ∞ , with respective multipliers $1/3, -1$, and 2 .
- Since -2 is attracting and the parabolic fixed point at $0 \in \mathcal{J}$ has two attracting petals, \mathcal{F} has infinitely many components by Lemma A.1.
- Since R is quadratic, and \mathcal{J} is not a Cantor set, \mathcal{J} is connected using A.4; therefore each component $\mathcal{O} \subset \mathcal{F}$ is simply connected.
- Let $\mathcal{O}^\#$ denote the component of \mathcal{F} containing the attracting point at -2 . Then since $\mathcal{O}^\#$ contains a critical point we see that $R : \mathcal{O}^\# \rightarrow \mathcal{O}^\#$ is a 2-fold branched covering of itself, and since R is degree 2, there are no other components of $R^{-1}(\mathcal{O}^\#)$ so it is completely invariant.
- The geometric finiteness comes from the fixed point at -2 forcing a critical point in \mathcal{F} , and similarly the fixed point at 0 attracts the other critical point, which is also in \mathcal{F} .

This shows the hypotheses of Theorem 1.5 (B2) are satisfied; therefore $\lambda(\partial\mathcal{O}^\#) = 1$, and for any component \mathcal{O} mapping onto the components of \mathcal{F} containing the petals, we have $\lambda(\partial\mathcal{O}) = 0$.

Remark. An interesting contrast between Examples 1 and 2 lies in the different natures of their respective completely invariant Fatou components.

Example 3. We turn to a hyperbolic polynomial example. The map $P(z) = z^2 - 1$ is a hyperbolic polynomial with a superattracting period 2 cycle:

$$0 \mapsto -1 \mapsto 0.$$

Since $\mathcal{J} = \mathcal{O}^\infty$, $\lambda(\mathcal{O}^\infty) = 1$, and there are two bounded components in the immediate basin of attraction for the attracting 2-cycle $\{0, -1\}$. Lemma A.1 implies that \mathcal{F} has infinitely many components. By Proposition 2.3, $\lambda(\partial\mathcal{O}) = 0$ for all components except \mathcal{O}^∞ since \mathcal{J} is connected and locally connected by hyperbolicity. There are many similar examples.

Example 4. There are many cubic polynomials to which Theorem 1.5 (A) applies. In particular, there are cubic polynomials with the property that one critical point is attracted to ∞ and the other stays bounded. For these we frequently see that the Julia set is disconnected but not Cantor. Then Proposition 3.3 applies in this case.

REFERENCES

- [1] A. Beardon, *Iteration of Rational Functions*, Springer-Verlag, New York, 1991.
- [2] C. Curry, J. Mayer, J. Meddaugh and J. Rogers, [Any counterexample to Makienko's conjecture is an indecomposable continuum](#), *Ergodic Theory and Dynam. Sys.*, **29** (2009), 875–883.
- [3] C. Curry, J. Mayer and E. Tymchatyn, [Topology and measure of buried points in Julia sets](#), *Fund. Math.*, **222** (2013), 1–17.
- [4] A. Freire, A. Lopes and R. Mañé, [An invariant measure for rational maps](#), *Bol. Soc. Brasil. Mat.*, **14** (1983), 45–62.

- [5] J. Hawkins, [Lebesgue ergodic rational maps in parameter space](#), *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **13** (2003), 1423–1447.
- [6] M. Lyubich, [Entropy properties of rational endomorphisms of the Riemann sphere](#), *Ergodic Theory and Dynam. Sys.*, **3** (1983), 351–385.
- [7] C. McMullen, *Complex Dynamics and Renormalization*, Princeton Univ. Press, 1994.
- [8] J. Milnor, *Dynamics in One Complex Variable (3rd ed.)*, Princeton Univ. Press, 2006.
- [9] S. Morosawa, [On the residual Julia sets of rational functions](#), *Ergodic Theory and Dynam. Sys.*, **17** (1997), 205–210.
- [10] S. Morosawa, Y. Nishimura, M. Taniguchi and T. Ueda, *Holomorphic Dynamics*, Cambridge Univ. Press, 2000.
- [11] J. Qiao, [Topological complexity of Julia sets](#), *Sci. China Ser. A*, **40** (1997), 1158–1165.
- [12] L. Tan and Y. Yin, Local connectivity of the Julia set for geometrically finite rational maps, *Sci. China Ser. A*, **39** (1996), 39–47.

Received for publication August 2017.

E-mail address: jmh@math.unc.edu

E-mail address: met@math.unc.edu