

Stability of Cantor Julia sets in the space of iterated elliptic functions

Jane Hawkins

ABSTRACT. Starting with a real lattice $\Lambda \subset \mathbb{C}$ that is of one of several shapes, and the Weierstrass elliptic \wp function with poles at lattice points, which we denote by \wp_Λ , we show that there are many maps of the form $a\wp_\Lambda + b$ that are stable under perturbation of all parameters (Λ, a, b) and have Cantor Julia sets. We use this and other stability results to describe moduli space of order 2 elliptic functions with poles at the lattice points.

1. Introduction

There is a well established theory of the dynamics of iterated meromorphic functions [1, 2, 6], and elliptic functions provide an interesting class of examples. An elliptic function is meromorphic and periodic with respect to a lattice $\Lambda \subset \mathbb{C}$; in particular the author and others have shown that the dynamics depend on the lattice Λ , both its shape and size, and also on the function, (see [3],[8]–[13],[15] – [18] for example). In this paper we discuss stability properties and moduli space of a large family of elliptic functions.

The building blocks of all elliptic functions are the Weierstrass elliptic \wp function and its derivative. In this paper we explore the moduli space of:

$$\mathfrak{D} = \{\text{order 2 elliptic functions with double poles at lattice points}\}.$$

We let \wp denote the classical Weierstrass \wp function, which will be defined and discussed in some detail below. The starting point of our study is that for any given lattice Λ , $\wp \in \mathfrak{D}$, and so are the maps $\wp + b$, and $a\wp + b$, for complex constants $a \neq 0$ and b . It is a classical result, as shown for example in [7], and proved in Proposition 2.3 below, that this is all that can occur.

A lattice Λ is a group of complex numbers generated by two linearly independent non-zero vectors in \mathbb{C} ; we write $\Lambda = [\lambda_1, \lambda_2]$, and $\Lambda = \{m\lambda_1 + n\lambda_2 : m, n \in \mathbb{Z}\} \subset \mathbb{C}$. Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere. The quotient space \mathbb{C}/Λ determines a torus. An *elliptic function* $f: \mathbb{C} \rightarrow \mathbb{C}_\infty$ is a meromorphic function in \mathbb{C} which is periodic with respect to a lattice Λ . If f is elliptic, $f(z + \lambda) = f(z)$ for all $\lambda \in \Lambda, z \in \mathbb{C}$, and the set $z + \Lambda = \{z + \lambda, \lambda \in \Lambda\}$ is called *the residue class of z* .

1991 *Mathematics Subject Classification*. Primary 37F45, 30D05.

Key words and phrases. Complex dynamics, elliptic functions, stability.

The author was supported in part by NSF Grant DMS 1600746 .

This means that \mathfrak{D} is locally homeomorphic to a ball in \mathbb{C}^4 ; there are identifications up to complex conjugacy that reduce the space, and one (complex) dimensional slices can be visualized. We discuss some of the structure of $\mathfrak{M} = \mathfrak{D}/\sim$, where $f \sim g$, $f, g \in \mathfrak{D}$ if and only if f is conformally conjugate to g ; this is what we refer to as *moduli space* or reduced parameter space for \mathfrak{D} . For example, the author and Moreno Rocha showed in [13] that b can be parametrized by \mathbb{C}/Λ , which is topologically a torus, and further reductions are possible when the lattice is triangular.

Throughout this paper, we write \wp_Λ when we want to denote the Weierstrass elliptic \wp function with period lattice Λ , except when the statement in which it appears holds independently of the lattice. Despite the complicated structure of moduli space, we prove that there are many regions in moduli space, i.e., open sets in \mathfrak{M} , with the property that for every $(\Lambda, a, b) \in U \subset \mathfrak{M}$, the Julia set $J(a\wp_\Lambda + b)$ is a Cantor set. In the last section we discuss hyperbolic components in \mathfrak{M} with connected Julia sets, typically with b near 0, extending existing results on \wp_Λ .

2. Preliminary background and definitions

We say two lattices $\Lambda_1 = [\lambda_1, \lambda_2]$ and $\Lambda_2 = [\omega_1, \omega_2]$ are *similar* if the ratio $\tau_1 = \lambda_2/\lambda_1$ is equal to $\tau_2 = \omega_2/\omega_1$; call the common value τ . Then $\Lambda_1 = \lambda_1[1, \tau]$ and $\Lambda_2 = \omega_1[1, \tau]$; since there are many choices of generators, we choose a generator so that $\text{Im}(\tau) > 0$. Clearly $\Lambda_1 = k\Lambda_2$ for some nonzero $k \in \mathbb{C}$ exactly when they are similar. Similarity is an equivalence relation between lattices, and an equivalence class of lattices is called a *shape*. In Figure 1 we show a region S of the upper half plane with the property that every lattice shape is represented exactly once, by $[1, \tau]$, $\tau \in S$ (see [11] or [7] for more details).

For any Λ , the *Weierstrass elliptic function* \wp , is defined by

$$(2.1) \quad \wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right),$$

$z \in \mathbb{C}$; it is an even elliptic function with poles of order 2. Its derivative is an odd elliptic function which is also periodic with respect to Λ . Analogous to the case for sine and cosine functions, the elliptic functions \wp and its derivative \wp' , are related to each other via a differential equation, namely:

$$(2.2) \quad \wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

where

$$(2.3) \quad g_2(\Lambda) = 60 \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-4} \text{ and } g_3(\Lambda) = 140 \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-6}.$$

That is, $y = \wp(z)$ is a solution to the differential equation:

$$(2.4) \quad \left(\frac{dy}{dz} \right)^2 = 4y^3 - g_2y - g_3.$$

By differentiating both sides of Eqn (2.2) and solving for \wp'' , we have the identity:

$$(2.5) \quad \wp''(z) = 6\wp^2(z) - \frac{1}{2}g_2.$$

The numbers $g_2(\Lambda)$ and $g_3(\Lambda)$ are invariants of the lattice Λ in the following sense: if $g_2(\Lambda_1) = g_2(\Lambda_2)$ and $g_3(\Lambda_1) = g_3(\Lambda_2)$, then $\Lambda_1 = \Lambda_2$. Furthermore given

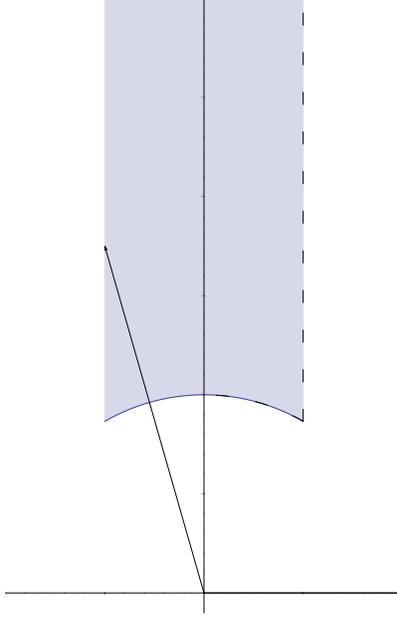


FIGURE 1. Every lattice Λ is similar to one of the form $\Omega = [1, \tau]$ where τ is in the shaded region S . The dotted boundary curves are not in S .

any g_2 and g_3 such that $g_2^3 - 27g_3^2 \neq 0$ there exists a lattice Λ having $g_2 = g_2(\Lambda)$ and $g_3 = g_3(\Lambda)$ as its invariants [7]. For $\Lambda_\tau = [1, \tau]$, the functions $g_i(\tau) = g_i(\Lambda_\tau)$, $i = 2, 3$, are analytic functions of τ in the open upper half plane $\text{Im}(\tau) > 0$ ([7], Theorem 3.2).

From Eqn (2.3) we have the following homogeneity in the invariants g_2 and g_3 .

LEMMA 2.1. For lattices Λ_1 and Λ_2 , $\Lambda_2 = k\Lambda_1 \Leftrightarrow$

$$g_2(\Lambda_2) = k^{-4}g_2(\Lambda_1) \quad \text{and} \quad g_3(\Lambda_2) = k^{-6}g_3(\Lambda_1).$$

A lattice Λ is said to be *real* if $\Lambda = \bar{\Lambda} := \{\bar{\lambda} : \lambda \in \Lambda\}$, where \bar{z} denotes the complex conjugate of $z \in \mathbb{C}$, and the next result is standard.

PROPOSITION 2.2. The following are equivalent:

- (1) Λ is a real lattice;
- (2) $\wp_\Lambda(\bar{z}) = \overline{\wp_\Lambda(z)}$;
- (3) $g_2, g_3 \in \mathbb{R}$.

Given any Λ , for $k \in \mathbb{C} \setminus \{0\}$, using Eqn (2.1), the following homogeneity properties hold:

$$(2.6) \quad \begin{aligned} \wp_{k\Lambda}(ku) &= \frac{1}{k^2}\wp_\Lambda(u) \\ \wp'_{k\Lambda}(ku) &= \frac{1}{k^3}\wp'_\Lambda(u) \end{aligned}$$

Each lattice shape determines an entire parameter space of lattices, obtained by varying either $k \in \mathbb{C}$, or g_2 and g_3 (staying within the shape class), and this

has been studied by many authors, including [8]–[12], [15, 16, 23]. Those studies focus on the diversity and bifurcations of the dynamics of iterating the function \wp_Λ , as Λ varies. In this paper we change the focus to look at stable regions in a larger space of elliptic functions; in other words, are there open sets in \mathfrak{D} for which we see the same dynamics? The following shows that family \mathfrak{D} provides a natural class to consider; details of the proof can be found in ([7], Chapter 1) for example.

PROPOSITION 2.3. Every elliptic function of order 2 with double poles at lattice points is of the form: $f(z) = a\wp_\Lambda(z) + b$, $a, b \in \mathbb{C}$, $a \neq 0$.

PROOF. We assume by hypothesis that f is elliptic with double poles at the lattice points, so that each pole is in the residue class $0 + \Lambda$. Additionally, $f(z)$ must be even, so vanishes in a pair of residue classes $\pm\kappa + \Lambda$ for some κ . Then the function $g(z) = \wp_\Lambda(z) - \wp_\Lambda(\kappa)$ is elliptic with the same zeros and (double) poles as f . Every nonconstant elliptic function must have poles by Liouville's Theorem, so since $f/g(z)$ is an elliptic function with no poles, it is therefore a nonzero constant. This means that $f(z) = a(\wp_\Lambda(z) - \wp_\Lambda(\kappa)) = a\wp_\Lambda(z) + b$ as claimed. \square

The space of elliptic functions of order 2 with double poles has complex dimension 4; for the lattice Λ we need two coordinates, a point in the region shown in Figure 1 and a complex multiple of that. We can denote this pair by (τ, k) to correspond to the lattice $\Lambda = k[1, \tau]$. In addition we need a pair $(a, b) \in \mathbb{C}^2$, with $a \neq 0$; there is also a further reduction since it is enough to choose b from one fundamental region for Λ . There are other identifications when we consider a reduced space (no two maps conformally conjugate), but the maps move holomorphically as we vary each parameter. For example, one can parametrize the lattices Λ by the invariants $(g_2, g_3) \in \mathbb{C}^2$, with singularities at the locus of points $g_2^3 - 27g_3^2 = 0$, or we could use the critical values e_1, e_2, e_3 , with the identifications: e_j 's are all distinct and $\sum e_j = 0$. Therefore we obtain a complex manifold M of complex dimension 4. Our focus will be in a neighborhood of $a = 1$ with Λ a real lattice (however, see also Theorem 7.3), since by varying only Λ and b , we already obtain a parameter space with two complex dimensions. Up to now, if we consider Julia sets associated to iterating the meromorphic function \wp_Λ , for Λ any lattice, all connectivity results show that $J(\wp_\Lambda)$ is connected. However the connectivity of $J(\wp_\Lambda)$ has not been established for all lattices yet.

The main results of this paper are summarized by the following. We give the needed definitions immediately below.

THEOREM 2.4 (Main Result). *If $\Lambda = [1, \tau]$ is a lattice that is square, real rectangular, or real triangular, then there exists some $k \in \mathbb{R}$ and $b \in \mathbb{C}$ such that in a neighborhood of $(\tau, k, a, b) \subset \mathbb{C}^4$ all Julia sets are Cantor and $a\wp_{k\Lambda} + b$ is J -stable. That is, the map lies in a hyperbolic component of moduli space.*

THEOREM 2.5. *There is no square lattice Λ for which \wp_Λ is J -stable.*

We now turn to definitions and proofs of the main results.

2.1. Real period lattices for \wp_Λ . For most of this paper we assume that $\Lambda = [\lambda_1, \lambda_2]$, with $\lambda_1 > 0$ and λ_2 lying in the upper half plane. A closed, connected subset Q of \mathbb{C} is a *fundamental region* for Λ if for each $z \in \mathbb{C}$, Q contains at least one point in the same Λ -orbit as z , and no two points in the interior of Q are in

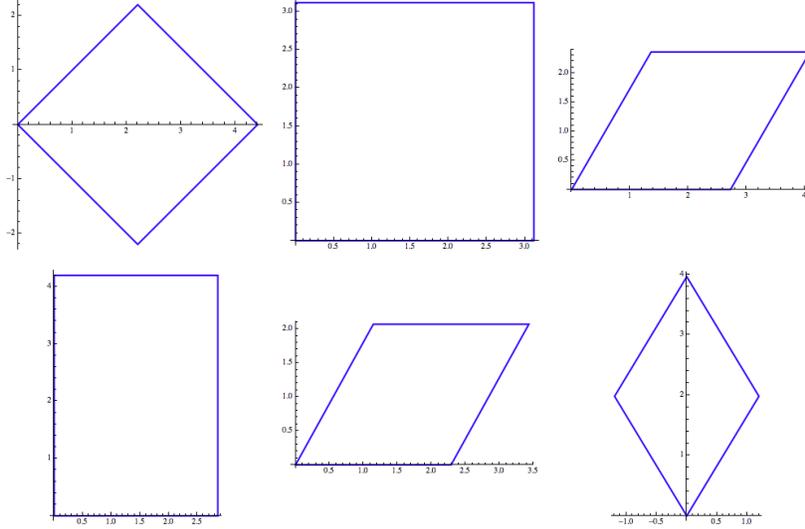


FIGURE 2. A fundamental region for each of the possible real lattice shapes, going clockwise from upper left: rhombic square ($g_2 < 0, g_3 = 0$), rectangular square ($g_2 > 0, g_3 > 0$), triangular rhombic ($g_2 = 0, g_3 > 0$), vertical rhombic ($g_2 < 0, g_3 > 0$), horizontal rhombic ($g_2 > 0, g_3 > 0$), and real rectangular ($g_2 > 0, g_3 > 0$),

the same Λ -orbit of $z \mapsto z + \lambda$, with $\lambda \in \Lambda$. If Q is a fundamental region for Λ , then for any $s \in \mathbb{C}$, the set

$$Q + s = \{z + s : z \in Q\}$$

is also a fundamental region. If Q is a parallelogram, in we call Q a *period parallelogram* for Λ .

If Λ is a real lattice, a fundamental region Q can be chosen to be a rectangle with two sides parallel to the real axis and two sides parallel to the imaginary axis, or a rhombus with sides of the form $(0, \lambda), (0, \bar{\lambda})$ for some nonzero $\lambda \in \mathbb{C}$ [7]. While the property of being a real lattice is not invariant under the similarity relation, multiplication by i , or any real or purely imaginary number, preserves this property. We find the following result useful in what follows.

LEMMA 2.6. *If Λ is a real lattice with invariants (g_2, g_3) , then Λ is similar to a lattice Ω with $g_3 = g_3(\Omega) \geq 0$ and $g_2(\Omega) = g_2(\Lambda)$.*

PROOF. Assume that $(g_2, g_3) \in \mathbb{R}^2 \setminus \mathcal{E}$, where \mathcal{E} is the curve given by $x^3 = 27y^2$. If $g_3 < 0$, then we set $\Omega = i\Lambda$. Then by Lemma 2.1, $g_3(\Omega) = i^{-6}g_3(\Lambda) = -g_3(\Lambda) > 0$, and g_2 is left unchanged by multiplication by i^{-4} . \square

REMARK 2.7. We make the following observations which come directly from classical identities. These have dynamical significance, these are also discussed in more detail in the sources mentioned above.

Critical points: For any lattice Λ , \wp_Λ has infinitely many simple critical points, one at each half lattice point, and we denote them by $c_1 + \Lambda, c_2 + \Lambda$,

and $c_3 + \Lambda$, where

$$c_1 = \frac{\lambda_1}{2}, c_2 = \frac{\lambda_2}{2}, c_3 = \frac{\lambda_1 + \lambda_2}{2}.$$

Since the generators are not unique we adopt the convention that when Λ is real rectangular, λ_1 is real and λ_2 is purely imaginary. When Λ is real rhombic we choose $\lambda_2 = \overline{\lambda_1}$, and λ_1 in the first quadrant. We denote the set of all critical points by $\text{Crit}(\wp_\Lambda)$.

Critical values: \wp_Λ has three critical values $e_j = \wp_\Lambda(c_j)$ satisfying, for Λ real, at least one $e_j \in \mathbb{R}$. Also, one of these hold: $e_2 < e_3 < 0$ (if $g_3 > 0$), $e_2 < 0 < e_3 < e_1$ (if $g_3 < 0$), or $e_3 = 0$ (if $g_3 = 0$). In the third case, $e_2 = -e_1 = \sqrt{g_2}/2$ and $e_3 = 0$, with e_2 and e_1 both real in the *rectangular square* case and complex conjugates in the *rhombic square* case.

Critical value relations: Since for any lattice Λ , e_1, e_2, e_3 are the distinct zeros of Equation (2.2), we have these critical value relations:

$$(2.7) \quad \wp'_\Lambda(z)^2 = 4(\wp_\Lambda(z) - e_1)(\wp_\Lambda(z) - e_2)(\wp_\Lambda(z) - e_3).$$

Equating like terms in Equations (2.2) and (2.7), we obtain

$$(2.8) \quad \begin{aligned} e_1 + e_2 + e_3 &= 0, \\ e_1e_3 + e_2e_3 + e_1e_2 &= \frac{-g_2}{4}, \\ e_1e_2e_3 &= \frac{g_3}{4}. \end{aligned}$$

If we consider the polynomial coming from Equation (2.2),

$$(2.9) \quad q(u) = 4u^3 - g_2u - g_3,$$

a cubic polynomial of the form (2.9) has discriminant:

$$(2.10) \quad \Delta(g_2, g_3) = g_2^3 - 27g_3^2.$$

Real rectangular: The lattice Λ is real rectangular if and only if $\Delta(g_2, g_3) > 0$ (which forces $g_2 > 0$).

Rectangular square: Λ is real rectangular square if and only if the roots of q are $0, \pm\sqrt{g_2}/2$, and then we have: $e_3 = 0$ and $e_1 = \sqrt{g_2}/2 = -e_2 > 0$.

Non-square Rectangular: For real rectangular with $g_3 > 0$, we have $e_1 > 0$, $e_2 < 0$, and $e_2 < e_3 < 0$.

Real rhombic: The lattice Λ is real rhombic if and only if $\Delta(g_2, g_3) < 0$.

Vertical real rhombic: Setting c_1 to be a real critical point, and e_1 the corresponding critical value, $e_1 > 0$, $e_2 = e_1/2 + \zeta i$, for some $\zeta \in \mathbb{R}$ (non-zero), and $e_3 = \overline{e_2}$.

Rhombic square: In this case, $e_1 = 0$ and $e_2 = ib$, with $e_3 = \overline{e_2}$ and $b > 0$.

Our results center around Λ being a real lattice, and g_3 is non-negative by Lemma 2.6. The possible lattice shapes under consideration are shown in Figure 2, though the results we obtain imply these shapes can be perturbed so that the symmetry with respect to the real axis is lost. Despite the fact that a nonconstant analytic function f in a region D does not have any interior maximum points, it is useful for some examples to find the critical values of \wp' . The function \wp' is also elliptic, having the same poles as \wp , so the moduli of the critical values only provide a maximum value of the derivative in some local settings; these can give useful extreme values of \wp' on one real dimensional compact sets.

PROPOSITION 2.8. The critical values of \wp' are all the values of

$$\left[-g_3 \pm \left(\frac{g_2}{3}\right)^{3/2}\right]^{\frac{1}{2}}.$$

PROOF. The critical points of \wp' are the points $u \in \mathbb{C}$ such that $\wp''(u) = 0$, or equivalently by (2.5), where $\wp^2(u) = g_2/12$.

Setting $y = \wp(u)$; we have, using (2.2)

$$(2.11) \quad (\wp'(u))^2 = 4y^3 - g_2y - g_3$$

and $12y^2 = g_2$. Therefore we can write $y = \sqrt{\frac{g_2}{12}} = \pm \frac{1}{2} \left(\frac{g_2}{3}\right)^{1/2}$, which we plug into the right side of (2.11) to see that

$$\wp'(u) = \left[-g_3 \pm \left(\frac{g_2}{3}\right)^{3/2}\right]^{\frac{1}{2}}.$$

□

In [13] Proposition 2.8 was used to determine the maximum modulus of \wp_Λ' along some one real dimensional lines, but it is not a tool whose usefulness is apparent in general, due to the Maximum Modulus Principle.

3. Julia sets

3.1. Fatou and Julia sets for elliptic functions. Definitions and properties of Julia sets for meromorphic functions are discussed in [1, 2, 5] and [6]. Let $f: \mathbb{C} \rightarrow \mathbb{C}_\infty$ be a meromorphic function where $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. The *Fatou set* $F(f)$ is the set of points $z \in \mathbb{C}_\infty$ such that $\{f^n: n \in \mathbb{N}\}$ is defined and normal in some neighborhood of z . The *Julia set* is the complement of the Fatou set on the sphere, $J(f) = \mathbb{C}_\infty \setminus F(f)$. Since $\mathbb{C}_\infty \setminus \overline{\bigcup_{n \geq 0} f^{-n}(\infty)}$ is the largest open set where all iterates are defined and $f(\mathbb{C}_\infty \setminus \overline{\bigcup_{n \geq 0} f^{-n}(\infty)}) \subset \mathbb{C}_\infty \setminus \overline{\bigcup_{n \geq 0} f^{-n}(\infty)}$, then Montel's theorem implies that

$$J(f) = \overline{\bigcup_{n \geq 0} f^{-n}(\infty)}.$$

If f is any elliptic function with period lattice Λ , the *singular set* $Sing(f)$ of f is the set of critical values of f and their limit points. A function is called *Class S* if f has only finitely many critical (and asymptotic) values; for each lattice Λ , every elliptic function with period lattice Λ is of Class *S*. If f is elliptic with critical values $\{v_1, v_2, \dots, v_m\}$, then the *postcritical set* of f is:

$$P(f) = \overline{\bigcup_{n \geq 0} f^n(v_1 \cup v_2 \cdots \cup v_m)}.$$

DEFINITION 3.1. If any elliptic function f with period lattice Λ has a component $W \subset F(f)$ which contains a simple closed loop which forms the boundary of a fundamental region for Λ , then W is a *double toral band*.

The definition of a *hyperbolic elliptic function* is the same as that for a rational map, namely, $J(f) \cap P(f) = \emptyset$, and this is equivalent to uniform expansion on $J(f)$. [11].

THEOREM 3.2 ([11], Sec. 3). *For an elliptic function f , if f is hyperbolic and all the critical values of f are contained in one Fatou component, then all of the following hold:*

- $J(f_\Lambda)$ is a Cantor set;
- W is a double toral band;
- there is exactly one attracting fixed point for f_Λ .

3.2. Cantor Julia Sets and J-stability for $a\wp_\Lambda + b$. Given a lattice Λ , we are interested in the dynamics and Julia sets of order two elliptic functions with poles at every $\lambda \in \Lambda$. Therefore, by Proposition 2.3 we consider the family of maps, for any triple $m = (\Lambda, a, b)$, with Λ a lattice:

$$(3.1) \quad f_m(z) = a\wp_\Lambda(z) + b, \quad a, b \in \mathbb{C}, \quad a \neq 0.$$

By [13] it is enough to consider b coming from one fundamental region of Λ , as $a\wp_\Lambda(z) + b + \lambda$ is conformally conjugate to f_m for any $\lambda \in \Lambda$.

The theory of stable families of holomorphic families, started by Mañé, Sad, and Sullivan [19] and analyzed in [20] was generalized to the setting of meromorphic maps with finite singular set by Keen and Kotus [14]. It is discussed in the elliptic setting in [11]. We summarize it in terms of our current setting.

THEOREM 3.3. *Let f_m be a holomorphic family of elliptic functions, parametrized by $m \in \mathfrak{M}$, and let m_0 be a point in \mathfrak{M} . Then the following are equivalent:*

- (1) *The number of attracting cycles of f_m is locally constant at m_0 .*
- (2) *The maximum period of an attracting cycle of f_m is locally bounded at m_0 .*
- (3) *The Julia set moves homomorphically at m_0*
- (4) *For all m sufficiently close to m_0 , every periodic point of f_m is attracting or repelling (or persistently indifferent).*
- (5) *In the Hausdorff topology, the Julia set $J(f_m)$ depends continuously on m in a neighborhood of m_0 .*
- (6) *For $i = 1, 2, 3$, letting $c_i(m)$ denote the residue class of the critical point c_i for f_m (each c_i depends continuously on Λ), the maps*

$$m \mapsto f_m^k(c_i(m)), \quad k = 0, 1, \dots$$

form a normal family at m_0 .

- (7) *There is a neighborhood $U \subset \mathfrak{M}$ of m_0 such that for all $m \in U$, $c_i(m) \in J(f_m)$ if and only if $c_i(m_0) \in J(f_{m_0})$.*

The set $\mathfrak{M}^{\text{stab}} \subset \mathfrak{M}$ denotes the *J-stable set* of parameters in the sense that any one of the above conditions are satisfied. From now on we use the term *stable* for *J-stable* parameters or maps.

THEOREM 3.4 ([14]). *For any holomorphic family of elliptic functions defined over the complex manifold \mathfrak{M} , $\mathfrak{M}^{\text{stab}}$ is open and dense in \mathfrak{M} .*

We give some key corollaries to Theorem 3.3, the first of which uses Theorem 3.2.

COROLLARY 3.5. For $m = (\Lambda, a, b)$ and f_m as in (3.1), if f_{m_0} is hyperbolic and all the critical values of f_{m_0} are contained in one Fatou component, then $f_{m_0} \in \mathfrak{M}^{\text{stab}}$ and $J(f_m)$ is a Cantor set for m near m_0 .

Fixed	$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$	$\{\mathbf{g}_2, \mathbf{g}_3\}$	Side length	$\wp_\Lambda(\mathbf{c}_3/2)$
Standard	$\{-i, i, 0\}$	$\{-4, 0\}$	γ	1
e_1	$\{-ci, ci, 0\}$	$\{-4c^2, 0\}$	$\frac{\gamma}{\sqrt{c}}$	c
g_2	$\left\{\frac{-\sqrt{-c}}{2}, \frac{\sqrt{-c}}{2}, 0\right\}$	$\{-c, 0\}$	$\left(\frac{4}{c}\right)^{1/4} \gamma$	$\frac{\sqrt{c}}{2}$
side length	$\left\{\frac{-\gamma^2}{c^2}i, \frac{\gamma^2}{c^2}i, 0\right\}$	$\left\{\frac{-4\gamma^4}{c^4}, 0\right\}$	c	$\frac{\gamma^2}{c^2}$

TABLE 1. The relationships among values and parameters of \wp_Λ for rhombic square lattices, using $c > 0$; the last column shows the real quarter lattice point values. $\gamma \approx 2.62206$ is a lemniscate constant (see e.g., [9])

COROLLARY 3.6. For $m = (\Lambda, a, b)$ and f_m as in (3.1), if there are 3 distinct attracting cycles for f_{m_0} , then $f_{m_0} \in \mathfrak{M}^{\text{stab}}$ and $J(f_m)$ is connected for m near m_0 .

PROOF. This follows from Theorem 3.2, Theorem 3.3 and the fact that a perturbation will not destroy the property of a periodic point being attracting. \square

More generally, the following holds.

COROLLARY 3.7. For $m = (\Lambda, a, b)$ and f_m as in (3.1), if f_{m_0} is hyperbolic, then $f_{m_0} \in \mathfrak{M}^{\text{stab}}$.

4. Rhombic square lattices

We first look at the setting where Λ is rhombic square. In this case $g_2 < 0$ and $g_3 = 0$. We note that the map \wp_Λ is quite unstable; in [8] it was shown that the only real critical value is the prepole 0, and this implies that $J(\wp_\Lambda) = \widehat{\mathbb{C}}$, the whole sphere. The next result illustrates the instability of this setting; Table 1 shows the interrelations among side length and critical values.

THEOREM 4.1 (Instability Theorem). *Given $g_2 < 0$, and the associated lattice $\Lambda = \Lambda(g_2, 0) = [\lambda_1, \overline{\lambda_1}]$, and any $\epsilon > 0$ we can find a constant b , $|b| < \epsilon$, such that*

- (1) $J(\wp_\Lambda) = \widehat{\mathbb{C}}$, and
- (2) $J(\wp_\Lambda + b)$ has a super-attracting cycle.
- (3) In (2), we can choose the value $b \in (0, \epsilon)$.

PROOF. The fact that $J(\wp_\Lambda) = \widehat{\mathbb{C}}$ is just Theorem 1.2 of [8]. Using the labelling of the critical points from Remark 2.7, given any small $\epsilon > 0$, since $e_3 = 0$ is a pole of \wp_Λ , \wp_Λ maps $B_\epsilon(0)$ to a set which contains a ball at ∞ ; i.e., there exists $R > 0$ such that $B_R(\infty) \equiv \{z : |z| > R\} \subset \wp_\Lambda(B_\epsilon(0))$.

Similarly, if we define the meromorphic map $\widetilde{\wp}_\Lambda(t) = \wp_\Lambda(t) + t$, then there exists $R > 0$ such that $B_R(\infty) \subset \widetilde{\wp}_\Lambda(B_\epsilon(0))$ as well; moreover for ϵ small enough, $\widetilde{\wp}_\Lambda|_{\mathbb{R}} : (-\epsilon, \epsilon) \rightarrow (A, \infty]$ for some $A > 0$, and the mapping is surjective. We now consider any $t_N = N\lambda + c_3 \in B_R(\infty)$, (a very large critical point in the residue class of c_3), $\lambda \in \Lambda$. We can then find a point $\varepsilon_0 \in B_\epsilon(0)$ such that $\widetilde{\wp}_\Lambda(\varepsilon_0) = t_N$. If $\lambda \in \mathbb{R}$, then $t_N, \varepsilon_0 \in \mathbb{R}$ as well.

For the map $\wp_\Lambda + \varepsilon_0$, we have the following critical orbit:

$$c_3 \mapsto 0 + \varepsilon_0 \mapsto \wp_\Lambda(\varepsilon_0) + \varepsilon_0 = \widetilde{\wp}_\Lambda(\varepsilon_0) = t_N = N\lambda + c_3 \mapsto \varepsilon_0,$$

since $\wp_\Lambda(N\lambda + c_3) = \wp_\Lambda(c_3) = 0$.

Therefore $t_N = N\lambda + c_3$ is a superattracting periodic point for $\wp_\Lambda + \varepsilon_0$, of period 2, and choosing $b = \varepsilon_0$ gives the result. \square

The same proof gives the following result.

COROLLARY 4.2. Assume we have any nonzero $g_2 \in \mathbb{C}$, and the associated lattice $\Lambda = \Lambda(g_2, 0) = [\lambda_1, i\lambda_1]$; equivalently suppose Λ is any square lattice. For any $\epsilon > 0$ we can find a b , $|b| < \epsilon$ such that

- (1) c_3 maps under \wp_Λ to a pole;
- (2) the orbit of c_3 terminates in a superattracting cycle for $\wp_\Lambda + b$.

PROOF. Since Λ is square, $e_3 = 0$. Setting $c_1 = \lambda_1/2$, $c_2 = i\lambda_1/2$, then $c_3 = c_1 + c_2$. Since 0 is a lattice point, $\wp_\Lambda(c_3)$ is a pole. The proof of (2) is identical to that of Theorem 4.1, except that $\widetilde{\wp}_\Lambda(t) = \wp_\Lambda(t) + t$ is a map on \mathbb{C} , and does not leave \mathbb{R} invariant.

The rest of the proof is the same and the superattracting cycle is of the form $\{\varepsilon_0, t_N\}$ for a large critical point $t_N \in B_R(\infty)$ and a small $\varepsilon_0 \in B_\epsilon(0)$. \square

EXAMPLE 4.3. Following the constructive algorithm coming from the proof of Theorem 4.1, if we choose Λ to be the standard rhombic square lattice with $(g_2, g_3) = (-4, 0)$ and $\epsilon = .1$, we can find $\varepsilon_0 \approx .08044 + .0328856i$ so that we have the following critical orbit:

$$c_3 \mapsto \varepsilon_0 \mapsto c_3 + 50\lambda_1 \mapsto \varepsilon_0.$$

We then have a superattracting period 2 orbit containing the critical point $c_3 + 50\lambda_1$; since the basin of attraction for this cycle is extremely small, it is quite difficult to write a computer program accurately showing the Julia set.

Computer experimentation shows that there are many more types of bifurcations occurring for parameters b near 0, so we look for stability away from the poles in this setting. In particular, we look specifically at the constants that will give us superattracting fixed points, keeping in mind that we need g_2 small enough that all the critical values will lie in the immediate attracting basin of that point (see Table 1).

In [9], the following result was proved using the basic homogeneity equations and helps determine the entries in Table 1.

PROPOSITION 4.4 ([9], Prop. 5.6). If Λ_1 is rectangular square and Λ_2 is rhombic square, both of side length γ , then:

$$\wp_{\Lambda_2}(e^{\pi i/4}z) = -i\wp_{\Lambda_1}(z) \text{ and } \wp'_{\Lambda_2}(e^{\pi i/4}z) = e^{-3\pi i/4}\wp'_{\Lambda_1}(z),$$

$$\text{so } |\wp'_{\Lambda_2}(e^{\pi i/4}z)| = |\wp'_{\Lambda_1}(z)|.$$

For $\Lambda_2 = [\alpha + \alpha i, \alpha - \alpha i]$ rhombic square, we label the critical points $c_1 = \frac{\alpha + \alpha i}{2}$, $c_2 = \frac{\alpha - \alpha i}{2}$, and $c_3 = \alpha$, $\alpha \in \mathbb{R}$. Then $e_1, e_2 = \pm\sqrt{g_2}/2 = \pm i\sqrt{|g_2|}/2$ are purely imaginary and $e_3 = 0$.

We note that if $\Lambda = [\alpha + \alpha i, \alpha - \alpha i]$, $\alpha > 0$, then real quarter lattice points occur at $\alpha/2 + 2m\alpha$, $m \in \mathbb{Z}$.

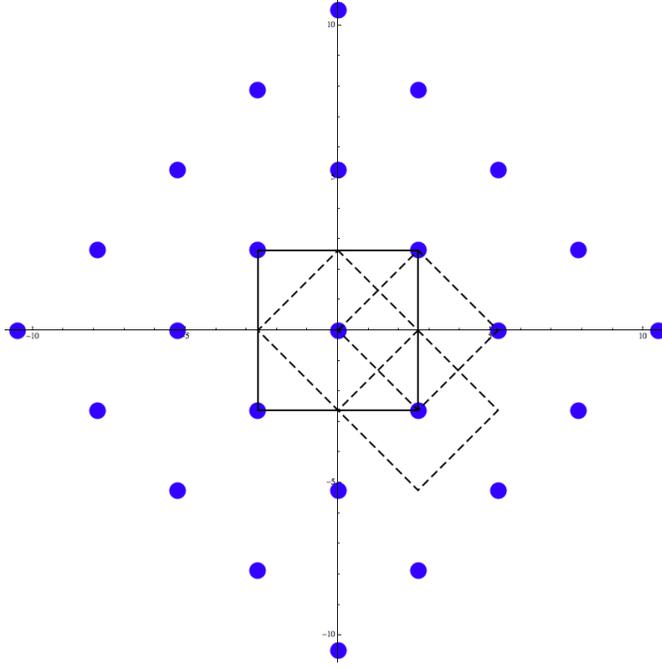


FIGURE 3. We show the lattice points for a real rhombic lattice; the dotted lines show where the map \wp_Λ is purely imaginary and the solid lines, including the axes, show where it is real-valued.

LEMMA 4.5. *Let $\Lambda = [\alpha + \alpha i, \alpha - \alpha i]$, $\alpha > 0$. Then the following hold:*

- (1) *If $g_2 = -4$, then we have the standard lattice $\Lambda(-4, 0) = \Gamma = [\beta + \beta i, \beta - \beta i]$ with square side length γ . Therefore $\beta = \gamma/\sqrt{2}$, $\wp_\Lambda(\beta/2) = 1$ and $\wp'_\Lambda(\beta/2) = -2\sqrt{2}$.*
- (2) *For an arbitrary negative real g_2 the real quarter lattice value of the corresponding lattice is $\wp_\Lambda(\alpha/2) = \sqrt{-g_2}/2 = |e_1| = |e_2|$; moreover if the lattice is $k > 0$ times the standard lattice, $\wp_\Lambda(\alpha/2) = 1/k^2$, and $\wp'_\Lambda(\alpha/2) = (-1)(-g_2)^{3/4} = -2\sqrt{2}k^{-3}$.*

PROOF. These statements follow from the classical identities given by Equations (2.6) and (2.8). \square

REMARK 4.6. From the classical identities, we have the following for rhombic square lattices $\Lambda = [\alpha + \alpha i, \alpha - \alpha i]$; the dotted lines in Figure 3 illustrate where the map \wp_Λ is purely imaginary and the solid lines show where it is real-valued.

- With $c_3 = \alpha \in \mathbb{R}$, and $c_1, c_2 = \frac{1}{2}(-1 \pm i)\alpha$, \wp_Λ is real valued on lines through the diagonals that pass through lattice points.
- \wp_Λ is purely imaginary on the sides of the squares whose vertices are lattice points.
- Let $u \in \mathbb{C}$. Then

$$(4.1) \quad \wp_\Lambda(u \pm c_3) = \frac{e_1 \cdot e_2}{\wp_\Lambda(u)} = -\frac{g_2}{4\wp_\Lambda(u)}.$$

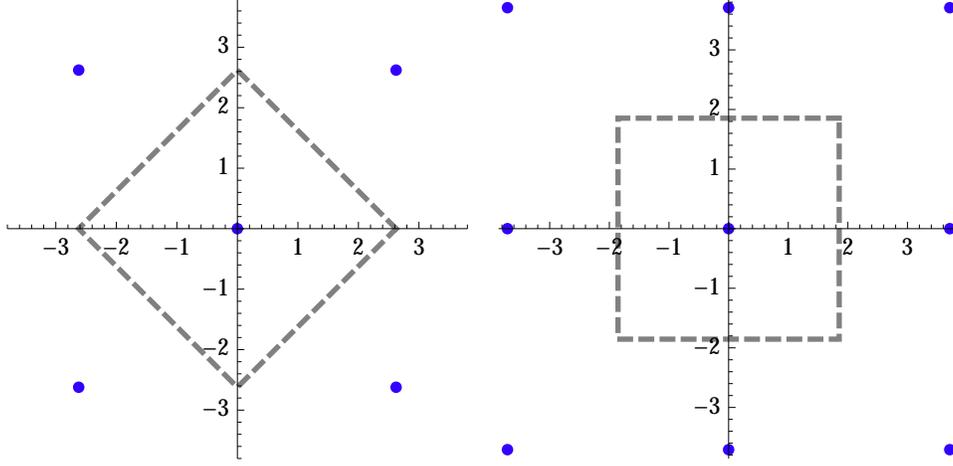


FIGURE 4. On the left, one period square for \wp_Λ for Λ rhombic bounded by ζ ; ζ gets mapped to the imaginary axis by \wp_Λ . On the right, δ bounds a fundamental region of $\Lambda_1 = e^{\pi i/4}\Lambda$.

PROPOSITION 4.7. For any $g_2 \in (-3, 0)$, let ζ denote the closed square boundary of a fundamental region of $\Lambda = \Lambda(g_2, 0)$ formed by line segments from c_3 to ic_3 , ic_3 to $-c_3$, $-c_3$ to $-ic_3$, and $-ic_3$ to c_3 . Then $|\wp'_\Lambda(z)|_{z \in \zeta} < 1$.

PROOF. Denote by Λ_1 the lattice $e^{\pi i/4}\Lambda$, and by δ the square fundamental region with sides parallel to the axes, with vertices at half lattice points. The curve ζ can be written as $e^{\pi i/4}\delta$, and it was shown in [13], and follows from Proposition 2.8, that $|\wp'_{\Lambda_1}(z)|_{z \in \delta} < 1$, so the result follows from Proposition 4.4. \square

Figure 4 illustrates Proposition 4.7.

COROLLARY 4.8. Given any $g_2 < 0$, and the corresponding curve ζ given in Proposition 4.7, set $b = c_3$. Then the image of ζ under $\wp_\Lambda + b$ is the vertical line segment from $c_3 - i\frac{\sqrt{|g_2|}}{2}$ to $c_3 + i\frac{\sqrt{|g_2|}}{2}$.

PROOF. We denote the map $\wp_\Lambda + b = f_m$ with $m = (\Lambda, 1, b)$. We can write ζ as $\zeta_1 \cup \zeta_2 \cup \zeta_3 \cup \zeta_4$ where each ζ_i is a line segment joining the points in the order in which we list them: $\zeta_1 = [ic_3, c_3]$, $\zeta_2 = [c_3, -ic_3]$, $\zeta_3 = [-ic_3, -c_3]$, $\zeta_4 = [-c_3, ic_3]$. Since $\zeta_3 = -\zeta_1$, their images are the same under f_m ; similarly we have $f_m(\zeta_2) = f_m(\zeta_4)$.

The midpoint of each ζ_j is a critical point in the residue class of c_1 or c_2 , and the endpoints of each ζ_j map under f_m to $b = c_3$. From Proposition 4.7 (and its proof) we have that the image of ζ_1 is purely imaginary under \wp_Λ , so its image under f_m lies along a vertical line $c_3 + iy$, $y \in \mathbb{R}$. Each half segment of ζ_1 maps injectively onto $[c_3 + e_1, c_3]$ and each half segment of ζ_2 maps injectively onto $[c_3, c_3 + e_2]$, which, from Table 1 gives the result. \square

Using Lemma 4.5 and Table 1 we have the following.

LEMMA 4.9. *If $(g_2, g_3) = (-1, 0)$ then $|\wp_\Lambda'(z)| < 1$ on the line segment $(c_3/2, 3c_3/2) \subset \mathbb{R}$, $\wp_\Lambda'(c_3/2) = -1$, $\wp_\Lambda'(c_3) = 0$, and $\wp_\Lambda(3c_3/2) = 1$. Moreover, $c_3 = \gamma$, so $c_3/2 > 1$.*

PROOF. We apply Lemma 4.5 (2) with $k = \sqrt{2}$. Since the precise value of γ is known: $\gamma = \Gamma(1/4)^2/(2\sqrt{2}\pi)$, standard approximations give the last statement (cf. [21]). \square

We use these results to obtain a concrete map with a stable Cantor Julia set.

PROPOSITION 4.10. *If $(g_2, g_3) = (-1, 0)$ using $a = 1$ and $b = c_3 \in \mathbb{R}$, then for the map $f(z) = a\wp(z) + b$, the following hold:*

- (1) The critical point c_3 is fixed.
- (2) The diagonal line segment from ic_3 to c_3 (of length $\sqrt{2}\gamma$) is mapped 2-to-1 onto $V = [c_3 - .5i, c_3]$, the vertical line segment. There is one branch point at the midpoint, which is the critical point c_1 . Note that one endpoint is $f(c_1) = c_3 - .5i$ and the other is $f(c_3) = c_3$.
- (3) The set V is mapped two-to-one by f onto $J = [c_3 - \alpha, c_3] \subset \mathbb{R}$, where $\alpha \in (0, 1/12)$;
- (4) The entire interval J converges to c_3 under iteration of f .

PROOF. Since $\Lambda = \Lambda(g_2, g_3)$ is rhombic square, we have that $g_3 = e_3 = 0$; (1) follows from our choice of b , and (2) follows from Corollary 4.8 above.

To prove (3), since $\wp_\Lambda(bi) < 0$ for any nonzero $b \in \mathbb{R}$, and $c_3 \in \mathbb{R}$, f maps V onto the interval in \mathbb{R} of the form $(c_3 - \frac{g_2}{4\wp_\Lambda(.5i)}, c_3)$ by applying (4.1) from Remark 4.6. Using the Laurent series expansion about $u = 0$, which simplifies since $g_3 = 0$, we have that:

$$(4.2) \quad \wp_\Lambda(u) = \frac{1}{u^2} + \frac{g_2}{20}u^2 + \frac{g_2^2}{1200}u^6 + O(z^{10}).$$

Under our hypotheses, we have the following:

$$(4.3) \quad \wp_\Lambda(.5i) = -4 + 0.0125 - \frac{1}{1200}.015625 + O(z^{10}),$$

and since it is an alternating series, $-4 < \wp_\Lambda(.5i) < -3.9875 < -3$; and hence (multiplying by 4 and inverting, using $-g_2 = 1$),

$$-1/16 > -g_2/4\wp_\Lambda(.5i) > -1/4(3.9875) > -1/12,$$

so

$$c_3 - \frac{g_2}{4\wp_\Lambda(.5i)} > c_3 - 1/12.$$

This proves (3).

To prove (4), we note that f maps $B = (c_3/2, 3c_3/2)$ onto (c_3, η) for some $\eta > c_3$, and by Lemma 4.9 and (3), $J \subset B$ and $|f'(z)| < 1$ for all $z \in B$. The result now follows since by (2.5), \wp_Λ'' is always positive on \mathbb{R} , so \wp_Λ' is strictly increasing on \mathbb{R} , and is increasing from a negative value with modulus < 1 to 0 on J , using Lemma 4.9. Then the Mean Value Theorem on \mathbb{R} gives that for all $z \in J$, $\lim_{n \rightarrow \infty} f^n(z) = c_3$. \square

We now give the first main theorem of this section.

THEOREM 4.11 (Stability theorem 1). *Let $\Lambda = \Lambda(g_2, 0)$ be a (real) square rhombic lattice (so $g_2 < 0$). Then there exist $k > 0$ and $b \in \mathbb{R}$ so that the map $f_{m_0} = \wp_{k\Lambda} + b$ is hyperbolic, stable in \mathfrak{M} , and $J(f_{m_0})$ is a Cantor set in a neighborhood of $m_0 = (k\Lambda, 1, b)$.*

PROOF. Since $g_2 \neq 0$, we set $k = 1/|g_2| > 0$. Then $\Lambda(kg_2, 0)$ is still real rhombic square and satisfies all the hypotheses of Proposition 4.10. Then choosing $b = c_3$, c_3 is a superattracting fixed point and there is a double toral band containing all the critical values. By Theorem 3.2, $J(f)$ is a Cantor set.

Using $m = (\ell, \tau, a, b)$ near m_0 (keeping the lattice $\ell[1, \tau]$ near $k\Lambda$) will not affect the presence of an attracting fixed point, under small perturbations of $m_0 \in \mathfrak{M}$, the map f_m remains hyperbolic and $J(f_m)$ remains a Cantor set. \square

4.1. Arbitrary square lattices. We can generalize Theorem 4.11 to any square lattice using the same method, the homogeneity equation (2.6), and applying (4.1).

THEOREM 4.12 (Stability theorem 2). *Let $\Lambda = \Lambda(g_2)$ be any square lattice (so $g_2 \in \mathbb{C}$, $g_2 \neq 0$, and $g_3 = 0$). Then we can find $k > 0$ and $b \in \mathbb{C}$ so that for $m_0 = (k\Lambda, 1, b)$, the map $f_{m_0} = \wp_{k\Lambda} + b$ is hyperbolic, stable in \mathfrak{M} , and $J(f_m)$ is a Cantor set for all m in a neighborhood of m_0 .*

PROOF. We write $g_2 = r \cdot \exp(2\pi i\theta)$, $\theta \in [0, 1)$ and we show the theorem holds for $k = 1/r$ and writing $k\Lambda = \Lambda_1 = [\lambda_1, i\lambda_1]$. We then choose $b = c_3$, where c_3 is the critical point corresponding to the given value of $g_2(\Lambda_1) = 1/r^4 g_2(\Lambda)$, using the technique and notation above. By our choice, $|g_2(\Lambda_1)| = 1$, and the homogeneity identities give us that $|e_1| = 1/2$, $e_2 = -e_1$, and $e_3 = 0$. For simplicity, writing f for f_{m_0} , the same steps as in Proposition 4.10 hold, subject to minor modifications.

- The critical point c_3 is fixed by f ; this is by our choice of b .
- The boundary of a fundamental region formed by the diagonal line segment S from ic_3 to c_3 (of length $\sqrt{2}\gamma$), T from c_3 to $-ic_3$, and $-S, -T$, is mapped onto $V = [c_3 + e_1, c_3] \cup [c_3 + e_2]$, 2 line segments of length .5 meeting at c_3 as follows. Each of S and T gets mapped 2-to-one onto half of V . There is one branch point at the midpoint of S , which is the critical point c_1 (or c_2 , depending on the labels). One endpoint is $f(c_1) = c_3 + e_1$ and the other is $f(c_3) = c_3$.
- The set V is mapped two-to-one onto on interval $J = [c_3 - \alpha, c_3] \subset \mathbb{C}$, with $|J| < 1/12$.
- The entire interval J converges to c_3 under iteration of f .

Using this, we obtain the result since all 3 critical points are in the immediate attracting basin of c_3 . \square

5. Triangular lattices and stability

We now turn to the case of triangular lattices, where some new techniques are required. Let Λ be a real triangular lattice, so we assume $\Lambda = [\lambda e^{2\pi i/3}, \lambda e^{4\pi i/3}]$, $\lambda > 0$. Because every triangular lattice is similar to a lattice of this form, we assume that $g_3 \in \mathbb{R} \setminus \{0\}$ and $g_2 = 0$. There are various convenient ways to generate the lattice besides the one given above; we summarize a few properties of real triangular

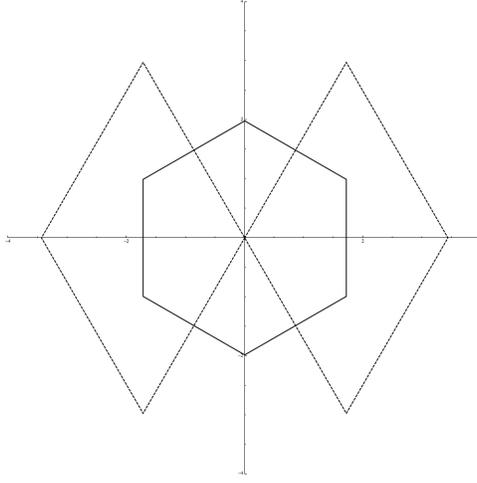


FIGURE 5. Three fundamental regions are shown for real triangular lattice: two are pairs of equilateral triangles and one is a hexagonal period parallelogram.

lattices here. We denote the cube roots of unity by $\omega = \exp(2\pi i/3)$, so $\omega^2 = \omega^{-1}$, and 1. Then $\Lambda = \omega\Lambda = \omega^2\Lambda$. Then a key identity is, for any $u \in \mathbb{C}$,

$$(5.1) \quad \wp_{\Lambda}(\omega u) = \omega \wp_{\Lambda}(u).$$

1. **Critical points:** There are 3 residue classes of critical points for \wp_{Λ} . By convention $c_3 > 0$, then $c_3 + \Lambda$ all lie along lines parallel to the real axis. Then $c_2 = (1/2)\omega\lambda$, and $c_3 = \bar{c}_2$ are a conjugate pair of points, each generating a residue class.
2. **Critical values:** The critical values for \wp_{Λ} are $e_j = \left(\frac{g_3}{4}\right)^{1/3}$, where e_3 is the real root, and the other two are chosen to correspond with the labelling of c_j above. Under our hypotheses, $e_3 \in \mathbb{R}$. We also have that $\text{Re}(e_2) = -e_3/2$, which means that all the critical values have explicit formulas when g_3 is known. In our main example we use small values of g_3 ; e.g., choosing $g_3 = 1/2$, yields $e_3 = 1/2$, and $e_1, e_2 = -1/4 \pm i\sqrt{3}/2$.
3. **Scaling the lattice:** Starting with the *standard lattice* which corresponds to $g_3 = 4$ and $e_3 = 1$, there is an explicit formula for the side length, but we label it as $\lambda^* \approx 2.42865$. Then the scaling works as follows: for a lattice with generators of length $k\lambda^* > 0$ (usually we are interested in $k > 1$), we have $(g_2, g_3) = (0, 4k^{-6})$, which results in critical points of modulus $k\lambda^*/2$, and the new critical values \tilde{e}_j will be $\tilde{e}_j = (1/k^2)e_j$.
4. **Derivatives:** Along vertical lines of the form $(2n+1)c_3 + ti$, $t \in \mathbb{R}, n \in \mathbb{Z}$, we have that \wp_{Λ} is real, and \wp'_{Λ} is purely imaginary. Also, the maximum value of \wp_{Λ} along these lines is e_3 , occurring at c_3 , and \wp_{Λ} decreases from e_3 to $-\infty$, periodically, along the lines. The map \wp'_{Λ} is 0 at the residue class $[c_3]$, and has critical points, or stationary values, where $\wp''_{\Lambda}(z) = 0$ along these lines. This is discussed below.
5. **Current goal:** In ([10], Corollary 3.3) the authors showed that for any triangular lattice, $J(\wp_{\Lambda})$ is connected. We want to reduce g_3 to shrink

the derivative, which is equivalent to stretching the side length of the corresponding lattice; we then choose a value of b so that the symmetry of the triangular lattice forces all three critical points to be in the basin of attraction of a single attracting fixed point.

When $g_3 > 0$, a fundamental region for Λ is made up of two equilateral triangles, with a common side along the real axis, the line segment $[0, 2c_3]$ and there are other natural choices (see Figure 5). Setting $c_4 = c_1 - c_2$, (purely imaginary and in the residue class of c_3), centers of the equilateral triangles occur at $(\pm 2/3)c_4$.

LEMMA 5.1. *For Λ triangular with $g_3 > 0$, we have that $\wp_\Lambda(u) = 0$ at $z_1, z_2 = \pm \frac{2}{3}c_4$, $z_3, z_4 = \pm \omega \frac{2}{3}c_4$, and $z_5, z_6 = \pm \omega^2 \frac{2}{3}c_4$. At each of the zeros of \wp_Λ , we have $\wp_\Lambda'(z_j) = -\sqrt{g_3}i$, and z_j is a stationary point for \wp_Λ' (i.e., $\wp_\Lambda''(z_j) = 0$); moreover $\wp_\Lambda^{(3)}(z_j) = 0$ as well.*

PROOF. These statements follow from ([7], Section 21) and (2.6). By (2.5), $\wp_\Lambda''(z_j) = 6\wp_\Lambda(z_j)^2 = 0$. Differentiating (2.5) gives that $\wp_\Lambda^{(3)}(z_j) = 12\wp_\Lambda(z_j)\wp_\Lambda'(z_j) = 0$ for $j = 1, 2, \dots, 6$. \square

There is an easy variation of Lemma 5.1 when $g_3 < 0$, replacing c_4 by c_3 . We now construct a stable map with Cantor Julia set. We give proofs only of the steps that have not already been proved.

THEOREM 5.2. *Let $(g_2, g_3) = (0, 1/2)$ so Λ is a real triangular lattice, and write $\Lambda = [a + \sqrt{3}ai, a - \sqrt{3}ai]$, $a > 0$. Let $p_0 = z_1 = (2\sqrt{3}a/3)i$ as in Lemma 5.1. Then for the map $f(z) = \wp_\Lambda + p_0$, the following hold.*

- (1) $\wp_\Lambda(p_0) = 0$, so $f(p_0) = p_0$;
- (2) $\wp_\Lambda'(p_0) = -(\sqrt{2}/2)i$, so p_0 is an attracting fixed point.
- (3) \wp_Λ maps the hexagon whose endpoints are z_1, \dots, z_6 onto three line segments of length $1/2$ sharing a common endpoint at p_0 . The other endpoints are $e_1 + p_0$, $e_2 + p_0$, and $e_3 + p_0$.
- (4) All three critical values of f lie in the immediate attracting basin of p_0 .

PROOF. (1) and (2) follow from Lemma 5.1 since $p_0 = z_1$. Using the notation from the lemma, the boundary of a fundamental region formed by the hexagon H with 3 distinguished segments: H_1 from z_1 to z_3 , H_3 , from z_3 vertically down to z_5 , and H_2 from z_5 to $z_2 = -z_1$. They are labelled so that $c_j \in H_j$. Each H_j is mapped by f two-to-one onto a line segment L_j , a line segment of length $.5$ with a common endpoint of $b \in \mathbb{R}$. There is one branch point at the midpoint of H_j , which is the critical point c_j . One endpoint of each image segment L_j is $f(z_k) = b$, and there are 3 spokes coming out with endpoints $p_0 + e_3$, $p_0 + \omega e_3$, and $b + \omega^2 e_3$.

We apply Eqn (5.1) with $u = z - p_0$; then

$$(5.2) \quad \wp_\Lambda(\omega(z - p_0)) = \omega \wp_\Lambda(z - p_0).$$

By induction on n , we show that $|f^n(c_j) - p_0| = \rho_n$ for $j = 1, 2, 3$. For $n = 1$, this is just the statement that $|e_j| = 1/2$ for each j . Assume that $|f^{n-1}(c_j) - p_0| = \rho_{n-1}$ for $j = 1, 2, 3$; then

$$\begin{aligned} f^n(c_j) &= f(f^{n-1}(c_j)) \\ &= \wp_\Lambda(f^{n-1}(c_j)) + p_0; \end{aligned}$$

therefore

$$\begin{aligned} |f^n(c_1) - p_0| &= |\wp_\Lambda(f^{n-1}(c_1))| \\ &= |\omega| \cdot |\wp_\Lambda(f^{n-1}(c_2))| = |\omega^2| \cdot |\wp_\Lambda(f^{n-1}(c_3))| \\ &= \rho_n, \end{aligned}$$

which is independent of j . Since the attracting fixed point p_0 must contain one of the critical values e_{j_0} , $\lim_{n \rightarrow \infty} \rho_n = 0$, and by the symmetry of the critical orbits about p_0 , they all lie in the same attracting basin of p_0 . □

We obtain the next stability result as a corollary.

THEOREM 5.3 (Stability Theorem 3.). *Let $\Lambda = \Lambda(0, g_3)$ be a real triangular lattice. Then there exists $k \in \mathbb{R}$ and $b \in \mathbb{C}$ so that the map $\wp_{k\Lambda} + b$ is hyperbolic, stable in \mathfrak{M} , and $J(\wp_{k\Lambda} + b)$ is a Cantor set in a neighborhood of $m_0 = (k\Lambda, 1, b)$.*

PROOF. Since $g_3 \neq 0$, we set $k = (2g_3)^{1/6}$, taking the real root. Then $k\Lambda = k\Lambda(0, g_3) = \Lambda(0, 1/2)$, by Lemma 2.1. We apply Theorem 5.2, choosing $b = p_0$, the corresponding zero for $\wp_{k\Lambda}$. We have a double toral band for the map $f(z) = \wp_{k\Lambda}(z) + p_0$ containing all the critical values, so by Theorem 3.2, $J(f)$ is a Cantor set.

Under small perturbations around the point: $m_0 = (k\Lambda, 1, p_0)$ the map f_m remains hyperbolic and $J(f_m)$ remains a Cantor set. □

Since minor modifications give Theorem 5.2 for $(g_2, g_3) = (0, -1/2)$, using the same techniques and identities, we can always choose the scaling constant to be a positive real number; this is illustrated in Figure 7. We note that when $g_3 = 1/2$, the fixed point p_0 is purely imaginary, and when $g_3 = -1/2$, p_0 is real.

THEOREM 5.4 (Stability Theorem 4.). *Let $\Lambda = \Lambda(0, g_3)$ be any triangular lattice. Then there exists $k > 0$ and $b \in \mathbb{C}$ so that the map $\wp_{k\Lambda} + b$ is hyperbolic, stable in \mathfrak{M} , and $J(\wp_{k\Lambda} + b)$ is a Cantor set in a neighborhood of $m_0 = (k\Lambda, 1, b)$.*

PROOF. Since $g_3 \neq 0$, we set $k = (2|g_3|)^{1/6}$, choosing the positive real root. The rest of the proof is as in Theorem 5.3. □

EXAMPLE 5.5 (Stable example and conjecture). In this example, for a real triangular lattice, $\Lambda = [0, g_3]$, $g_3 \neq 0$, we describe a method to obtain a triangular lattice such that the map $f = \wp_{k\Lambda} + b$, with $b = c_3 - e_3$ is hyperbolic, has a super-attracting fixed point at $c_3 \in \mathbb{R}$, and $J(f)$ is a Cantor set. Assuming this example exists, and we show one in Figure 8, the map f lies in a hyperbolic component of \mathfrak{M} . In particular, the value of b is quite different from the earlier examples.

Idea of the proof: We scale Λ and use the lattice $k\Lambda$ given by $k = (2g_3)^{1/6}$, so that we can assume that $g_3 = 1/2$. Therefore without loss of generality, it suffices to prove the result for $g_3 = 1/2$. We follow the steps as in Proposition 4.10, with modifications for the lattice shape:

- The critical point c_3 is fixed; this is by our choice of b .
- The boundary of a fundamental region formed by the hexagon H with 3 distinguished segments: H_1 from z_1 to z_3 , H_3 , from z_3 vertically down to z_5 , and H_2 from z_5 to $z_2 = -z_1$. They are labelled so that $c_j \in H_j$. Each H_j is mapped by f two-to-one onto a line segment L_j , a line segment of length .5 with a common endpoint of $b \in \mathbb{R}$. There

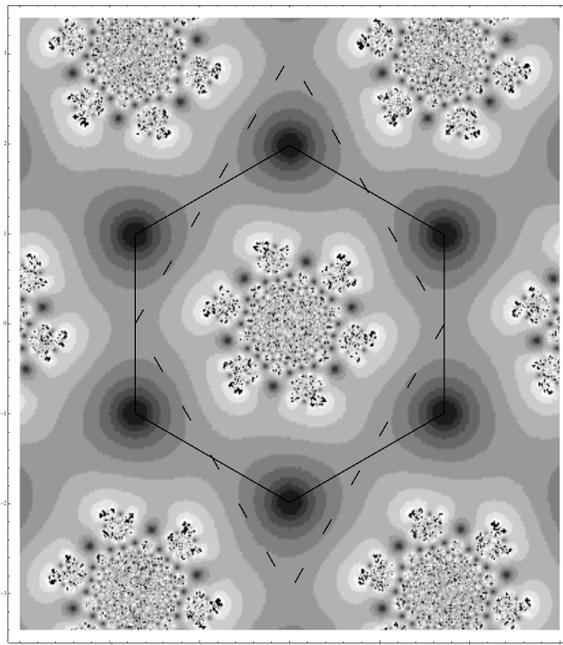


FIGURE 6. A hexagonal fundamental region for a real triangular lattice and a quadrilateral region (dashed). The 6 points of intersection between the two fundamental regions are points where $\varphi_{\Lambda}'(c_j) = f'(c_j) = 0$, with f as in Thm 5.2; $J(f)$ is the Cantor set surrounded by white halos.

is one branch point at the midpoint of H_j , which is the critical point c_j . One endpoint of each image segment L_j is $f(z_k) = b$, and there are 3 spokes coming out with endpoints c_3 , $b + \omega c_3$, and $b + \omega^2 c_3$.

- The set $L_1 \cup L_2 \cup L_3$ converges to c_3 under iteration of f .

The difficulty: It remains to justify the last statement above. We have

$e_2 = -\frac{1}{4} + i\frac{\sqrt{3}}{4}$ and $e_1 = -\frac{1}{4} - i\frac{\sqrt{3}}{4}$. The map $f = \varphi_{\Lambda} + b$ is holomorphic in a neighborhood of c_3 and its series expansion can be calculated term by term. However, it is difficult to prove explicitly that a disk of radius r , with $r > .5$ lies in the immediate basin of attraction.

Assuming the numerics are correct, the map f is stable, and $J(f)$ with the Julia set of a perturbation are shown in Figure 8.

EXAMPLE 5.6. More generally it seems that this technique works for every real lattice. If we consider $(g_2, g_3) = (1.74063, -0.442781)$, we have a real rhombic lattice. We can repeat the technique of previous proofs to find Cantor Julia sets for $f = \varphi_{\Lambda} + b$, with $b = c_3 - e_3$. The Julia set $J(f)$ is shown in Figure 10. The lattice Λ is exactly $2\Lambda_0$, one where $J(\varphi_{\Lambda_0})$ is connected; it appears in Section 7.2, Figure 16.

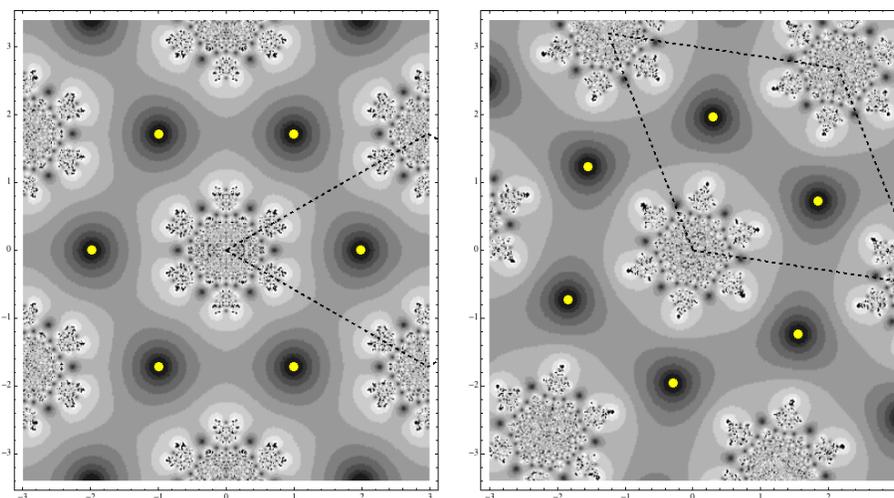


FIGURE 7. Illustration of Thm 5.3 showing Cantor Julia sets for triangular lattices. On the left, $g_3 = -.5$ and on the right, $g_3 = .5 \exp(2\pi i/7)$. The centers of the triangles are marked in yellow (white) and the Julia set is the Cantor set of darker points with white halos.

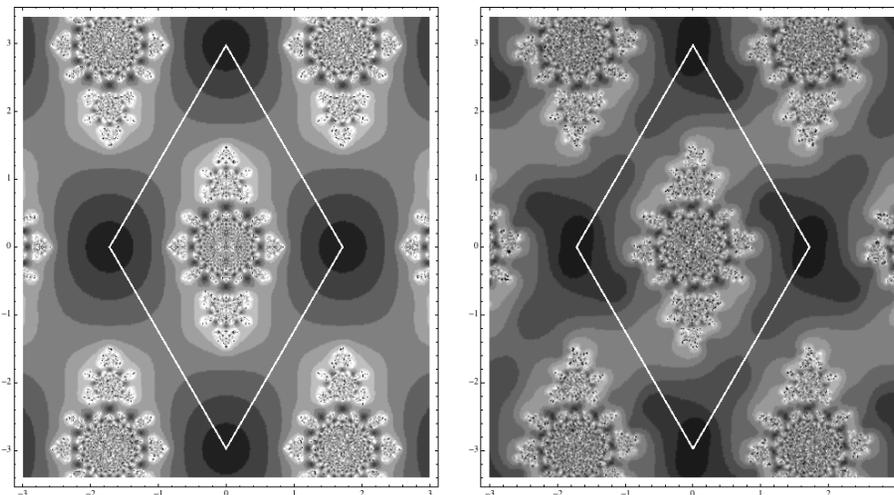


FIGURE 8. $J(\varphi_\Lambda + b)$ from Example 5.5, with $b = 1.21732$ (left), and $J(a\varphi_\Lambda + \tilde{b})$ (right), both with the real triangular lattice from Fig 6; $a = 1.05 + .15i$ and $\tilde{b} \approx 1.2782 + 0.1826i$.

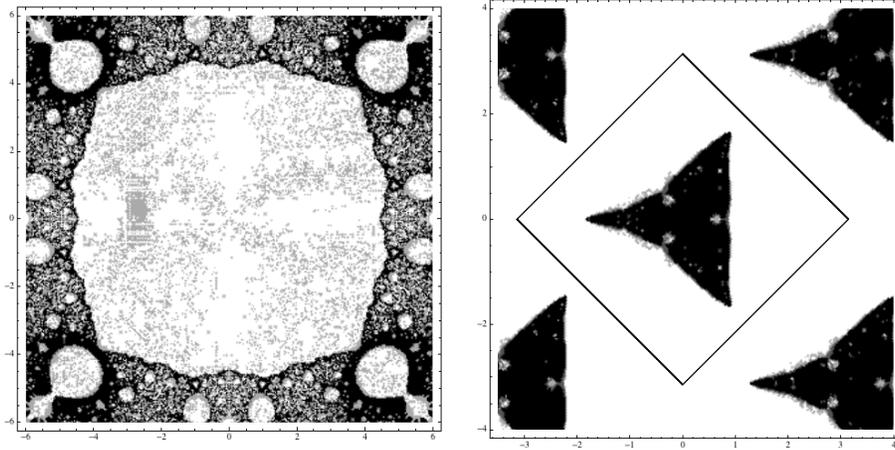


FIGURE 9. On the left we have a -space for the function $a\varphi_\Lambda + b$ with $b = 3.11817$ and on the right, b -space, with $a = 1$. We use $(g_2, g_3) = (-1/2, 0)$ so we expect stability to show up around the value 1 on the left and the value $b = 3.11817$ on the right. Black is associated to unstable parameters, and white is associated to stable ones, with Cantor Julia set.

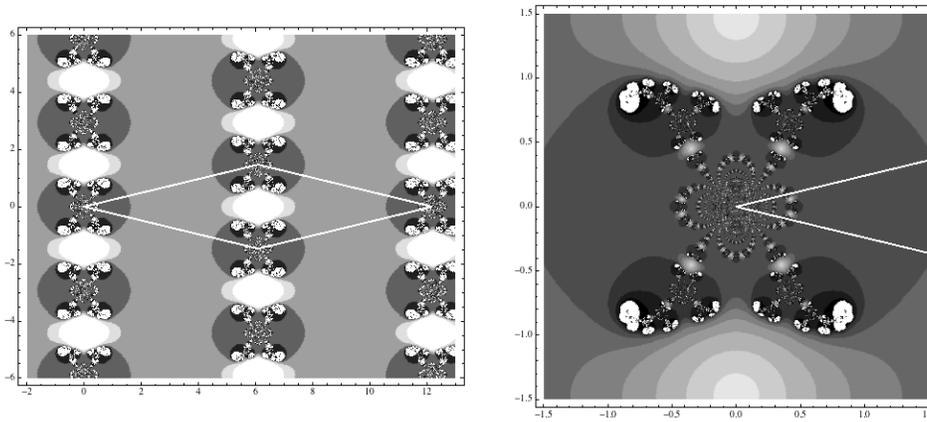


FIGURE 10. $J(\varphi_{(2\Lambda_0)} + b)$ from Example 5.6, a Cantor set, for a real rhombic lattice Λ_0 . Before scaling, $J(\varphi_{\Lambda_0})$ is connected and $F(\varphi_{\Lambda_0})$ has a single toral band, as shown in Figure 16.

6. Real rectangular lattices

In [13], the authors give sufficient conditions for Cantor Julia sets to occur for maps of the form $f(z) = \varphi_\Lambda + b$, with Λ a real rectangular lattice, and b a constant lying on the horizontal half lattice line. They also present a number of examples showing that Cantor Julia sets often occur in this setting. The techniques used

are very particular to real rectangular lattices and do not easily extend to other lattices. We summarize those results here.

Assume $\Lambda = [\lambda_1, i\lambda_2]$, with $\lambda_1, \lambda_2 > 0$; as usual $c_1 = \lambda_1/2, c_2 = i\lambda_2/2$, and $c_3 = c_1 + c_2$. Then $\wp([c_j]) \in \mathbb{R}$, $e_1 > 0$ and $e_2 < 0$.

LEMMA 6.1. *With the labeling as above, using $b = c_3 - e_3$, the function $f = \wp + b$ has a superattracting fixed point at c_3 .*

This is an obvious statement, so it remains to give conditions for which all the critical values are in the attracting basin of the fixed point c_3 of $\wp_\Lambda + b$. We note that $b = \alpha + i\lambda_2/2$ for α real; the key observation is that b lies on the horizontal half lattice line, denoted:

$$(6.1) \quad L = \{z \in \mathbb{C} : z = t + i\lambda_2/2, t \in \mathbb{R}\}.$$

The line L contains all critical values of $\wp_\Lambda + b$ (since $e_j \in \mathbb{R}$). With b fixed at $c_3 - e_3$, and defining $f(z) = \wp_\Lambda + b$, we have the following results from [13].

LEMMA 6.2. *For any real rectangular lattice, the function \wp_Λ maps L into \mathbb{R} and f maps L into L . Moreover, $\wp_\Lambda'(z) \in \mathbb{R}$ if $z \in L$ and reaches a real maximum and minimum on every periodic interval of L .*

PROOF. On L , we have $\wp_\Lambda : L \rightarrow [e_2, e_3]$ ([7], Chap 1.19). For any $t \in \mathbb{R}$, $\wp_\Lambda : \mathbb{R} \rightarrow [e_1, \infty)$, so $f(t + c_2) = \wp_\Lambda(t + c_2) + \alpha + c_2$, which is then of the form $s + i\lambda_2/2 \in L$, since $s \in \mathbb{R}$. A simple argument that the stationary points of \wp_Λ' , which are the zeros of \wp_Λ'' occur on L as well (see e.g. [7]); that is, using (2.5), there is always a real root of $12\wp(z)^2 - g_2 = 0$ between e_2 and e_3 . \square

LEMMA 6.3. *The function f given above maps the line $V = \{\lambda_1 + iy : y \in \mathbb{R}\}$ and $-V$ into L .*

PROOF. \wp_Λ takes \mathbb{R} and L to \mathbb{R} , and \wp_Λ maps V and $-V$ into \mathbb{R} [7], so f maps \mathbb{R}, V , and $-V$ into L whenever $b \in L$. \square

We have therefore reduced the problem to one on L ; moreover $f|_L : L \rightarrow J$, where J is the compact line segment from $e_2 + c_2 \in L$ to $e_3 + c_2 \in L$.

THEOREM 6.4. *If the basin of attraction of $c_3 \in L$ contains $[0, \lambda_1/2] + i\lambda_2/2$, then $J(f)$ is a Cantor set.*

PROOF. The symmetry of \wp_Λ about critical points gives the proof. \square

We can now give the result from [13], stated here in the current context.

THEOREM 6.5 (Stability theorem 4.). *If $\Lambda = \Lambda(g_2, g_3)$ is real rectangular, and satisfies $\left[-g_3 \pm \left(\frac{g_2}{3}\right)^{3/2}\right]^{\frac{1}{2}} < 1$, then $J(f_m)$ is a Cantor set, for $m = (\Lambda, 1, c_3 - e_3)$ and f_m is stable.*

PROOF. By (2.10), Λ real rectangular implies, that $g_2^3 - 27g_3^2 > 0$; equivalently we have

$$\left(\frac{g_2}{3}\right)^3 > g_3^2,$$

so taking either square root, $\left[-g_3 \pm \left(\frac{g_2}{3}\right)^{3/2}\right]^{\frac{1}{2}}$ is real. Then Proposition 2.8 implies that $|\wp_\Lambda'|$ restricted to L is always strictly less than 1, so the entire line L , or equivalently the interval $[e_2, e_3]$ is attracted to e_3 under iteration of f . An application of Corollary 3.5 gives the result. \square

7. Hyperbolic components of \mathfrak{M} with connected Julia sets

There are many results on \wp_Λ and $J(\wp_\Lambda)$ that imply stability for small values of b (e.g., [9, 10, 11]); in other words, $J(a\wp_\Lambda + b)$ moves holomorphically in a neighborhood of $(\Lambda, 1, 0)$. For the examples we consider in this section, $J(\wp_\Lambda)$ is connected. The connectivity of $J(\wp_\Lambda)$ was studied in [11] where the following was proved.

THEOREM 7.1. *$J(\wp_\Lambda)$ is connected if one (or more) of the following holds:*

- (1) *Each critical value e_1, e_2, e_3 lies in its own Fatou component;*
- (2) *There are 3 distinct attracting cycles.*
- (3) *Every periodic Fatou component is completely contained in one fundamental period of Λ .*

These are sufficient but not necessary conditions; for Λ square or triangular, we always have that $J(\wp_\Lambda)$ is connected even if there are no attracting cycles or no Fatou components, both of which can occur ([3, 4, 9, 11]). In the square lattice case, since $e_3 = 0$ is a pole, small perturbations can easily move e_3 to a component of the Fatou set as shown in Theorem 4.1. However, Theorem 7.1(2) will yield stable maps.

PROPOSITION 7.2. *If $\Lambda = k[1, \tau]$ and \wp_Λ has 3 attracting cycles, then $m = (\Lambda, 1, 0)$ lies in a hyperbolic component of moduli space \mathfrak{M} .*

PROOF. There are 3 distinct critical values and each attracting cycle must have at least one critical value in its immediate attracting basin. Since there are no free critical values and the attracting cycles persist under small perturbations, the result follows. \square

Figure 11 shows $J(\wp_\Lambda)$ for Λ real triangular, and Figure 15 shows $J(a\wp_\Lambda)$ for a (different) real triangular Λ , both with three distinct superattracting fixed points; they are discussed below.

7.1. Triangular lattices with hyperbolic maps $a\wp_\Lambda + b$ near $b = 0$. The symmetry in the lattice and the resulting dynamics of \wp_Λ when Λ is a triangular lattice lead to many hyperbolic maps of the form \wp_Λ , and these lie in hyperbolic components of the family $a\wp_\Lambda + b$. We describe the setting briefly here.

In ([9], Theorem 8.3) the authors showed that there are infinitely many values of $g_3 \in \mathbb{R}$ corresponding to triangular lattices with \wp_Λ having three superattracting fixed points, $p_1, p_2 = e^{2\pi i/3}p_1$, and $p_3 = e^{4\pi i/3}p_1$. It is clear that these maps will be in hyperbolic components of \mathfrak{M} since the fixed points and their multipliers move holomorphically near $(\Lambda, 1, 0)$. We show a typical Julia set in Figure 11; there are bifurcations to stable maps with higher period attracting cycles as shown in Figures 12 (with $g_3 \approx 3.082$), coming from the period 2 limb of the Mandelbrot-like set shown in Figure 14, and a perturbation in \mathfrak{M} shown in Figure 13. Examples of the type shown in Figures 11 and 12 were first shown in [9].

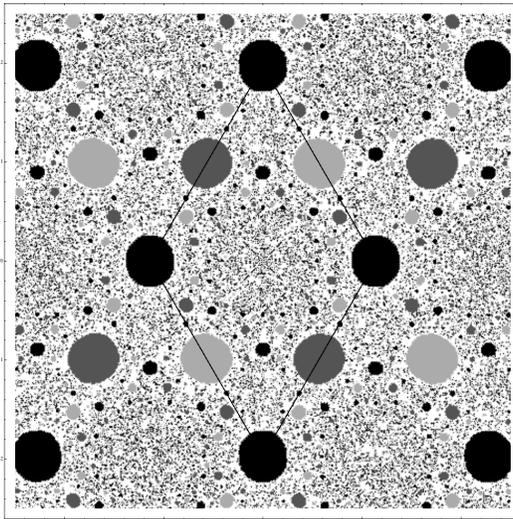


FIGURE 11. $J(\varphi_{\Lambda_0})$ for Λ_0 a triangular lattice chosen according to ([10], Thm 8.3). There are 3 superattracting fixed points which move stably in Λ , a , and b for $a\varphi_{\Lambda} + b$ near $(\Lambda_0, 1, 0)$. The different fixed point basins are colored distinct shades of gray.

We also have the following result which holds more generally in the triangular lattice setting. An example illustrating Theorem 7.3 appears in Figure 15.

THEOREM 7.3. *Suppose Λ is any triangular lattice. If c_1, c_2, c_3 are critical points with corresponding critical values e_1, e_2, e_3 , then choosing $a = c_j/e_j$ yields a hyperbolic map of the form $a\varphi_{\Lambda}$, with three superattracting fixed points.*

PROOF. We set $\omega = \exp(2\pi i/3)$, and label the residue classes of critical points to be the smallest critical points, and such that c_1 makes the smallest positive angle with the real axis. We then proceed counterclockwise to label $c_2 = \omega c_1$ and $c_3 = \omega^2 c_1$. We also have that $e_j = (g_3/4)^{1/3}$, where the roots are labelled by the critical points. Since $\varphi_{\Lambda}(c_j) = e_j$, fix say $j = 1$ and choose $a = c_1/e_1$; then $a\varphi_{\Lambda}(c_1) = ae_1 = c_1$. For this value of a , we have $a\varphi_{\Lambda}(c_2) = (c_1/e_1)e_2$, but $e_2/e_1 = \omega$, so $a\varphi_{\Lambda}(c_2) = \omega c_1 = c_2$.

Similarly $a\varphi_{\Lambda}(c_3) = (c_1/e_1)e_3 = \omega^2 c_1 = c_3$. Therefore we have three superattracting fixed points so by Proposition 7.2 (2), $f(z) = a\varphi_{\Lambda} + b$ is stable in a neighborhood of $(\Lambda, a, 0)$. \square

7.2. Toral bands for hyperbolic maps $a\varphi_{\Lambda} + b$ near $b = 0$. A *toral band* is any Fatou component that is not completely contained in one period parallelogram; compare this definition with Definition 3.1.

In Section 3 we discussed how double toral bands give rise to hyperbolic components in \mathfrak{M} ; this is also the case for certain toral bands with connected Julia sets. Depending on the nature of the toral band, these types of Fatou components can persist in hyperbolic components of \mathfrak{M} . We show in Figure 16 an example of a lattice Λ with the following properties: (1) φ_{Λ} has a toral band; (2) φ_{Λ} has two attracting fixed points, one with a toral band in its immediate attracting basin, and another with small basins of attraction. (3) The Julia set is connected [11],

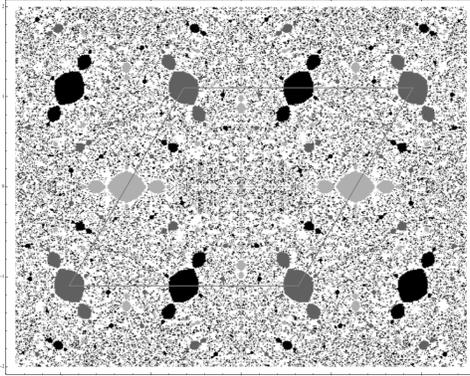


FIGURE 12. $J(\varphi_\Lambda)$ for Λ a triangular lattice and φ_Λ a hyperbolic map with three attracting period two orbits, each a different shade of gray.

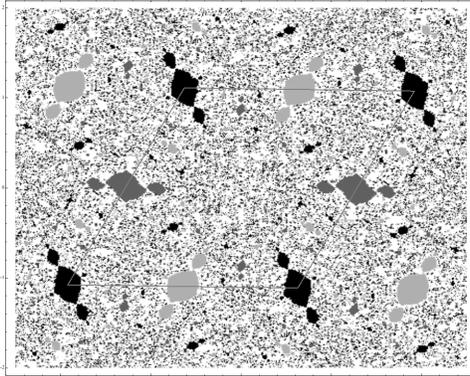


FIGURE 13. $J(\varphi_{\Lambda'} + b)$, with Λ' a non-real lattice near Λ in Fig. 12 and b close to 0. The attracting period two orbits persist.

and the map is hyperbolic. We show the Julia set of a perturbation of φ_Λ in \mathfrak{M} to the right of $J(\varphi_\Lambda)$.

References

- [1] BAKER, I. N.; KOTUS, J.; LÜ Y. Iterates of meromorphic functions I. Ergodic Theory Dynam. Systems **11**, (1991) 241–248.
- [2] BERGWEILER, W. Iteration of meromorphic functions. Bull. Amer. Math. Soc **29**, no. 2, (1993) 151–188.
- [3] CLEMONS, J. Connectivity of Julia sets for Weierstrass elliptic functions on square lattices. Proc. Amer. Math. Soc. **140**, 6, (2012) 1963–1972.
- [4] CLEMONS, J.; KOSS, L. Higher order elliptic functions with connected Julia sets. Topology Proc. **53** (2019), 5772.
- [5] DEVANEY, R.; KEEN, L. Dynamics of Tangent, Dynamical Systems, Springer Lecture Notes in Math, #1342, (1988) 105–111.
- [6] DEVANEY, R.; KEEN, L. Dynamics of meromorphic maps with polynomial Schwarzian derivative. Ann. Sci. Ecole Norm. Sup. (4), **22**, (1989) 55–81.
- [7] DU VAL, P. (1973) Elliptic Functions and Elliptic Curves: Cambridge University Press.

- [8] HAWKINS, J. Smooth Julia sets of elliptic functions for square rhombic lattices, *Topology Proc.* **30**, 1, (2006) 265–278.
- [9] HAWKINS, J.; KOSS, L. Ergodic properties and Julia sets of Weierstrass elliptic functions. *Monatsh. Math.*, **137**, 4, (2002) 273 – 300.
- [10] HAWKINS, J.; KOSS, L. Parametrized Dynamics of the Weierstrass Elliptic Function, *Conf. Geom. Dyn.* **8**, (2004) 1–35.
- [11] HAWKINS, J.; KOSS, L. Connectivity Properties of Julia sets of Weierstrass Elliptic Functions, *Topology Appl.* **152**, 1-2, (2005) 107–137.
- [12] HAWKINS, J.; KOSS, L.; KOTUS, J. Elliptic functions with critical orbits approaching infinity under iteration, *J. of Diff. Eqn and Appl.* **16**, 5–6, (2010) 613–630.
- [13] HAWKINS, J; MORENO ROCHA, M Dynamics and Julia sets of iterated elliptic functions, *New York Jour Math*, to appear.
- [14] KEEN L ; KOTUS J (1999) Ergodicity of some classes of meromorphic functions. *Ann. Acad. Scient. Fenn. Math.*, **24**: 133 - 145
- [15] KOSS, L. Examples of parametrized families of elliptic functions with empty Fatou sets, *New York J. Math.* **20**, (2014) 607–625.
- [16] KOSS, L.; ROY, K. Dynamics of vertical real rhombic Weierstrass elliptic functions, *Involve* **10**, 3, (2017) 361–378.
- [17] KOTUS, J. Elliptic functions with critical points eventually mapped onto infinity. *Monatsh. Math.* **149**, 2, (2006) 103–117.
- [18] KOTUS, J.; URBANŃSKI, M. Hausdorff dimension and Hausdorff measures of Julia sets of elliptic functions. *Bull. London Math. Soc.* **35**, (2003) 269-275.
- [19] MAÑÉ, R.; SAD, P.; SULLIVAN, D. On the dynamics of rational maps. *Ann. Sci. cole Norm. Sup.* (4) 16 (1983), no. 2, 193217.
- [20] MCMULLEN, C. (1994) *Complex Dynamics and Renormalization*. *Annals of Mathematics Studies*, 135. Princeton University Press, Princeton, NJ.
- [21] MILNE-THOMSON, L. (1950) *Jacobian Elliptic Function Tables*: Dover Publications, Inc.
- [22] MILNOR, J. (2006) *Dynamics in One Complex Variable*, Third edition. *Annals of Mathematics Studies*, 160. Princeton University Press, Princeton, NJ.
- [23] MORENO ROCHA, M.; PÉREZ LUCAS, P. A class of even elliptic functions with no Herman rings. *Topology Proc.* **48**, (2016) 151–162.

DEPARTMENT OF MATHEMATICS, CB #325), UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL, CHAPEL HILL, NC 27599

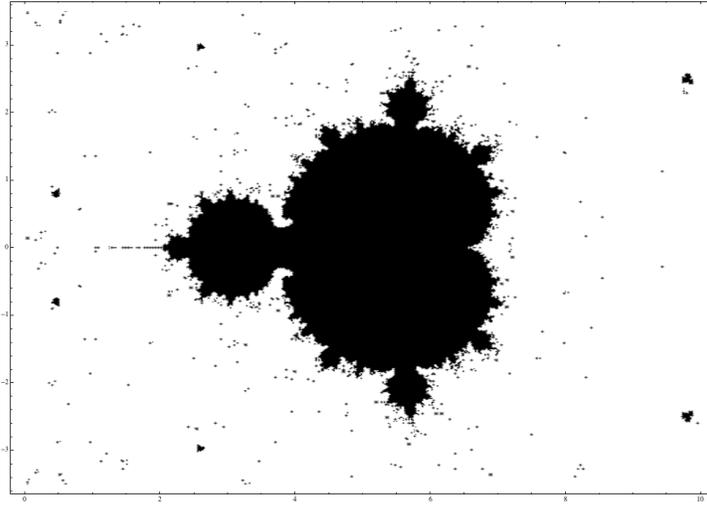


FIGURE 14. We show g_3 space for triangular lattices [10]. The periods of attracting cycles roughly match those in the Mandelbrot set; this shows that there are many hyperbolic components as the cycles move homomorphically for $a\varphi_\Lambda + b$ near $(\Lambda_0, 1, 0)$ so the cycles persist.

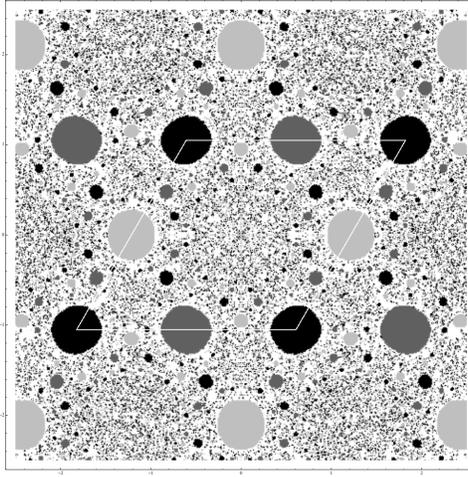


FIGURE 15. $J(a\varphi_\Lambda)$ for Λ a “random” real triangular lattice, then scaled by $a = c_j/e_j$ for any of $j = 1, 2$, or 3 . This results in 3 superattracting fixed points which move stably in Λ , a , and b for $a\varphi_\Lambda + b$ near $(\Lambda, a, 0)$. The different fixed points and their attracting basins are colored distinct shades of gray.

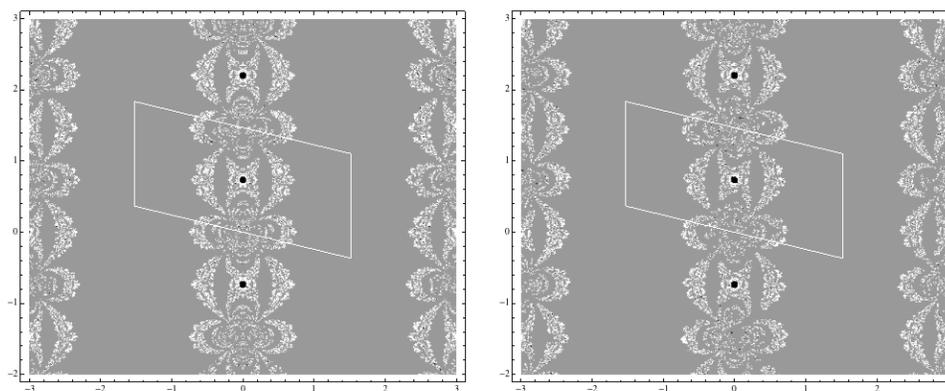


FIGURE 16. On the left, $J(\varphi_\Lambda)$ for Λ a real rhombic lattice, and on the right, $J(a\varphi_\Lambda + b)$ for $a = 1 + .02i$, $b = .1i$, show the stability. The maps each have an attracting fixed point with a black basin, and an attracting fixed point whose gray basin is a toral band.