DYNAMICS OF MATHEMATICAL GROUPS

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What is a mathematician? Not who, but what. Are there axioms or is there a lemma whose hypotheses, if satisfied, allow you to conclude "then you are a mathematician"? Clearly not; regardless there are some conjectural characteristics that come to mind. Most of us majored in math, and in doing so decided before the age of 20 that we wanted to study math. At some point most of us studied math to the exclusion of all but closely allied subjects. However this is a fairly modern definition; the history of math and mathematicians is much sloppier than that and filled with fortuitous events that led to mathematical discoveries we take for granted today. Nicole Oresme was a Catholic bishop in France in the 1300's during a time when astrologers were members of the intelligentsia; he also held the title of Secretary of the King and wrote popular articles about science by translating Aristotle's work into French. At that time courts often kept on hand astrologers to help predict significant events, but Oresme was strongly against the acceptance of astrology as a science. Philosopher, priest, mathematician and all-around scientist, Oresme used probabilistic arguments in his analysis of nature, long before probability existed. This led to the following admission that he wrote more than once: "..(except for the knowledge of true faith)... I indeed know nothing except that I know that I know nothing." He also came up with the earliest notion of graphs appearing in *Tractatus de latitudinibus formarum* and shown in Figure 1, which plot velocity against time of a moving object. Three hundred years later, Pascal gave the formal mathematical underpinnings of probability theory in order to solve a problem of how to calculate odds correctly when a gambling game is interrupted.

So early mathematicians were not focussing exclusively on math. Oresme's motivation for working out properties of irrational rotation on the circle is well-documented, and brings to mind this hummable song from the late 60's musical *Hair*:

> When the moon is in the Seventh House
> And Jupiter aligns with Mars
> Then peace will guide the planets
> And love will steer the stars

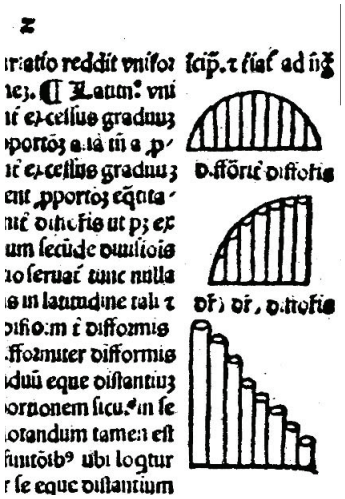


FIGURE 1. Oresme's graphs of velocity vs. time from the 1300's

Indeed Oresme developed the notion of incommensurability of rotations of the circle, irrational rotations, and specifically discussed their aperiodic orbits in order to debunk the ideas of astrology. In other words, to prove that Jupiter will never again align with Mars while the moon is in the seventh house, my favorite research area of ergodic theory was born. This was no fluke; Oresme knew math and physics; while he predicted that it was quite "probable" that the earth moved, it would be another 200 years before Copernicus proved that the sun, not the earth, was the center of the universe.

These are some origins of the math I work with daily, and why it is important that we not try to define what it means to be a mathematician. Nevertheless the purpose of this essay is to give some handwaving arguments and proof-by-example descriptions of what makes mathematicians tick. Translating the mathematical parlance of the previous sentence it means I claim little, will prove nothing, but hope to explain a few of our professional tics.

Mathematicians from the history books conjure up images of men, and rightfully so. Most of them were men. By now we all have heard a few of the depressing stories of women mathematicians before and in the twentieth century. Hypatia, daughter of a well-known Greek mathematician, was a respected philosopher and mathematician. She was accused of being a pagan enchantress and was stripped and torn apart limb from limb by a mob of Christian zealots in the streets of Alexandria in 415 AD. She is alleged to have said: "Reserve your right to think, for even to think wrongly is better than not to think at all",

(but since many believe that none of her writings survived it may not be accurate).

Maria Agnesi, born in 1718 in Milan, analysed an important planar curve which due to a so-called translation error earned her and the curve the nickname Witch of Agnesi.

In 18th century France, Sophie Germain sent comments to her math professors, sometimes including original proofs, using a male pseudonym,"M. le Blanc", so they would be read.

The first woman to get the degree at Cambridge University, Charlotte Scott's name was left off the list of top graduates that year.

Some of the stories have cheerful details. Mary Everest Boole was a self-taught mathematician and a mother of four daughters, married to George Boole. She wrote several books that were attempts to make mathematics more accessible to children. Some of them changed approaches to education completely.

From her 1909 book *Philosophy and Fun of Algebra*, she makes this slightly offbeat plug for learning algebra.

> Many people think that it is impossible to make Algebra about anything except number. This is a complete mistake. We make an Algebra whenever we arrange facts that we know round a centre which is a statement of what it is that we want to know and do not know; and then proceed to deal logically with all the statements, including the statement of our own ignorance.

She's famous for teaching children math through sewing - in fact they were making string envelopes of curves. Rumor has it that she co-wrote a lot of George's papers.

I prefer stories of mathematicians' intellectual valor, I will relate one of my favorites below, but we should not ignore the humbling accounts of women in math. The anecdotes of discrimination against women in math are still legion; who among us hasn't witnessed a few or experienced many? I could tell you stories that could shock you but you might not believe them, and in any case, that's a different essay.

Having broached the subject, here is a sample of what has been said to me.

(1) We don't think you're serious about math so we want you to apply for graduate schools in Europe so you're not competing with the men here. (I received my Ph.D. from the University of Warwick on a Marshall Scholarship.)
(2) Women always sleep with their male coauthors.

(3) Does your having a child mean you're never going to do research again?
(4) Women shouldn't be allowed to attend conferences with men. Too distracting.
(5) I'd like to have you as my Ph.D. advisor but I don't think women should be working outside the home; my wife stays home.

Followed by a sample of what else I've heard said, some of it within the past year.

(1) She is much too good looking to be doing math. She should find a better profession.
(2) I don't think I have ever seen a blond woman doing math before. Are you serious about math?
(3) There is no way I will even consider hiring someone's girlfriend no matter how good she is.
(4) What if we hire them both, they break up, and we're stuck with her?
(5) What area of math do you work on related to this conference, as if I care?

Interspersed with comments of this sort has been a thread, an unbreakable one, of encouragement and support and this is probably true for most mathematicians and probably for all the women whose research continues beyond their Ph.D. There was a male math teacher in each of my elementary, middle and high schools who singled me out for my math ability. In third grade before I knew that life has hardships in it, my teacher announced to the class one day "I think writing down answers to our weekly Times Table Quiz is hampering the progress of some students." The goal was to know single digit multiplication so well we could complete timed random worksheets in 5 minutes or less. He then asked me to stand up and call out the answers, and showed how my best time was cut in half from 3 minutes to one and a half minutes without a pencil. He asked a few others to do likewise, and they fumbled around a bit and did poorly, and then the exercise was quickly ended. I shared a moment with him at that instant when we realized I was different. My grandmother, a second grade teacher, bought me a special Life Science Library book called Mathematics, published in 1963 before I turned 10, that I pored through on my visits to her house. The book is a treasure trove of ideas from calculus, probability, and the newly emerged field of topology, written for the layman; I was completely intrigued by the ideas presented.

Even so, I did not take calculus in high school and declared math as my undergraduate major only as a back-up plan since I knew I was

good at it. I was looking for "a better fit" for something to study; I never found one. Mathematics really started to come to life as a potential career in graduate school at Warwick. I had never met people my age who sat around discussing math for fun instead of to prep for an exam - and professionals came from all over the world to sit in the Common Room sipping coffee and tea for the same purpose. The late 70's was a very exciting time to be at Warwick and working in ergodic theory and dynamics; however the good mentoring and advice I received there came from many directions. Ian Stewart, Dusa McDuff, David Epstein, Christopher Zeeman all advised me, and most importantly the advisor I ended up working with, Klaus Schmidt, was highly encouraging and came as part of a robust and enthusiastic dynamical systems and ergodic theory package at Warwick. Even at the time I felt fortunate to have landed at Warwick so the path there mattered little to me then.

Considering that educating undergraduates is what gets us our paycheck, most mathematicians have an oscillating view of teaching. Some days student interaction is the highlight of a long week with nary an original idea. Other times it is taxing to go into a room and compete with rows of laptops, smart phones, and iPads, while a research paper lies bubbling with energy but untouched. Professional attention to undergraduate instruction has gone in and out of fashion during my years in the field. There is a consensus agreement that we deliver instruction in math so students pick up on the excitement of the field and want to learn more, and to prepare them for a profession in which good analytic math skills will translate quickly into a successful worker. But there are as many ways to achieve that goal, or fail at it, as there are practitioners of the trade.

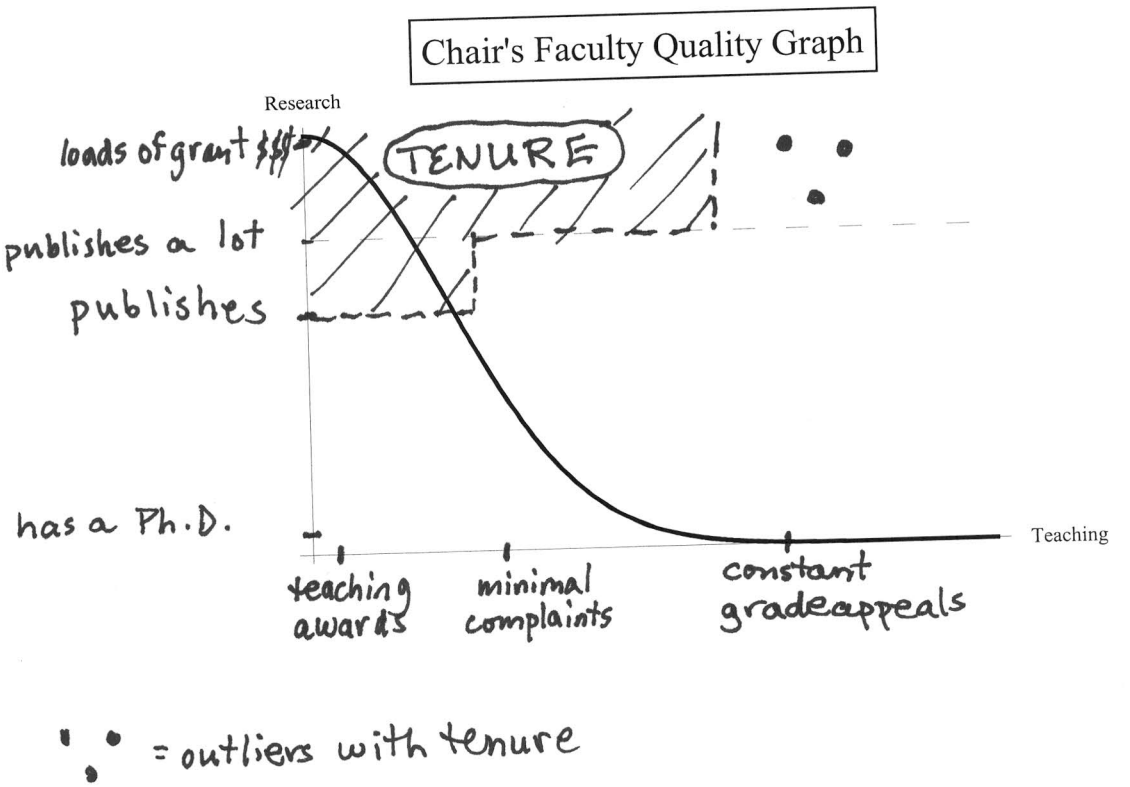
My own introduction to teaching was a baptism by fire. Since I did all of my graduate work in England, I had little idea that in the United States graduate students had teaching duties and hence training. Moreover, as a Marshall scholar, apart from auditing some undergraduate classes, I had little contact with undergrads. The classes I sat in on were courses in measure theory and functional analysis, taught on the level and in the manner that we see in the first few years of American grad school, taken only by advanced British third year undergrads. There was no dialogue with students and very little eye contact; the professor went through the syllabus on the board, pausing briefly to take a breath, and off he or she went again. (I took three or four courses from women faculty.) There were no student evaluations and no pauses during class to see how we were doing out there. We were taking notes,listening, and sometimes thinking; that's what we were

doing. If you thought the professor made an error, you did not interrupt; it was typical to meet with some classmates and your notes after class, go through the material carefully, find errors and determine if they were yours or the prof's. If there was a genuine error then one or more students would carefully approach the instructor at tea or in their office. Should I be ashamed to admit that I thought the system worked? The daily tea attended by virtually all faculty and grads was a key feature, but I thought it worked.

Upon my arrival at Stony Brook for my first job, I was assigned to teach two courses: Galois theory, which I had never taken, and probability theory which had 125 enrolled students and met in an amphitheater. I was 25, lefthanded (this related to my messy use of chalk), too short for the boards in the large classrooms, and simply a terrible instructor. I followed the only model I knew: entered the room and nodded to the class, turned my back to the room, and proceeded to reel off the material from my notes, much of which I had learned the week before. I had terrible stage fright, and also since most of my math learning took place in England, stumbled over "zee's" and "zed's", the pronunciation of Greek letters, and in combinatorics problems attracted snickers by saying anticlockwise instead of counterclockwise. It wasn't long before my undergrads asked the Stony Brook grad students what was wrong with me, thinking I must be a student. Then the advice came trickling in. Most of it was quite helpful too - stop regularly and look at the students, see if they have questions, explain what you're about to do and/or what you've just done. Before each class my stomach would clench up and I would start perspiring profusely; to say that I had no sense of humor while teaching is an understatement. One day one of the grad students who was dating a math major in my class, came to offer more specialized advice. By this time my teaching abilities were legendary among the math majors and graduate students - Writes Fast and Never Turns Around. He said "You need to have more fun in the class. You should actually say funny things once in a while; make the class laugh. You have a sense of humor outside of class. One of my friends wears a gorilla suit from time to time to teach in. You should think about that." That definitely was funny, thinking of myself sweating in a gorilla suit with my back to the class, copying field extension properties from my notebook to the board. Still, I took his point and relaxed a bit and started making eye contact with the class even though I'm quite sure he meant the suggestion literally.

After two years of painful learning by doing, I was a halfway decent teacher, at least that's what the Undergraduate Director thought. My early teaching disasters had gone unnoticed by the faculty except for

the visiting prof from Rice whose office was next door to mine. (He offered a sympathetic ear and we got married a few years later.) The Undergrad Director called me into his office and said "I hear you're a good teacher. That's good, but not so good. Didn't anyone tell you we fire our good teachers?" It was said half in jest, but after checking around I found out that it was generally accepted that if your teaching was good enough to attract compliments, then you weren't spending enough time on research. Given the outcome of that job, all modesty aside, my teaching wasn't really all that good. My years at Stony Brook helped form my current world view on getting tenure in a research department; I've visited and taught in many departments since then and been in my current job for decades. Instead of giving you the thousand words, I refer you to my Faculty Performance Evaluation Charts in Figure 2.



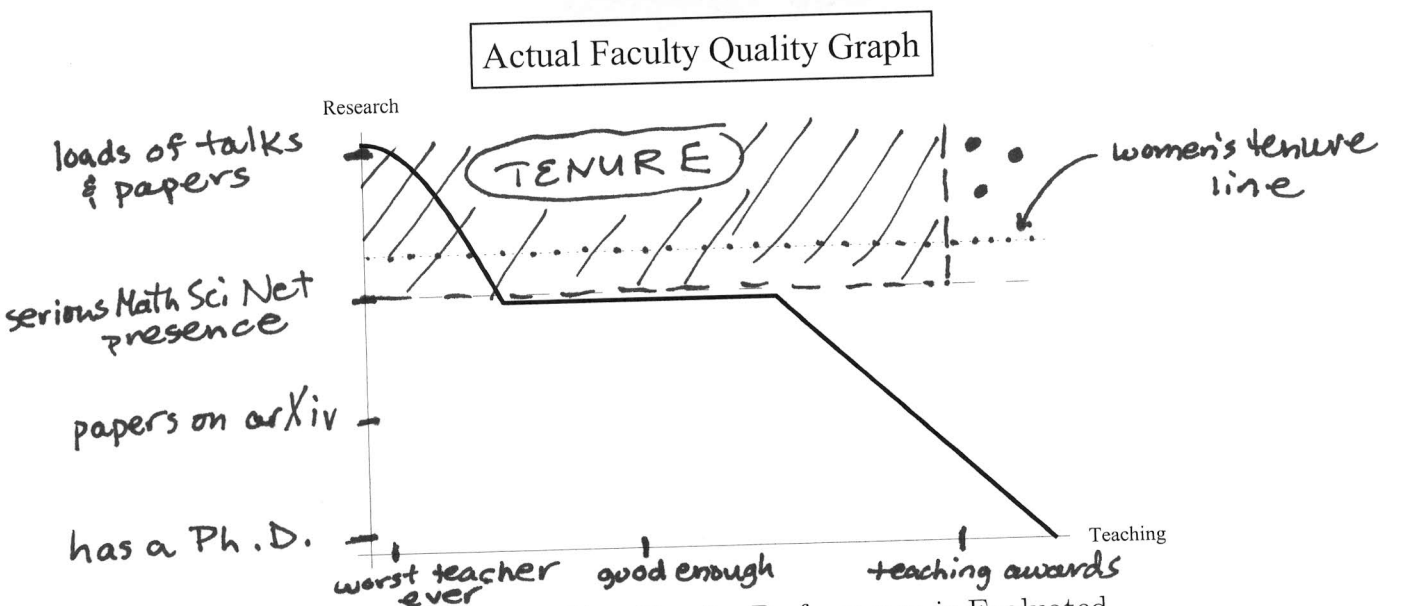


FIGURE 2. How Faculty Performance is Evaluated

= outliers with tenure

Most of us do not have deep mathematical knowledge by the time we start our first job. Becoming a competent research mathematician involves a long apprenticeship working with the best people in our field for as long as we can until we are ready to launch our own research program. What math we know and how we learned it has changed quite a bit in the 3+ decades I have been studying math, and it is divided a bit by generation. Philosopher Marshall McLuhan was prescient in his 1960's book about the impact of media on our culture. At the time he wrote *The Medium is the Massage*, a play on his phrase "the medium is the message", the first minicomputer had been invented. While it is likely that he was referring to television more than computing, now that the line between TV and computer has been blurred it doesn't matter; he was right. There is no doubt but that computers have transformed all areas of pure and applied mathematics. Probably each of us has some relationship to at least one computer, and by relationship I mean that if the device isn't working or available we suffer withdrawal symptoms. But what does a computer do for a mathematician? The answer varies: sometimes it retrieves math papers others have written, sometimes we use it as a desktop publisher, and a computer does a lot of fast calculating, estimating, and visualizing. Let us not overlook the symbolic and logical manipulation too. For me, computers suggest theorems to prove. I love to iterate mappings on my laptop to obtain output in visual form. From there I can guess what is probably going on mathematically and start to prove it.

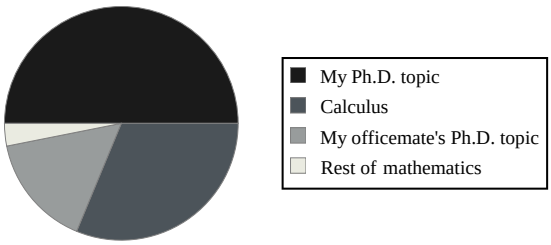


FIGURE 3. Mastery of Mathematics by New Ph.D.'s in the 20th century

Choosing a good math problem to think about and solve is simultaneously very personal and often a byproduct of a chance encounter with another mathematician at a conference, a colloquium, or even in an airport. A propos of working on a good problem, I am a fan of Henri Lebesgue. It takes real guts to write a Ph.D. thesis where you invent a

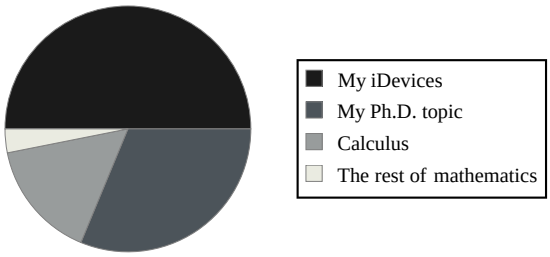


FIGURE 4. Mastery of Mathematics by New Ph.D.'s in the 21st century

new type of integration. There's an apochryphal story about his Ph.D. defense in Nancy, France in the early 1900's. Henri is standing at the blackboard being barraged by questions about his newly defined integral. He is sweating profusely and one of his questioners asks: "But why do you want a new integral? What is wrong with the Riemann integral?" Lebesgue pulls his handkerchief from his pocket and wipes his sweating brow. He crumples it up to return it to his pocket and pauses. He holds up the crumpled cloth and says "So we can integrate functions with graphs like this." Okay, surely you are thinking that unless he tore his handkerchief, it still formed a continuous if not differentiable surface, so the Riemann integral would have been just fine. Nevertheless his sweaty kerchief represented a nasty function, so I like the story.

I was on a panel with women mathematicians a few years ago and a young successful panelist had this advice for the audience comprised mainly of graduate students and new faculty, all female. Never be afraid to try the hard problems. Hard problems that interest you a lot are ones you can likely solve. To embellish what she said a bit, there are problems in any field that attract a lot of time and attention. They resonate on some level due to their simplicity, their demand for technical prowess, or their duality to something you know. Go ahead and look at some of those; maybe you can solve one or maybe you can get in on some interesting action that will keep you busy for a while. You could end up with a Mathematical Object named after you like Lebesgue did.

There is no shortage of jokes about mathematicians being weirdos and social misfits, with punch lines about whose shoes the guy is staring at (his, introvert; yours, extrovert). In my experience we're not

as strange as all that. In fact, most mathematicians are quite well-travelled, well-read or musically inclined, and while not exactly well-dressed, have strong opinions about what they wear. I have dined with colleagues at restaurants sought out for their outstanding pierogi, bulgogi, or barramundi, and witnessed sommeliers being quizzed before a bottle of wine could be uncorked at the table. I have walked miles to a pub in England for the perfect and only acceptable ale. While I'm too squeamish to eat any raw fish with the property that an error in preparation asphyxiates the diner, I've been present for many a conversation where the virtues of fugu are extolled. From what I have seen if a mathematician is staring at another's feet it is to determine if the soles of their shoes give better support than the ones being worn, in which case they will change brands. So why do people insist that we are strange? Because as a group we are extreme in our preferences. I have noticed that many mathematicians, myself included, have an algorithm for dressing that isn't used by the general population. I mean the average person just doesn't use an algorithm for dressing, but hear me out on this. It makes sense to establish a dress code that minimizes the amount of time deciding what to wear while maximizing the ability to teach, give a public talk, and walk comfortably. So we each have a "uniform", and the men are just as particular as the women on this point. Uniforms run along the lines of: black pants, polo shirt, Nikes; or jeans, plaid shirt, New Balance. And it is often the case that the mathematician has a closet full of them. I set up my own wardrobe at the beginning of the semester: line up cotton t-shirts, pants, cardigans, clogs. Choose one from each category each day; pretty simple.

At a recent AMS governance meeting I got into a conversation with a few women mathematicians I hardly know, all of us dressed "one level up" for the administrative occasion, and we discovered we were all wearing the same brand of clothes: clothes with simple lines and colors that are easy to wash and wear, so worth their high price. I was surprised that I adhere to the stereotype, but why not? Saying that mathematicians rarely change our clothes is an oversimplification; if black pants match everything, why buy red ones?

Another way in which we mathematicians are often perceived to be antisocial is in our every day language. By training we use intense focus to produce and solidify ideas that can be communicated only through precise language. Woe unto anyone using superlatives in our presence, family and friends included. If someone says "This is the best movie ever made" to a mathematician, it opens the door to hearing lists of other films considered by the arguer and critics, dating back to before talkies, Oscar winners and nominees, all that might possibly be

better. An alternative response might be simply "I don't care about films" and the subject is changed. It's an amazingly unpopular trait at family gatherings; I see how it gets us a bad reputation. Putting this tendency towards literal interpretation to good use, at a math conference I sometimes make a statement along the lines of "Don't you think that was the best talk on xxx ever given?" It's my way of flushing out information on all talks given in the past 40 years on that topic, because that will always come out immediately and in earnest with no talk left undiscussed.

Many mathematicians are addicted to travel; some will accept every invitation to speak anywhere, and funds permitting, every invitation to attend everything. It is because of this that we have been able to develop our views on food, music, and dress. However this personality trait can wreak havoc with home life and also takes its toll on your health. Deep vein thrombosis is not terribly uncommon among research mathematicians age around 60 or up; nor is divorce. A few years ago, returning from a large overseas conference I was on the same flight as a conference colleague. He had been diagnosed with deep vein thrombosis so every hour or so I could see him walking around the plane, a wild-haired bearded man bobbing up and down above the seats, doing deep knee bends in the middle of the aisle. Even though the airline tells us it's the proper thing to do, most don't, so it attracted more attention than the food carts. We then changed planes stateside to head to our respective homes and once again were in the terminal together. He asked me if I would keep an eye on his backpack, then lay down on his back on the floor, and shutting his eyes he put his feet up on a bench where travellers sit. This made a bit of a scene in our security cautious times. I sat a few seats away reading a great preprint of his, and at some point an airport official came over, walked around his large frame taking up a significant piece of floor space, and then stood near him. The uniformed worker then wondered aloud if anyone knew him or what was wrong with him. Even though I didn't know him well, I jumped up to defend his behavior, explained he was fine as far as I could tell, and was sleeping off the overseas trip while taking care of his circulatory problem. The uniforms starting to gather dispersed, my mathematical friend simply opened one eye and said "Thanks". After a bit of non-conversation we headed off to our respective gates.

There are many ways in which I think mathematicians are cool and wonderful people, a primary reason being that they love math as much as I do and will talk about it at almost any occasion. I've been on some great hikes with math people, I have had some fascinating conversations over good beer. It is quite liberating to discuss math while hiking since

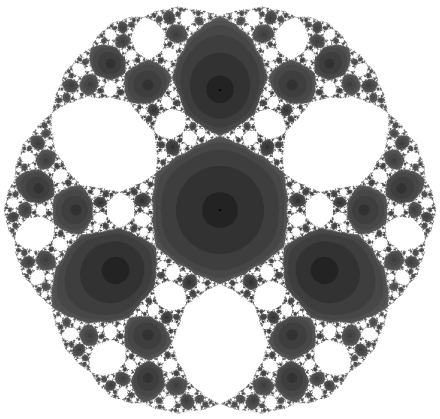


FIGURE 5. A Julia set in $\mathbb{C}$, with symmetries

you can't write anything on paper so you have to keep all the ideas afloat in your head. Over good beer math is great, but sometimes it seems like my head is afloat more than my ideas. I don't have deep vein thrombosis (yet) and I'm still married to the same guy I met at Stony Brook 32 years ago, so I don't think I'm addicted to travel. However I can pack a carry-on bag for two weeks in about an hour so definitely show those tendencies.

You might be a mathematician if your doodling is mathematical, you wake up and think about your problem before getting out of bed sometimes, and your down time is spent working on smaller or back burner math problems. If one computer is perpetually running programs that help suggest theorems, draw pictures, plot things, or compute, you may be one of us. If in addition you have pads of paper lying around with math on them, and if a delayed flight means maybe another step in a proof can be understood, you're probably there. Some of my recent computer doodlings are inspired by the connections between complex dynamics and algebra talked about by Joe Silverman. I am now looking at Julia sets with a lot of symmetry that give well-defined Julia sets on the real projective plane as realized for example by a Steiner surface or Boys surface. I am working with topologist Sue Goodman and we have consulted Mathematica guru Mark McClure to get the pictures, now it's a question of correct statements of the theorems; but these are now jumping up and down waiting to be written down.

The blending of analysis, topology, algebra, and computer algorithms keeps me amused for days on end - actually decades. Returning to the

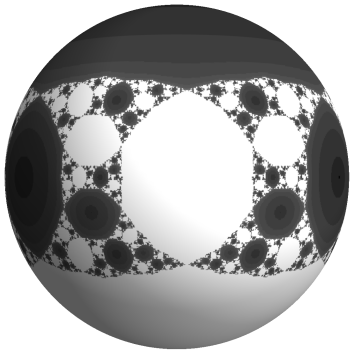


FIGURE 6. The same Julia set on the sphere

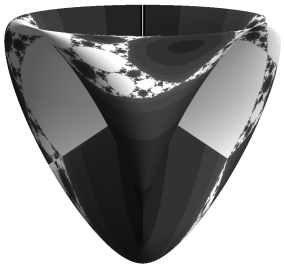


FIGURE 7. Here is that same Julia set, well-defined on a Steiner surface

question that I posed in my opening sentence, I have no answer. And I need to get back to my math.