PARAMETERIZED DYNAMICS FOR THE WEIERSTRASS ELLIPTIC FUNCTION OVER SQUARE PERIOD LATTICES

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ABSTRACT. We iterate the Weierstrass elliptic $\wp$ function in order to understand the dependence of the dynamics on the underlying period lattice $\mathbf{L}$. We focus on square lattices and use the holomorphic dependence on the classical invariants $(g_2, g_3) = (g_2, 0)$ to show that in parameter space (g_2-space) one sees both quadratic-like attracting orbit behavior and pre-pole dynamics. In the case of pre-pole parameters all critical orbits terminate at poles and the Julia set of $\wp_{\mathbf{L}}$ is the entire sphere. We show that both the Mandelbrot-like dynamics and the pre-pole parameters accumulate on pre-pole parameters of lower order providing results on the dynamics occurring in parameter space "between Mandelbrot sets".

1. Introduction

We parametrize a family of periodic meromorphic maps from $\mathbb{C}$ onto the Riemann sphere $\mathbb{C}_\infty$ in order to analyze the asymptotic behavior changes under iteration, as the parameter moves holomorphically. In particular starting with a maximal discrete subgroup of points in $\mathbb{C}$, a square lattice denoted $\mathbf{L}$, generated by a nonzero pair $\lambda, \lambda i \in \mathbb{C}$, we iterate the Weierstrass elliptic $\wp$ function, the basic building block for maps periodic with respect to each element of $\mathbf{L}$. We denote the map by $\wp_{\mathbf{L}}$ and by $\wp_{\mathbf{L}}^n$ we mean $\wp_{\mathbf{L}} \circ \wp_{\mathbf{L}} \cdots \circ \wp_{\mathbf{L}}$ n times; note that $\wp_{\mathbf{L}}^n$ is not defined at every $z \in \mathbb{C}_\infty$. This is because any doubly periodic function f must have poles; if not, then by Liouville's Theorem f is bounded and hence constant in $\mathbb{C}$. The Riemann sphere splits in the classical way into the open set of normal points, points z such that the family $\{\wp_L^n\}$ is a normal family near z, called the Fatou set and the "chaotic" complement called the Julia set. For the meromorphic setting a good background exposition can be found in [Bergweiler, 1993].

We study parameter space for Weierstrass elliptic $\wp$ functions with square period lattices since this restriction gives a family of elliptic functions which can be parametrized by a single nonzero complex parameter. On the other hand this class already exhibits most of the richness of behavior typical of elliptic functions as shown for example in [Hawkins & Koss, 2002, 2004, 2005] and [Hawkins & Look, 2005].

In this paper we answer a question posed by Bob Devaney about the parameter space. It is evident that Mandelbrot-like bifurcations occur and indeed many of the maps in the family under consideration have been shown to be quadratic-like [Hawkins

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& Look, 2005]. A simple computer algorithm searching for attracting periodic orbits gives a picture like that shown in Fig. 4 [Hawkins & Koss, 2004, 2005]. There are large white areas visible in Fig. 4 and the natural question posed by Devaney is: what occurs in parameter space "between Mandelbrot sets"?

In [Lei, 1990] it was shown that certain points in parameter space of $z^2 + c$, $c \in \mathbb{C}$, show a resemblance between the Julia set (dynamical space) and parameter space. Here we prove a similar phenomenon occurs. All Julia sets of Weierstrass elliptic $\wp$ functions have poles occurring at lattice points and pre-poles evident throughout the Julia set. In the spirit of [Lei, 1990] we might expect to see "pre-poles" in parameter space for an elliptic function. We show this to be the case for the Weierstrass elliptic function with a square period lattice.

Section 2 presents the needed background material about the dynamics of iterated meromorphic functions. In Sec. 3 we show that there are natural analogs of poles and pre-poles in parameter space and that all the types of dynamics accumulate around each pole just as in the dynamical space; that is, patterns analogous to those appearing in Julia sets appear in parameter space.

2. Overview of the dynamics of the Weierstrass $\wp$ function

We define a lattice of points in the complex plane by $\mathbf{L} = [\lambda_1, \lambda_2] \colon = \{m\lambda_1 + n\lambda_2 \colon m, n \in \mathbb{Z}\}$, where $\lambda_1, \lambda_2 \in \mathbb{C}\backslash\{0\}$ satisfy $\lambda_2/\lambda_1 \notin \mathbb{R}$. There are several equivalent characterizations. A lattice $\mathbf{L}$ is a maximal discrete subgroup of $\mathbb{R}^2$. Equivalently, $\mathbf{L}$ is the $\mathbb{Z}$-linear span of a set of 2 linearly independent vectors in $\mathbb{R}^2$, and the vectors $v_1 = (\Re(\lambda_1), \Im(\lambda_1))$ and $v_2 = (\Re(\lambda_2), \Im(\lambda_2))$ are a basis for $\mathbf{L}$. Lattices have many bases; i.e., the generators of $\mathbf{L}$ are not unique.

We will consider $\mathbf{L} \subset \mathbb{C}$, and define a $\mathbf{L}$ to be *square* if $i\mathbf{L} = \mathbf{L}$. A *fundamental domain* for the quotient $\mathbb{R}^2/\mathbf{L} \cong \mathbb{C}/\mathbf{L}$ is the set:

$$\mathcal{F}(\mathbf{L}) = \{s\lambda_1 + t\lambda_2 : 0 \leq s, t < 1\};$$

This proposition is easily proved.

Proposition 2.1. The following are equivalent for a lattice Λ.

(1) $\mathbf{L}$ is a square lattice.
(2) There exists a $\lambda > 0$ and $\theta \in (-\pi/2, \pi/2]$ such that $\mathbf{L} = [\lambda e^{i\theta}, \lambda e^{-i(\theta+\pi/2)}]$.
(3) There exist real numbers $a \geq 0, b \geq 0$, not both 0, such that $\mathbf{L} = [a+b\,i, b-a\,i]$.
(4) A fundamental domain $\mathcal{F}(\mathbf{L})$ is a square.

The ratio $\tau = \lambda_2/\lambda_1$ is an important feature of a lattice. If $\mathbf{L} = [\lambda_1, \lambda_2]$, and $k \neq 0$ is any complex number, then $k\mathbf{L}$ is the lattice defined by taking $k\lambda$ for each $\lambda \in \mathbf{L}$; $k\mathbf{L}$ is said to be *similar* to $\mathbf{L}$. A square lattice corresponds to $\tau = i$ and is similar to the lattice $\mathbf{L} = [1, i]$. Similarity is an equivalence relation between lattices, and an equivalence class of lattices is called a *shape*. Clearly the shape of a square lattice gives it its name.

Definition 2.2. For any lattice $\mathbf{L}$, we define:

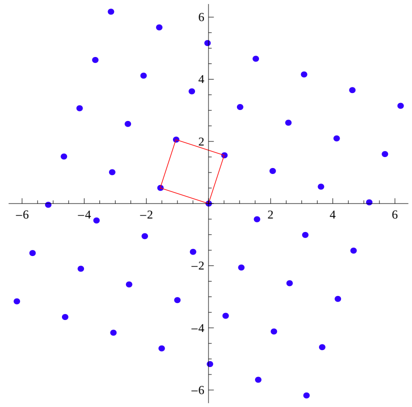

FIGURE 1. A Fundamental Region $\mathcal{F}$ of a square lattice $\mathbf{L}$

(1)
$$\overline{\mathbf{L}} = \{\overline{\lambda} : \lambda \in \mathbf{L}\},$$
and say $\mathbf{L}$ is *real* if $\overline{\mathbf{L}} = \mathbf{L}$.

(2) $\mathbf{L} = [\lambda_1, \lambda_2]$ is *real rectangular* if there exist generators such that λ_1 is real and λ_2 is purely imaginary. Any lattice similar to a real rectangular lattice is *rectangular*.

(3) $\mathbf{L} = [\lambda_1, \lambda_2]$ is *real rhombic* if there exist generators such that $\lambda_2 = \overline{\lambda_1}$. Any similar lattice is *rhombic*.

In each of these cases the period parallelogram with vertices $0, \lambda_1, \lambda_2$, and $\lambda_3 := \lambda_1 + \lambda_2$ can be chosen to look rectangular or rhombic respectively.

2.1. **Square lattices and elliptic Functions.** Throughout $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere and all meromorphic functions are considered to be functions with domain $\mathbb{C}$ and range $\mathbb{C}_\infty$. We give a brief overview of elliptic functions here; there is a review including details in [Duval, 1973].

Definition 2.3. An *elliptic function* is a meromorphic function which is periodic with respect to a lattice $\mathbf{L}$.

For any $z \in \mathbb{C}$ and any lattice $\mathbf{L}$, the *Weierstrass elliptic function* is defined by

$$\wp_{\mathbf{L}}(z) = \frac{1}{z^2} + \sum_{w \in \mathbf{L} \setminus \{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right). \tag{2.1}$$

Replacing every z by $-z$ in the definition we see that $\wp_{\mathbf{L}}$ is an even function. The map $\wp_{\mathbf{L}}$ is meromorphic, periodic with respect to $\mathbf{L}$, and has poles of order 2 at every lattice point.

The derivative of the Weierstrass elliptic function is also an elliptic function which is periodic with respect to $\mathbf{L}$ defined by

$$\wp'_{\mathbf{L}}(z) = -2 \sum_{w \in \mathbf{L}} \frac{1}{(z-w)^3}.$$

The Weierstrass elliptic function and its derivative are related by the differential equation:

$$\wp_{\mathbf{L}}'(z)^2 = 4\wp_{\mathbf{L}}(z)^3 - g_2\wp_{\mathbf{L}}(z) - g_3, \tag{2.2}$$

where $g_2(\mathbf{L}) = 60\sum_{w\in\mathbf{L}\setminus\{0\}} w^{-4}$ and $g_3(\mathbf{L}) = 140\sum_{w\in\mathbf{L}\setminus\{0\}} w^{-6}$.

The numbers $g_2(\mathbf{L})$ and $g_3(\mathbf{L})$ are invariants of the lattice in the following sense: if $g_2(\mathbf{L}) = g_2(\mathbf{L}')$ and $g_3(\mathbf{L}) = g_3(\mathbf{L}')$, then $\mathbf{L} = \mathbf{L}'$. Furthermore given any g_2 and g_3 such that $g_2^3 - 27g_3^2 \neq 0$ there exists a lattice $\mathbf{L}$ having $g_2 = g_2(\mathbf{L})$ and $g_3 = g_3(\mathbf{L})$ as its invariants [Duval, 1973].

The next result is classical and leads to a holomorphic parametrization of many families of elliptic functions [Duval, 1973].

Theorem 2.4. (1) For $\mathbf{L}_\tau = [1,\tau]$, the functions $g_i(\tau) = g_i(\mathbf{L}_\tau), i = 2,3,$ are analytic functions of τ in the open upper half plane $\Im(\tau) > 0$.

(2) For lattices $\mathbf{L}$ and $\mathbf{L}'$, $\mathbf{L}' = k\mathbf{L} \Leftrightarrow$

$$g_2(\mathbf{L}') = k^{-4}g_2(\mathbf{L}) \quad \text{and} \quad g_3(\mathbf{L}') = k^{-6}g_3(\mathbf{L}).$$

For any lattice $\mathbf{L}$, the Weierstrass elliptic function and its derivative satisfy the following properties: for $k \in \mathbb{C}\setminus\{0\}$,

$$\wp_{k\mathbf{L}}(ku) = \frac{1}{k^2}\wp_{\mathbf{L}}(u), \ \text{(homogeneity of } \wp_{\mathbf{L}}), \tag{2.3}$$

$$\wp_{k\mathbf{L}}'(ku) = \frac{1}{k^3}\wp_{\mathbf{L}}'(u), \ \text{(homogeneity of } \wp_{\mathbf{L}}'). \tag{2.4}$$

Verification of these homogeneity properties can be seen by substitution into the series definitions.

If $\wp_{\mathbf{L}}'(z_0) = 0$ then z_0 is a *critical point* and $\wp_{\mathbf{L}}(z_0)$ is a *critical value*. The critical values of the Weierstrass elliptic function on an arbitrary lattice $\mathbf{L} = [\lambda_1, \lambda_2]$ are as follows. For $j = 1, 2$, notice that $\wp_{\mathbf{L}}(\lambda_j - z) = \wp_{\mathbf{L}}(z)$ for all z. Taking derivatives of both sides we obtain $-\wp_{\mathbf{L}}'(\lambda_j - z) = \wp_{\mathbf{L}}'(z)$. Substituting $z = \lambda_1/2, \lambda_2/2$, or $\lambda_3/2$, we see that $\wp_{\mathbf{L}}'(z) = 0$ at these values. We use the notation

$$e_1 = \wp_{\mathbf{L}}(\frac{\lambda_1}{2}), \quad e_2 = \wp_{\mathbf{L}}(\frac{\lambda_2}{2}), \quad e_3 = \wp_{\mathbf{L}}(\frac{\lambda_3}{2})$$

to denote the critical values. Since e_1, e_2, e_3 are the distinct zeros of Eq. 2.2, we also write

$$\wp_{\mathbf{L}}'(z)^2 = 4(\wp_{\mathbf{L}}(z) - e_1)(\wp_{\mathbf{L}}(z) - e_2)(\wp_{\mathbf{L}}(z) - e_3). \tag{2.5}$$

Equating like terms in Eqs. 2.2 and 2.5, we obtain

$$e_1 + e_2 + e_3 = 0, \quad e_1e_3 + e_2e_3 + e_1e_2 = \frac{-g_2}{4}, \quad e_1e_2e_3 = \frac{g_3}{4}. \tag{2.6}$$

Consider the polynomial $p(x) = 4x^3 - g_2x - g_3$. Let $\triangle = g_2^3 - 27g_3^2 \neq 0$ denote its discriminant, and $\mathbf{L}$ the unique lattice associated with p via Theorem 2.4 and Eq. (2.2).

Prescribed Parameter	$\{e_1, e_2, e_3\}$	$\{g_2, g_3\}$	**L**-generator
Standard rectangular	$\{1, -1, 0\}$	$\{4, 0\}$	γ
Standard rhombic	$\{-i, i, 0\}$	$\{-4, 0\}$	γ
e_1	$\{c, -c, 0\}$	$\{4c^2, 0\}$	$\frac{\gamma}{\sqrt{c}}$
g_2	$\{\frac{\sqrt{\omega}}{2}, -\frac{\sqrt{\omega}}{2}, 0\}$	$\{\omega, 0\}$	$\left(\frac{4}{\omega}\right)^{1/4}\gamma$
L-generator	$\{\frac{1}{k^2}, -\frac{1}{k^2}, 0\}$	$\{\frac{4}{k^4}, 0\}$	$k\gamma$

TABLE 1. Parameter relationships for $\wp_{\mathbf{L}}$ on a square lattice

Proposition 2.5. [Duval, 1973] If $\mathbf{L}$ is a square lattice in $\mathbb{C}$, then the following hold.

(1) $g_3 = 0$ and $g_2 \neq 0$;
(2) $g_2 > 0$ if and only if $\triangle > 0$. (We call $\mathbf{L}$ *real rectangular (square)*).
(3) $g_2 < 0$ if and only if $\triangle < 0$. (We call $\mathbf{L}$ *real rhombic (square)*).
(4) $\mathbf{L} = [\lambda e^{i\theta}, \lambda e^{i\theta+\pi/2}]$ for some $\lambda > 0$ and $\theta \in (-\pi/2, 0]$. $\mathbf{L}$ is real rhombic iff $\theta = -\pi/4$ and real rectangular iff $\theta = 0$
(5) $e_3 = 0$, $e_1 = \sqrt{g_2}/2 = -e_2$ are the three critical values of the corresponding map $\wp_{\mathbf{L}}$.

Proof. The last property is obtained using Eqs. (2.2) and (2.6). The rest are classical and follow from the series definition of $\wp_{\mathbf{L}}$. □

We define the *standard (rectangular square) lattice* to be the unique lattice corresponding to $g_2 = 4$, ($g_3 = 0$), and giving $e_1 = 1$ and $e_2 = -1$. For the standard lattice we denote the side length by $\gamma \approx 2.62206$ (see eg. [Milne-Thomson, 1950]), and we call the standard lattice $\Gamma = [\gamma, i\gamma]$. The constant γ is called the *standard* side length. We summarize the connections between the various invariants of square lattices and the associated Weierstrass $\wp$ function in Table 1. Starting from the standard lattice defined eg. in [Duval, 1973], all other entries of Table 1 follow from the homogeneity equation (2.3).

We turn to the iteration of Weierstrass elliptic $\wp_{\mathbf{L}}$ functions and the effects on the long term dynamics of iterating $\wp_{\mathbf{L}}$ as $\mathbf{L}$ varies. A lot of expansion occurs in a single application of any $\wp_{\mathbf{L}}$, since if $\mathcal{F}$ is a fundamental domain for $\mathbf{L}$, $\wp_{\mathbf{L}}(\mathcal{F}) = \mathbb{C}_\infty$. On the other hand attracting periodic orbits are known to exist for any shape lattice [Hawkins & Koss, 2005]. These two facts, coupled with the highly nonlinear dependence of $\wp_{\mathbf{L}}$ and $\wp'_{\mathbf{L}}$ on a change in $\mathbf{L}$ as shown in Eqs. (2.3), (2.4), and Table 1 contribute to the complexity of parameter space.

2.2. **Fatou and Julia sets for elliptic functions.** We review the basic dynamical definitions and properties for meromorphic functions which appear in [Baker *et al*, 1991], [Bergweiler, 1993], [Devaney & Keen, 1988], and [Devaney & Keen, 1989]. Let $f\colon \mathbb{C} \to \mathbb{C}_\infty$ be a meromorphic function where $\mathbb{C}_\infty$ denotes the Riemann sphere. The *Fatou set* $F(f)$ is the set of points $z \in \mathbb{C}_\infty$ such that $\{f^n\colon n \in \mathbb{N}\}$ is defined and normal in some neighborhood of z. The *Julia set* is the complement of the Fatou set on the sphere, $J(f) = \mathbb{C}_\infty \backslash F(f)$. Notice that $\mathbb{C}_\infty \backslash \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}$ is the largest open set where all iterates are defined. Since $f(\mathbb{C}_\infty \backslash \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}) \subset \mathbb{C}_\infty \backslash \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}$, assuming there are more than three poles Montel's theorem implies that

$$J(f) = \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}.$$

Let $Crit(f)$ denote the set of critical points of f, *i.e.*,

$$Crit(f) = \{z\colon f'(z) = 0\}.$$

If z_0 is a critical point then $f(z_0)$ is a *critical value*. For each lattice, $\wp_{\mathbf{L}}$ has three critical values and no asymptotic values. The *singular set* $Sing(f)$ of f is the set of critical and finite asymptotic values of f and their limit points. A function is called *Class S* if f has only finitely many critical and asymptotic values; for each lattice $\mathbf{L}$, every elliptic function with period lattice $\mathbf{L}$ is of Class S. The *postcritical set* of $\wp_{\mathbf{L}}$ is:

$$P(\wp_{\mathbf{L}}) = \overline{\bigcup_{n\geq 0} \wp_{\mathbf{L}}^n(e_1 \cup e_2 \cup e_3)}.$$

When $\mathbf{L}$ is square we have from Proposition 2.5 that $\wp_{\mathbf{L}}(e_1) = \wp_{\mathbf{L}}(e_2)$ since they are negatives of each other, and that $\wp_{\mathbf{L}}(e_3) = \infty$ and we can iterate no further. In other words we really only have one critical orbit so we have a simple expression for $P(\wp_{\mathbf{L}})$:

$$P(\wp_{\mathbf{L}}) = \overline{\bigcup_{n\geq 0} \wp_{\mathbf{L}}^n(e_1)} \cup e_2 \cup \infty. \tag{2.7}$$

For a meromorphic function f, a point z_0 is *periodic* of period p if there exists a $p \geq 1$ such that $f^p(z_0) = z_0$. We also call the set $\{z_0, f(z_0), \ldots, f^{p-1}(z_0)\}$ a *p-cycle.* The *multiplier* of a point z_0 of period p is the derivative $(f^p)'(z_0)$. A periodic point z_0 is called *attracting, repelling,* or *neutral* if $|(f^p)'(z_0)|$ is less than, greater than, or equal to 1 respectively. If $|(f^p)'(z_0)| = 0$ then z_0 is called a *superattracting* periodic point. As in the case of rational maps, the Julia set is the closure of the repelling periodic points [Baker *et al*, 1991].

Suppose U is a connected component of the Fatou set. We say that U is *preperiodic* if there exists $n > m \geq 0$ such that $f^n(U) = f^m(U)$, and the minimum of $n - m = p$ for all such n, m is the *period* of the cycle.

Proposition 2.6. If p is an attracting fixed point or a rationally neutral fixed point for $\wp_{\mathbf{L}}$, then the local coordinate chart for the point is completely contained in one fundamental period of $\wp_{\mathbf{L}}$ (in fact in one half of one fundamental period).

Proof. This is due to the periodicity of $\wp_{\mathbf{L}}$; in each case the local form is invertible. If we spill into another half fundamental period or region, then injectivity fails.

□

Julia sets for square lattices exhibit additional symmetry. The following was proved in [Hawkins & Koss, 2002]; its proof uses the simple fact that by Eq. (2.3) we have $\wp_{\mathbf{L}}(iz) = -\wp_{\mathbf{L}}(z)$. The rotational symmetry expressed in the next result is shown in

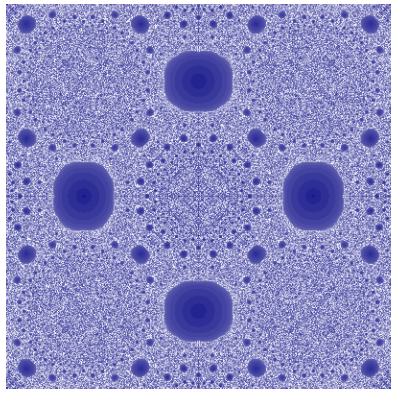

FIGURE 2. There is a lot of symmetry in $J(\wp_{\mathbf{L}})$ when $\mathbf{L}$ is square

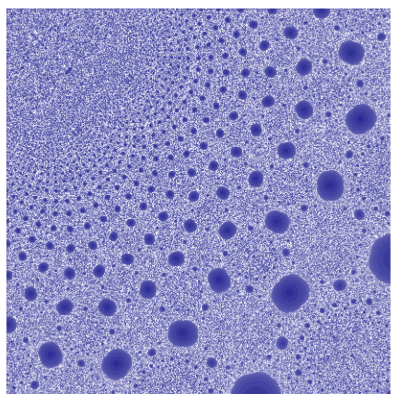

FIGURE 3. A blowup of a piece of the Julia set in Fig. 2 shows the pole and pre-pole structure

Fig. 2. In this figure the blue (or dark) points are in $F(\wp_{\mathbf{L}})$ and the white points approximate $J(\wp_{\mathbf{L}})$.

Theorem 2.7. If $\mathbf{L}$ is square, then $e^{\pi i/2}J(\wp_{\mathbf{L}}) = J(\wp_{\mathbf{L}})$ and $e^{\pi i/2}F(\wp_{\mathbf{L}}) = F(\wp_{\mathbf{L}})$.

In Fig. 3 we zoom in on a part of the Julia set shown in Fig. 2 which shows how the components of the basin of attraction for the attracting fixed point accumulate around poles and prepoles of the map.

3. Parameter Space for $\wp_{\mathbf{L}}$ with $\mathbf{L}$ a square lattice

This section contains the main results of the paper. We begin with a natural parametrization of $\wp_{\mathbf{L}}$ when $\mathbf{L}$ is a square lattice, which makes the family $\{\wp_{\mathbf{L}}\}_{\{\mathbf{L} \text{ a square lattice}\}}$ vary analytically with the parameter in the sense of Definition 3.4 below. There are various choices one can make, but it appears that one good way is to parametrize the Weierstrass elliptic functions with square period lattices using the parameter

$g_2 \in \mathbb{C} \setminus \{0\}$. Therefore it is convenient to simplify the notation by writing for a general lattice, the corresponding Weierstrass elliptic function as:

$$\wp(z, \{g_2, g_3\}) = \wp_{\mathbf{L}}(z) \Leftrightarrow \mathbf{L} = \mathbf{L}(g_2, g_3). \tag{3.1}$$

When $\mathbf{L}$ is square, by suppressing the extra 0 (since $g_3 = 0$) and setting $g_2 \equiv g$, we write

$$\wp(z, g) \equiv \wp(z, \{g, 0\}) \equiv \wp_{\mathbf{L}(g)}(z). \tag{3.2}$$

We want to stress that the function $\wp$ depends both on $z \in \mathbb{C}$ and $g \in \mathbb{C} \setminus 0$ in our study. Then homogeneity equations (2.3) and (2.4) can be recast as follows: for any $u \in \mathbb{C}$

$$\wp(ku, \frac{g}{k^4}) = \frac{1}{k^2}\wp(u, g), \quad \wp'(ku, \frac{g}{k^4}) = \frac{1}{k^3}\wp'(u, g), \tag{3.3}$$

or equivalently,

$$\wp(l^{-1/4}u, lg) = l^{1/2}\wp(u, g), \quad \wp'(l^{-1/4}u, lg) = l^{1/3}\wp'(u, g) \tag{3.4}$$

for $k, l \in \mathbb{C} \setminus \{0\}$. We also note that in this notation the standard lattice gives $\wp_\Gamma(z) = \wp(z, 4)$.

In Fig. 4 we show g colored according to the following algorithm. If there is an attracting periodic orbit for $\wp(\cdot, g)$ we color the parameter $g = a + ib$ dark. Otherwise the parameters are left light. Three fundamental regions have their boundaries marked in red (dark gray), see Corollary 3.2 below. It was shown in [Hawkins, 2006, 2010] that for each value of g along the marked boundary ray, the corresponding map has Julia set the whole sphere.

The problem with the algorithm used to produce Fig. 4 is that it leaves large empty gray areas where the dynamical properties of $\wp$ are not evident; what happens there? As mentioned earlier this is the motivation for this paper. We turn now to an answer to this question and an improved algorithm.

3.1. **Symmetries and Stability.** The parametrization discussed here, using the invariant g_2 to uniquely define $\wp_{\mathbf{L}}$ when $\mathbf{L}$ is square, was introduced in [Hawkins & Koss, 2004] where it was proved that there are some clearly visible symmetries in g-space. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. We say that the holomorphic family of meromorphic maps $\wp_{\mathbf{L}}$ parametrized over the complex manifold M is *reduced* if for all $g \neq g'$ in M, $\wp(\cdot, g)$ and $\wp(\cdot, g')$ are not conformally conjugate.

Theorem 3.1 (Hawkins & Koss, 2004, Thm 9.1)**.** For square lattices $\mathbf{L}_1$ and $\mathbf{L}_2$, $\wp_{\mathbf{L}_1}$ is conformally conjugate to $\wp_{\mathbf{L}_2}$ if and only if $e^{-2\pi i/3}g(\mathbf{L}_1) = g(\mathbf{L}_2)$ or $e^{2\pi i/3}g(\mathbf{L}_1) = g(\mathbf{L}_2)$.

Corollary 3.2. For square lattices, the sector of g-space such that

$$-\frac{\pi}{3} < Arg(g) \leq \frac{\pi}{3}$$

is a reduced holomorphic family of meromorphic maps.

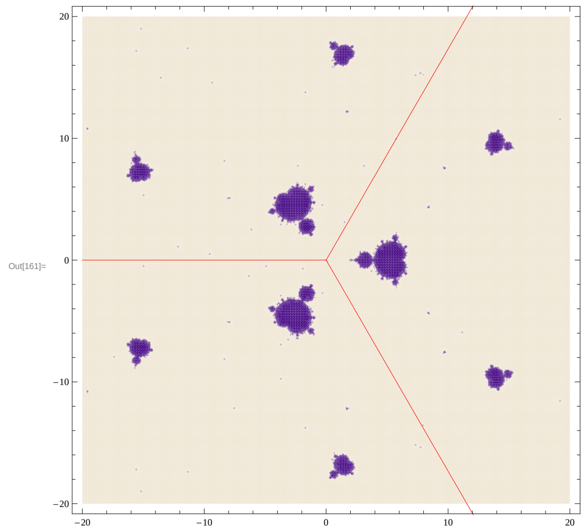

FIGURE 4. g-Parameter space for $\wp_{\mathbf{L}} = \wp(\cdot, \{g, 0\})$ for $\mathbf{L} = \mathbf{L}(g)$ a square lattice as shown in [Hawkins & Koss, 2004]

We call the region of g-space satisfying Corollary 3.2 the *primary fundamental region* of parameter space and denote it by M. It corresponds to the wedge on the right side of Fig. 4:

$$M = \{z = re^{i\theta} \,:\, r > 0, -\frac{\pi}{3} < \theta \leq \frac{\pi}{3}\} \tag{3.5}$$

There is one additional symmetry which does not result in conformal conjugacy, but instead uses the simplest anti-conformal mapping. The next result follows immediately from the series definition (2.1) of $\wp_{\mathbf{L}}$ [Hawkins & Koss, 2002].

Proposition 3.3. Define the map $\phi : \mathbb{C} \to \mathbb{C}$ by $\phi(z) = \overline{z}$; for any lattice $\mathbf{L}$ we have that $\phi \circ \wp_{\mathbf{L}} = \wp_{\overline{\mathbf{L}}} \circ \phi$ and $\phi \circ \wp'_{\mathbf{L}} = \wp'_{\overline{\mathbf{L}}} \circ \phi$. Therefore the map $\wp_{\mathbf{L}}$ is (C^1) conjugate to $\wp_{\overline{\mathbf{L}}}$, and the Julia sets are conjugate under ϕ as well.

The above results explain the visible symmetries in Figs. 4, 5, and 6, and we now turn to the finer structure within each of the six identical subregions of parameter space. In order to explain precisely what is being illustrated in the figure, we use the theory of holomorphic families of meromorphic maps introduced by Keen and Kotus [1997] and studied in this setting in [Hawkins & Koss, 2004]. For simplicity we use $M \subset \mathbb{C}^*$ for the complex manifold parametrizing square lattices, even though Corollary 3.2 shows that the reduced space M is a complex manifold homeomorphic to a cylinder. We give some definitions connecting parameter space to stability for Weierstrass $\wp$ functions with square period lattices.

Definition 3.4. Using the notation of Eq. 3.2:

(1) A *holomorphic family* of elliptic functions $\wp_{\mathbf{L}(g)}$ over a complex manifold M is a holomorphic map $\mathbb{C}_\infty \times M \to \mathbb{C}_\infty$, given by $(z, g) \mapsto \wp(z, g)$

(2) $M^{top} \subset M$ is the set of parameters g which have a neighborhood U with the property that $h \in U$ implies there is a homeomorphism $\phi : \mathbb{C}_\infty \to \mathbb{C}_\infty$ such that $\wp(\cdot, h) = \phi^{-1} \circ \wp(\cdot, g) \circ \phi$.
(3) The set $M^{qc} \subset M$ is defined similarly except that ϕ must be quasiconformal.
(4) We have for each $g \in M$ the set of critical values of each $\wp(\cdot, g)$ has 3 elements; moreover we also have that $\wp_{\mathbf{L}}(e_3) = 0$, which is a pole, and $\wp_{\mathbf{L}}(e_1) = \wp_{\mathbf{L}}(e_2)$ for all g. We label the single critical value $e_1(g)$; for $g \in M$, the critical value is the holomorphic function $e_1(g) = \frac{\sqrt{g}}{2}$. A *singular orbit relation* is a pair of integers $(m, n) \geq 0$ (or a pair of the form (m, ∞)) such that $\wp^m(e_1(g), g) = \wp^n(e_1(g), g)$ (respectively $\wp_1^m(e_1(g), g) = \infty$). The set $M^{post} \subset M$ of *postsingularly stable* parameters consists of all g such that the set of singular orbit relations is locally constant.
(5) A *holomorphic motion* of a set $J \subset \mathbb{C}_\infty$ over a connected complex manifold with basepoint (M, g_o) is a map $\phi : M \times J \to \mathbb{C}_\infty$ given by $(g, z) \mapsto \phi_g(z)$ satisfying:
   - for each fixed $z \in J, \phi_g(z)$ is holomorphic in g;
   - for each fixed g, $\phi_g(z)$ is an injective function of z;
   - $\phi_{g_o}(z) = z$; i.e., it is the identity function at the basepoint.

(6) A holomorphic motion over (M, g_o) *respects the dynamics* if
$$\phi_g(\wp(z, g_o)) = \wp(\phi_g(z), g)$$
whenever z and $\wp(z, g_o)$ both belong to J.
(7) The set $M^{stab} \subset M$ denotes the *J-stable set* of parameters such that the Julia set moves by a holomorphic motion respecting the dynamics.

The next theorem follows from a similar result proved in [Hawkins & Koss, 2004, Prop. 6.3].

Theorem 3.5. If $\mathcal{L} = \{\mathbf{L} = \mathbf{L}(g)\}$ is the set of square lattices parametrized by g, and forming a holomorphic family defined over the complex manifold M,
(1) $M^{qc} = M^{post} = M^{top} = M^{stab}$, and this set is open and dense in M.
(2) M^{stab} is the set of parameters g for which the total number of attracting and superattracting cycles of $\wp_{\mathbf{L}(g)}$ is constant in a neighborhood of g.

The focus of this paper is on the complement of M^{stab} in M.

Definition 3.6. The *bifurcation locus*, M^{bif} is defined to be $M \setminus M^{stab}$.

3.2. **Critical pre-poles in parameter space.** As usual $\mathbf{L}$ is assumed to be a square lattice; hence $e_3 = 0$ is always a pole. Our first observation is that whenever there exists an $m \in \mathbb{N}$ such that $\wp_{\mathbf{L}}^m(e_1) = \lambda \in \mathbf{L}$, then $J(\wp_{\mathbf{L}}) = \mathbb{C}_\infty$. This is because the condition forces all critical orbits to terminate at ∞ so there can be no Fatou component. It is well known that in this setting as for rational maps, each Fatou component requires an associated critical orbit, as is discussed for example in [Kotus, 2006]. The values of g for which this occurs form the "pre-poles" of parameter space.

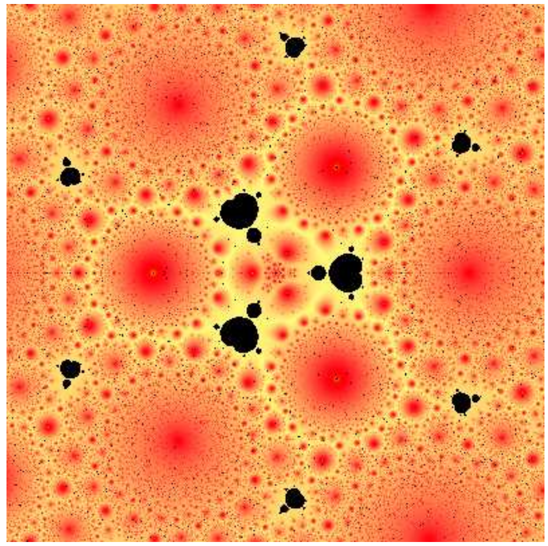

FIGURE 5. g-parameter space with $-20 \le \Re(g), \Im(g) \le 20$, showing parameter pre-poles and the presence of attracting orbits

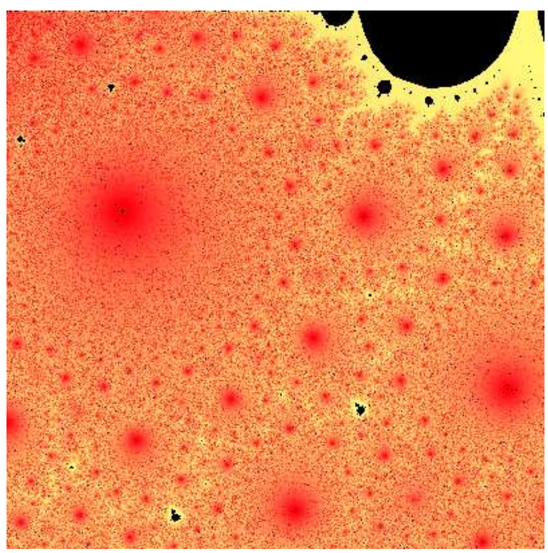

FIGURE 6. g-space with $0 \le \Re(g) \le 4, -4 \le \Im(g) \le 0$, showing parameter pre-pole details

Definition 3.7. (1) If $\mathbf{L}$ is a square lattice, $\mathbf{L} = \mathbf{L}(g)$, we say g is a *(critical) pre-pole parameter of order* k if $\wp^k_{\mathbf{L}}(e_1) = \infty$.

(2) We say that g is a *superattracting parameter of order* m if $\wp^{m-1}_{\mathbf{L}}(e_1) \in c_j + \mathbf{L}$, for some $j = 1, 2$, and $m \in \mathbb{N}$ is minimal.

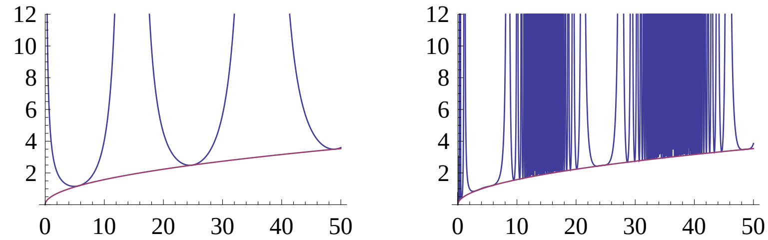

FIGURE 7. Graphs of $Q_1(g)$ and $Q_2(g)$

(3) An order 1 parameter pre-pole is also called a *parameter pole*, and since at an order 1 superattracting parameter the critical value is a critical point, it is also called a *superattracting parameter*.

Summary of the structure of parameter space. In the results that follow we show that parameter space is structured as follows:

(1) There are subhyperbolic regions where the free critical orbit tends to an attracting periodic orbit. These regions look like Mandelbrot sets for quadratic polynomials and contain stable parameters. These regions are colored black in Figs. 5 and 6.
(2) There are pre-pole parameters: parameters for which the free critical orbit lands on a pole. These parameters are colored red in Figs. 5 and 6.
(3) In every neighborhood of a pre-pole we find more tiny Mandelbrot sets and more poles. More precisely we prove that every pre-pole in parameter space of order n is an accumulation point for pre-poles of order $k > n$ and it is also an accumulation point for superattracting parameters of order $k > n$.
(4) It was shown in [Hawkins, 2006, 2010] that along the rays $\{g : g = re^{i\theta}, r > 0, \theta = \pm\pi/3, \pi\}$ the Julia set is the whole sphere. This forms the boundary of $M \subset \mathbb{C}_0$, or equivalently the "seam" of the cylinder M.

Definition 3.8. For each $n \in \mathbb{N}$ the n^{th} *order critical map* Q_n is defined on $\mathbb{C}^*$ by:

$$Q_n(g) = \wp^n_{\mathbf{L}}(e_1) = \wp^n\left(\left(\frac{\sqrt{g}}{2}\right), g\right), \tag{3.6}$$

where $\mathbf{L} = \mathbf{L}(g)$ and $e_1 = e_1(g) = \frac{\sqrt{g}}{2}$ is either nonzero critical value.

Using Table 1 we rewrite $e_1(g)$ as the image of the critical point $c_1 = c_1(g)$ to obtain:

$$Q_n(g) = \wp^{n+1}\left(\left(\frac{4}{g}\right)^{\frac{1}{4}}\frac{\gamma}{2}, g\right) = \wp^{n+1}(c_1, g) \tag{3.7}$$

Fig. 7 shows the graphs of Q_1 and Q_2 restricted to the positive real axis.

We denote by D_n the maximal domain on which Q_n is defined; $D_1 = \mathbb{C}^*$ and $D_{n+1} \subset D_n$ for all $n \geq 1$. If $\mathcal{P}_k$ denotes the set of poles of Q_k in D_k, then $D_{k+1} = D_k \setminus \mathcal{P}_k$.

Important Facts. We set up Definitions 3.7 and 3.8 so that the following hold. (1) Poles of Q_k are in one-to-one correspondence with critical pre-pole parameters of order k. (2) Superattracting parameters of order k correspond to parameters g such that the map $\wp(\cdot, g)$ has a superattracting orbit of period k, hence $Q_k(g) = \frac{\sqrt{g}}{2} = \pm e_1(g)$.

We make a convention about roots of complex numbers. For square roots we slit the plane along the negative real axis and choose a branch that gives a square root of a positive number along the positive real axis. Since we focus on $g \in M$, we take a branch of the cube root function to return a value in that region. The functions Q_n defined above are singly valued and different choices of branches in the identities for the functions give the same results we prove here.

Theorem 3.9. The following properties hold for the first order critical map.

(1) $Q_1(g) = \frac{\sqrt{g}}{2}\, \wp\left((\frac{g^{1/4}}{\sqrt{2}})^3, 4\right)$; therefore Q_1 is a meromorphic function of g on D_1. Q_1 has a pole at g if and only if $g = 4(m+in)^{4/3}\gamma^{4/3}$, where $\gamma \in \Gamma$ is the standard length and m, n is any nonzero integer pair. There are infinitely many order 1 real pole parameters (see Fig. 7).

(2) If $g = 4(m+in)^{4/3}\gamma^{4/3}$, (i.e., if g is a pole of Q_1), then $J(\wp_{\mathbf{L}(g)}) = \mathbb{C}_\infty$.

(3) For any $g \in D_1$, $Q_1(g) = \frac{\sqrt{g}}{2}$ if and only if $e_1(g) = \frac{\sqrt{g}}{2}$ is a critical point of $\wp(\cdot, g)$. For $g > 0$, the local minima of Q_1 occur at exactly at these points (see Fig. 7).

(4) If $g = 4^{1/3}((2m+1+i2n)\gamma)^{4/3}$, $m, n \in \mathbb{Z}$, (using any choice of cube root) then g is a superattracting parameter of order 1. All real order 1 superattracting parameters are positive.

Proof. (1): Fix any $z \in D_1$; Eq. (3.4) with $l = z/4$, $g = 4$, and $u = \sqrt{z}/2$, gives Q_1 relative to the standard lattice $\Gamma = \mathbf{L}(4)$ exactly as in (1). Taking $\frac{d}{dz}$ of $Q_1(z) = \frac{\sqrt{z}}{2}\, \wp\left((\frac{z^{1/4}}{\sqrt{2}})^3, 4\right)$ shows that Q_1 is meromorphic. The poles of Q_1 occur precisely when $\dfrac{g^{3/4}}{2\sqrt{2}} = (m+in)\gamma$, the right hand side giving all lattice points of Γ. We need $g \neq 0$ so we assume $(m, n) \in \mathbb{Z} \times \mathbb{Z} \setminus (0, 0)$. Solving for g gives the result. Setting $n = 0$ and letting m vary, choosing the real cube root gives a real pole parameter. (2): This follows from (1) and the remarks at the beginning of Sec. 3.2. (3): Using (1), we see that $Q_1(g) = \sqrt{g}/2$ if and only if $\wp_\Gamma(w) = 1$, where $w = (\frac{g^{1/4}}{\sqrt{2}})^3$. But $\wp_\Gamma(w) = 1$ if and only if w is a critical point of the form: $w = \frac{(2m+1+i2n)\gamma}{2}$, $m, n \in \mathbb{Z}$ or $w = \frac{(m+i2n+i)\gamma}{2}$, $m, n \in \mathbb{Z}$ (a half lattice point of Γ corresponding to c_1 or c_2, but not c_3) . Setting $(\frac{g^{1/4}}{\sqrt{2}})^3 = \frac{(2m+1+i2n)\gamma}{2}$ and solving for g gives the result. If we choose critical points of the form $c = \frac{(2m+i(2n+1))\gamma}{2}$, then $\wp(c, 4) = -1$, and solving for g as above does not give any new values if we allow all cube roots. For fixed $g_0 > 0$, the map $\wp(\cdot, g_0)$ takes all real numbers into positive real numbers, and in fact onto $[e_1, \infty] = [\sqrt{g_0}/2, \infty]$ (see [Hawkins & Koss, 2002, Lemma 4.7] for a proof). Therefore $Q_1(g_0) \geq \sqrt{g_0}/2$. Using (1), equality occurs when $\wp\left((\frac{g_0^{1/4}}{\sqrt{2}})^3, 4\right) = 1$, and this holds if and only if $\sqrt{g_0}/2$ is a critical point of $\wp(\cdot, g_0)$ by the homogeneity equations. (4): This statement was proved in [Hawkins & Koss, 2002, Corollary 6.9].

□

Similar proofs and induction on n give the following identities for the n^{th} order critical map.

Theorem 3.10. (1) For $n \geq 2$, $Q_n(g) = \frac{\sqrt{g}}{2}\wp\left((\frac{g^{1/4}}{\sqrt{2}})Q_{n-1}(g), 4\right)$.

(2) For $g > 0$, and every $n \in \mathbb{N}$ we have $Q_n \geq \frac{\sqrt{g}}{2}$. For $g \in D_n$, $Q_n(g) = \frac{\sqrt{g}}{2}$ precisely at the order n superattracting parameters.

We now turn to a discussion of the poles of Q_n. As discussed above, each $g \in D_n$ determines a lattice $\mathbf{L}(g)$ and a Weierstrass elliptic function $\wp(\cdot, g)$. Each lattice point of $\mathbf{L}(g)$ is a pole of $\wp(\cdot, g)$. To locate parameter poles and pre-poles in M we proceed inductively. For any $g_0 \in \mathbb{C}^*$, and $R, \varepsilon > 0$, let $B_\varepsilon(g_0) = \{z \in \mathbb{C} : |z - g_0| < \varepsilon\}$, and $B_R(\infty) = \{z \in \mathbb{C} : |z| > R\}$ denote balls about g_0 and ∞ respectively.

Critical pole parameters. There are infinitely many isolated poles of Q_1 giving parameter pre-poles of order one (i.e., critical points that are poles) in M, using Theorem 3.9(1). The formula also shows that the modulus of the poles of Q_1 is unbounded.

Pre-pole parameters of order 2. We apply Theorem 3.9 to obtain a pole of Q_1 as follows: fix any nonzero lattice point γ_0 of the standard lattice Γ, and then $p_1 = 4\gamma_0^{(4/3)}$ is a pole of Q_1. We consider the map on a small ball around p_1; i.e., $Q_1 : B_\varepsilon(g_0) \to \mathbb{C}_\infty$ where ε is chosen small enough so that p_1 is the only pole in the domain.

It follows that Q_1 is a nonconstant holomorphic map from its restricted domain into $\mathbb{C}_\infty$, hence is an open map. Therefore there exists an $R > 0$ such that $B_R(\infty) \subset Q_1(B_\varepsilon(p_1))$. The following lemma now follows immediately from this containment.

Lemma 3.11. For any large enough pole of Q_1, say $4(m\gamma)^{4/3} \equiv \gamma_m \in B_R(\infty)$ (using any cube root), there exists a parameter $h_1 \in B_\varepsilon(p_1)$ such that:

$$Q_1(h_1) = \wp\left(\frac{\sqrt{h_1}}{2}, h_1\right) = \gamma_m.$$

Corollary 3.12. p_1 is an accumulation point of poles of Q_2.

Proof. Given any $\varepsilon_0 > 0$, $\varepsilon_0 < \varepsilon$ above, Lemma 3.11 gives $h_1 \in B_{\varepsilon_0}(p_1)$ such that $Q_1(h_1) = \gamma_m$; then

$$Q_2(h_1) = \wp^2(e_1(h_1), h_1) = \wp(Q_1(h_1), h_1) = \wp(\gamma_m, h_1) = \infty$$

since γ_m is a lattice point for $L(h_1)$. $\square$

We set $h_1 \equiv p_2$ since it is an order 2 pre-pole parameter.

Corollary 3.13. The poles of Q_2 are unbounded in D_2.

Proof. The value of $|p_1|$ can be chosen arbitrarily large and there are poles of Q_2 arbitrarily close to p_1. $\square$

The same proof gives a related result for critical points of Q_1.

Lemma 3.14. For any large critical point of Q_1, say $\rho_m = 4^{1/3}(m\gamma)^{4/3} \in B_R(\infty)$, for m a large odd integer, there exists a parameter $s_1 \in B_\varepsilon(p_1)$ such that:

$$Q_1(s_1) = \wp\left(\frac{\sqrt{s_1}}{2}, s_1\right) = \rho_m = 4^{1/3}(m\gamma)^{4/3}.$$

Therefore the map $\wp_{\mathbf{L}(s_1)}$ has an order 2 superattracting periodic orbit within ε of p_1.

Corollary 3.15. Given any pole p of Q_1, in any arbitrary neighborhood N of p there exists an $h \in N$ which is an order 2 pre-pole parameter and an $s \in N$ which is an order 2 superattracting parameter.

Parameter poles of order k. Suppose we have constructed order j pre-pole parameters p_j, $j < k$, in a nested sequence of balls $B_\varepsilon(p_1) \supset \cdots \supset B_{\varepsilon_{k-2}}(p_{k-2}) \ni p_{k-1}$, and the set $A_{k-1} = \cup_{j<k-1}\mathcal{P}_j$ is the set of accumulation points of the set of parameter pre-poles of of order $k-1$. Also we assume that for any $M > 0$ there exists a pole p of Q_{k-1} such that $|p| > M$. Note that $D_{k-1} \cap A_{k-1} = \emptyset$, so each pole of Q_{k-1} is isolated. We choose $\varepsilon_{k-1} < \varepsilon_{k-2}$ small enough so that $B_{\varepsilon_{k-1}}(p_{k-1}) \subset B_{\varepsilon_{k-2}}(p_{k-2})$, and p_{k-1} is the only pole in the ball. On $B_{\varepsilon_{k-1}}(p_{k-1})$ we define the map: $Q^\Gamma_{k-1}(g) \equiv \dfrac{g^{1/4}}{\sqrt{2}} Q_{k-1}(g)$. Then Q^Γ_{k-1} is a nonconstant holomorphic map with the same poles as Q_{k-1}, hence is open. Therefore there exists an $R_k > R > 0$ such that $B_{R_k}(\infty) \subset Q^\Gamma_{k-1}(B_{\varepsilon_{k-1}}(p_{k-1}))$. We then have the following lemma.

Lemma 3.16. For any large enough $\gamma_k \in \Gamma$, in particular γ_k a pole of $\wp(\cdot, 4)$ and $\gamma_k \in B_{R_k}(\infty)$, there exists a parameter $h_{k-1} \in B_{\varepsilon_{k-1}}(p_{k-1})$ such that

$$Q^\Gamma_{k-1}(h_{k-1}) = \frac{h_{k-1}^{1/4}}{\sqrt{2}} Q_{k-1}(h_{k-1}) = \gamma_k.$$

Set $h_{k-1} \equiv p_k$. Clearly $\wp(Q^\Gamma_{k-1}(p_k), 4) = \wp(\gamma_k, 4) = \infty$.

Corollary 3.17. The order $k-1$ pre-pole p_{k-1} is an accumulation point of poles of Q_k.

Proof. We consider Q_k on $D_k = D_{k-1} \setminus \mathcal{P}_{k-1}$. For p_k chosen above, $Q_k(p_k)$ is defined and using Theorem 3.10,

$$Q_k(p_k) = \frac{\sqrt{p_k}}{2} \wp(Q^\Gamma_{k-1}(p_k), 4) = \infty.$$

Letting $\varepsilon_{k-1} \to 0$ gives the result. $\square$

Corollary 3.18. The order k pre-pole parameters are unbounded in $\mathbb{C}$.

We have just proved the following proposition.

Proposition 3.19. Let $\mathcal{P}_n$ denote the poles of Q_n. Then $\mathcal{P}_n$ has $\cup_{k<n}\mathcal{P}_k$ as its accumulation points.

Superattracting parameters of order k. Using the setup, notation and assumptions as immediately above, suppose in addition we have shown there are infinitely many order $k-1$ superattracting parameters forming an unbounded set in $\mathbb{C}$. Then we have the following lemma.

Proposition 3.20. The order $k-1$ pole p_{k-1} is an accumulation point of critical points of Q_k, and the order k superattracting parameters are unbounded.

Proof. Choose a large half lattice point of the standard lattice Γ along one of the sides of a square; say we have $C_k \in B_{R_k}(\infty)$ and $\wp(C_k, 4) = 1$. Therefore there exists a parameter $s_k \in B_{\varepsilon_k}(g_k)$ such that:

$$Q^{\Gamma}_{k-1}(s_k) = C_k.$$

Then $s_k \in D_k$ so $Q_k(s_k)$ is defined, and:

$$Q_k(s_k) = \frac{\sqrt{s_k}}{2}\wp(Q^{\Gamma}_{k-1}(s_k), 4) = \frac{\sqrt{s_k}}{2} \cdot 1,$$

so $Q_k(s_k) = e_1(s_k)$. By Theorem 3.10 (2) this implies s_k is an order k superattracting parameter.

□

Corollary 3.21. The order k superattracting parameters are unbounded in $\mathbb{C}$.

Using induction on n, this results in the following theorem, which explains what goes on in between the Mandelbrot sets and gives an idea of the complexity of M^{bif}.

Theorem 3.22. In g-space, the parameter space corresponding to square lattices with invariants $g_2 = g$ and $g_3 = 0$, there are infinitely many parameters corresponding to critical pre-poles of order n for each $n \in \mathbb{N}$. Each parameter pre-pole is an accumulation point of parameter pre-poles of higher order and also of superattracting parameters of order $k \geq n$.

The conclusion drawn from these results is that in every neighborhood of an order k pre-pole parameter we see both tiny Mandelbrot sets and more pre-pole parameters. Hence these points in parameter space are highly unstable. Figs. 5 and 6 illustrate Theorem 3.22. Red points are placed at the parameters values g such that the corresponding $\wp_{\mathbf{L}}$, with $\mathbf{L}(g_2, g_3) = (g, 0)$, has the property that all critical orbits terminate at poles. Black points are used to show parameters such that the associated $\wp_{\mathbf{L}}$ has an attracting orbit.

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