Journal of Difference Equations and Applications
Vol. 00, No. 00, January 2008, 1–15

RESEARCH ARTICLE

A family of elliptic functions with Julia set the whole sphere

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(*Received 00 Month 200x; in final form 00 Month 200x*)

We iterate the elliptic Weierstrass $\wp_\Lambda$ function with period lattice Λ and give a parametrized family such that the Julia set is the whole sphere. This contrasts with earlier results which showed that the dynamics and Julia sets depend on the shape and side length of Λ. While some square lattices have attracting periodic orbits, in this paper we prove that whenever the lattice Λ is real rhombic square, $J(\wp_\Lambda)$ is the whole sphere. These lattices are parametrized by a real invariant g_2 which ranges over all negative real numbers.

Keywords: complex dynamics, meromorphic functions, Julia sets, elliptic functions

AMS Subject Classification: Primary 54H20, 37F10; Secondary 37F20

In honor of Bob Devaney's 60^{th} *birthday.*

1. Introduction

Smooth Julia sets are somewhat of a rare occurrence in complex dynamics, and it is quite uncommon to find a parametrized family of meromorphic maps with empty Fatou set. The only holomorphic family of rational maps with Julia set the whole sphere is due to Lattès, and a thorough expository article about it can be found for example in [18].

In this study we focus on one specific lattice shape for the iterated Weierstrass elliptic $\wp$ function. While this is restrictive, it is already known there is a wide variety of dynamical behavior and topology among Julia sets resulting from parametrized families of elliptic functions with period lattice in the shape of a square [8–11]. We parametrize the family of maps $\wp_\Lambda$ with square period lattice Λ by $\mathbb{C} \setminus \{0\}$, and then show that there is an infinite ray of parameters for which $J(\wp_\Lambda) = \mathbb{C}_\infty$, where $\mathbb{C}_\infty$ denotes the Riemann sphere. More precisely, the set is the negative real axis without the origin. The reduced space (identifying conformally conjugate maps) is homeomorphic to an infinite cylinder and then the family with full Julia set forms a line which is the "seam" of the cylinder.

When the square lattice Λ is generated by conjugate vectors of the form $\lambda_1 = a + ai$, and $\lambda_2 = a - ai$, for some real number $a > 0$, then we say that the lattice is real rhombic square (shown in Figure 2). The starting point for this paper is the following proposition which appeared by the author and Koss in [9].

Proposition 1.1. *If Λ is rhombic square, then the Julia set is connected if and only if $J(\wp_\Lambda) = \mathbb{C}_\infty$.*

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ISSN: 1023-6198 print/ISSN 1563-5120 online

DOI: 10.1080/1023619YYxxxxxxxx
http://www.informaworld.com

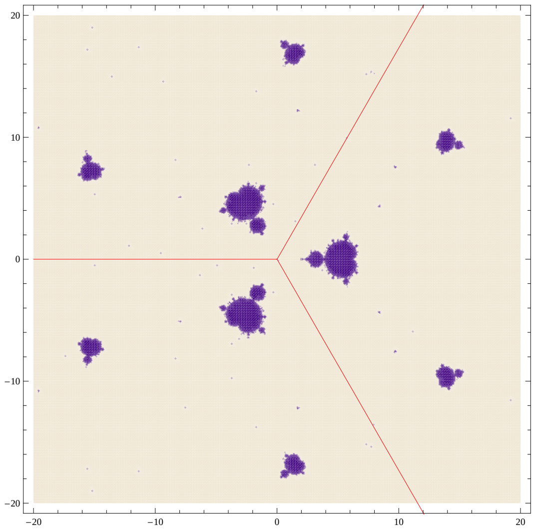

Figure 1. Parameter space for $\wp_\Lambda$ for Λ a square lattice

The main theorem in this paper was conjectured in [7] and proved in some special cases. We use different techniques, adapting results from real interval maps, to prove the general result.

THEOREM 1.2. *If Λ is a real rhombic square lattice, then $J(\wp_\Lambda) = \mathbb{C}_\infty$.*

Associated to any square lattice Λ and its corresponding Weierstrass elliptic function $\wp_\Lambda$ is a unique parameter $g_2 \in \mathbb{C} \setminus \{0\}$ appearing in Equation (1) and discussed below. Because of the symmetry in the parametrized space of square lattices as shown in [10], we have the following corollary.

COROLLARY 1.3. *For every $\lambda > 0$, the map $\wp_{\Lambda(g_2,0)} \equiv \wp$ satisfies $J(\wp) = \mathbb{C}_\infty$, if $g_2 = e^{\pm 2\pi i/3}\lambda$.*

Figure 1 shows an approximation to g_2-space; that is, each nonzero point of the form (a, b) in the space corresponds to the (square) lattice whose invariants are $(a + ib, 0)$. Writing this lattice as $\Lambda \equiv \Lambda(a, b)$, we consider the free critical orbit of $\wp_\Lambda$. The parameters with attracting periodic orbits are colored purple (dark gray). We have marked in red (dark gray) the rays corresponding to the real dimension-one families on which $J(\wp_\Lambda)$ is the entire sphere. Proving that this is the case is the main result of this paper.

In an earlier paper by the author and Koss, infinitely many values of g_2 we given that yield lattices corresponding to Weierstrass $\wp$ functions having Julia set the entire sphere ([10], Theorem 9.5); some of them are in the parametrized families of our main theorem and others are not, showing that this set of parameters does not exhaust the square lattices with this property.

While classical identities given below were used to solve the conjecture in a few special cases, and in the examples in [10], it is the more "modern theorem" of Singer [19] on the use of the Schwarzian derivative in the study of interval maps that leads to the general proof. The author thanks the organizers of the 2008 Dynamical Systems Topology Conference for their hospitality in Tossa de Mar, Spain.

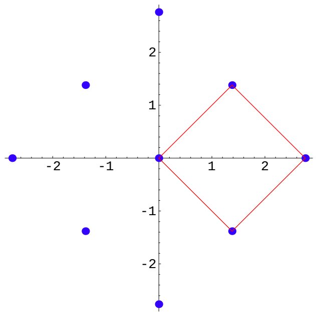

Figure 2. A real rhombic square lattice

2. Preliminary definitions and notation

Let $\lambda_1, \lambda_2 \in \mathbb{C}\backslash\{0\}$ such that $\lambda_2/\lambda_1 \notin \mathbb{R}$. A lattice is the group generated by λ_1 and λ_2 as follows: $\Lambda = [\lambda_1, \lambda_2] \equiv \{m\lambda_1 + n\lambda_2 \colon m, n \in \mathbb{Z}\}$. The generators of a lattice are not unique; if $\Lambda = [\lambda_1, \lambda_2]$, then all other generators λ_3, λ_4 of Λ are obtained by multiplying the vector (λ_1, λ_2) by the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = \pm 1$.

The group Λ acts on $\mathbb{C}$ by translation, each $\omega \in \Lambda$ inducing the transformation of $\mathbb{C}$:

$$T_\omega : z \mapsto z + \omega.$$

We write $z + \Lambda$ to denote a coset of $\mathbb{C}/\Lambda$ containing z.

Definition 2.1. A closed, connected subset Q of $\mathbb{C}$ is a *fundamental region* for Λ if

(1) for each $z \in \mathbb{C}$, Q contains at least one point in the same Λ-orbit as z;
(2) no two points in the interior of Q are in the same Λ-orbit.

If Q is any fundamental region for Λ, then for any $s \in \mathbb{C}$, the set

$$s + Q = \{z + s : z \in Q\}$$

is also a fundamental region. If Q is a parallelogram it is called a *period parallelogram* for Λ.

The ratio $\tau = \lambda_2/\lambda_1$ defines an equivalence relation on lattices. If $\Lambda = [\lambda_1, \lambda_2]$, and $k \neq 0$ is any complex number, then $k\Lambda$ is the lattice defined by taking $k\lambda$ for each $\lambda \in \Lambda$ and yields the same ratio τ; $k\Lambda$ is said to be *similar* to Λ. Similarity is an equivalence relation between lattices, and an equivalence class of lattices is called a *shape*. A lattice Λ is *real* if $\lambda \in \Lambda \Leftrightarrow \overline{\lambda} \in \Lambda$.

Definition 2.2.

(1) $\Lambda = [\lambda_1, \lambda_2]$ is *real rhombic* if there exist generators such that $\lambda_2 = \overline{\lambda_1}$. Any similar lattice is *rhombic*.
(2) A lattice Λ is *square* if $i\Lambda = \Lambda$. (Equivalently, Λ is square if it is similar to a lattice generated by $[\lambda, \lambda\, i]$, for some $\lambda > 0$.)

In each of these cases the period parallelogram with vertices $0, \lambda_1, \lambda_2$, and $\lambda_3 := \lambda_1 + \lambda_2$ can be chosen to be a rhombus or square respectively.

2.1 Real rhombic square lattices

The main result of this paper focuses on the specific type of lattice shown in Figure 2.

PROPOSITION 2.3. *The following are equivalent for a lattice Λ.*

(1) Λ is a real rhombic square lattice.
(2) There exists a $\lambda > 0$ such that $\Lambda = [\lambda e^{\pi i/4}, \lambda e^{-\pi i/4}]$.
(3) There exists $\gamma > 0$ such that $\Lambda = [2\gamma, \gamma + i\gamma]$.

Proof. By definition, Λ is real rhombic square if and only if: (a) $\Lambda = [\lambda_1, \lambda_2]$ with $\lambda_2 = \overline{\lambda_1}$; and (b) $i\Lambda = \Lambda$.

(2) implies (1): Obviously condition (a) is satisfied and writing $\lambda e^{\pi i/4}$ in its Cartesian form, we have $\Lambda = [\frac{\lambda}{\sqrt{2}}(1+i), \frac{\lambda}{\sqrt{2}}(1-i)]$. Then for any $\lambda \in \Lambda$, there exist $m, n \in \mathbb{Z}$ such that $\lambda = m(\frac{\lambda}{\sqrt{2}}(1+i)) + n(\frac{\lambda}{\sqrt{2}}(1-i))$. We have that

$$i\lambda = im(\frac{\lambda}{\sqrt{2}}(1+i)) + in(\frac{\lambda}{\sqrt{2}}(1-i))$$

$$= n(\frac{\lambda}{\sqrt{2}}(1+i)) - m(\frac{\lambda}{\sqrt{2}}(1-i)).$$

So $i\Lambda \subset \Lambda$; since

$$\Lambda = -\Lambda = i^2\Lambda \subset i\Lambda \subset \Lambda,$$

(b)is satisfied.

(2) and (3) are equivalent using $\gamma = \frac{\lambda}{\sqrt{2}}$ and changing the generator using the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

(1) implies (2): If (1) holds, then there exist generators satisfying $\lambda_2 = \overline{\lambda_1}$; in polar form, $\lambda_1 = re^{i\theta}$ and $\lambda_2 = re^{-i\theta}$ for some $\theta \in (0, \pi)$. Since in addition it is square, we use the remark in Definition 2.2, (3), to see that $\theta - (-\theta) = 2\theta = \pi/2$. Therefore $\theta = \pi/4$. □

2.2 Elliptic functions

We begin with $f : \mathbb{C} \to \mathbb{C}_\infty$ a meromorphic function where $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere.

Definition 2.4. An *elliptic function* is a meromorphic function in $\mathbb{C}$ which is periodic with respect to a lattice Λ.

For any $z \in \mathbb{C}$ and any lattice Λ, the *Weierstrass elliptic function* is defined by

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right).$$

Replacing every z by $-z$ in the definition we see that $\wp_\Lambda$ is an even function. The map $\wp_\Lambda$ is meromorphic, periodic with respect to Λ, and has order 2.

The derivative of the Weierstrass elliptic function is also an elliptic function which is periodic with respect to Λ defined by

$$\wp'_\Lambda(z) = -2 \sum_{w \in \Lambda} \frac{1}{(z-w)^3}.$$

The Weierstrass elliptic function and its derivative are related by the differential equation

$$\wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 - g_2\wp_\Lambda(z) - g_3, \tag{1}$$

where $g_2(\Lambda) = 60 \sum_{w \in \Lambda \setminus \{0\}} w^{-4}$ and $g_3(\Lambda) = 140 \sum_{w \in \Lambda \setminus \{0\}} w^{-6}$.

The numbers $g_2(\Lambda)$ and $g_3(\Lambda)$ are invariants of the lattice Λ in the following sense: if $g_2(\Lambda) = g_2(\Lambda')$ and $g_3(\Lambda) = g_3(\Lambda')$, then $\Lambda = \Lambda'$. Furthermore given any g_2 and g_3 such that $g_2^3 - 27g_3^2 \neq 0$ there exists a lattice Λ having $g_2 = g_2(\Lambda)$ and $g_3 = g_3(\Lambda)$ as its invariants [6].

THEOREM 2.5. *[6] For $\Lambda_\tau = [1, \tau]$, the functions $g_i(\tau) = g_i(\Lambda_\tau), i = 2, 3$, are analytic functions of τ in the open upper half plane $Im(\tau) > 0$.*

We have the following homogeneity in the invariants g_2 and g_3 [10].

LEMMA 2.6.
For lattices Λ and Λ', $\Lambda' = k\Lambda \Leftrightarrow$

$$g_2(\Lambda') = k^{-4}g_2(\Lambda) \quad \textit{and} \quad g_3(\Lambda') = k^{-6}g_3(\Lambda).$$

THEOREM 2.7. *[13] The following are equivalent:*

(1) $\wp_\Lambda(\overline{z}) = \overline{\wp_\Lambda(z)}$;
(2) Λ is a real lattice;
(3) $g_2, g_3 \in \mathbb{R}$.

For any lattice Λ, the Weierstrass elliptic function and its derivative satisfy the following properties: for $k \in \mathbb{C} \setminus \{0\}$,

$$\wp_{k\Lambda}(ku) = \frac{1}{k^2}\wp_\Lambda(u), \text{ (homogeneity of } \wp_\Lambda), \tag{2}$$

$$\wp'_{k\Lambda}(ku) = \frac{1}{k^3}\wp'_\Lambda(u), \text{ (homogeneity of } \wp'_\Lambda),$$

Verification of the homogeneity properties can be seen by substitution into the series definitions.

If $\wp'_\Lambda(z_0) = 0$ then z_0 is a *critical point* and $\wp_\Lambda(z_0)$ is a *critical value.* The critical values of the Weierstrass elliptic function on an arbitrary lattice $\Lambda = [\lambda_1, \lambda_2]$ are as follows.

For $j = 1, 2$, notice that $\wp_\Lambda(\lambda_j - z) = \wp_\Lambda(z)$ for all z. Taking derivatives of both sides we obtain $-\wp'_\Lambda(\lambda_j - z) = \wp'_\Lambda(z)$. Substituting $z = \lambda_1/2, \lambda_2/2$, or $\lambda_3/2$, we see that $\wp'_\Lambda(z) = 0$ at these values. We use the notation

$$e_1 = \wp_\Lambda(\frac{\lambda_1}{2}), \quad e_2 = \wp_\Lambda(\frac{\lambda_2}{2}), \quad e_3 = \wp_\Lambda(\frac{\lambda_3}{2})$$

to denote the critical values. Since e_1, e_2, e_3 are the distinct zeros of Equation 1, we also write

$$\wp'_\Lambda(z)^2 = 4(\wp_\Lambda(z) - e_1)(\wp_\Lambda(z) - e_2)(\wp_\Lambda(z) - e_3). \tag{3}$$

Equating like terms in Equations 1 and 3, we obtain

$$e_1 + e_2 + e_3 = 0, \quad e_1e_3 + e_2e_3 + e_1e_2 = \frac{-g_2}{4}, \quad e_1e_2e_3 = \frac{g_3}{4}. \tag{4}$$

Naturally, the lattice shape relates to the properties and dynamics of the corresponding Weierstrass elliptic function. Denote

$$p(x) = 4x^3 - g_2x - g_3, \tag{5}$$

the polynomial associated with Λ. Let $\triangle = g_2^3 - 27g_3^2 \neq 0$ denote its discriminant.

PROPOSITION 2.8. *[6] If Λ is rhombic square then $\triangle < 0$, $g_2 < 0$ and $g_3 = 0$, and the roots of p are $0, \pm\sqrt{g_2}/2$.*

The following corollary follows from Equations (1) and (4).

COROLLARY 2.9.
A lattice Λ is rhombic square if and only if $e_3 = 0$ and $e_1 = -\sqrt{g_2}/2 = -e_2$ are pure imaginary.

We define the *standard rhombic square lattice*, called the standard lattice from now on in this paper, to be the unique lattice corresponding to $g_2 = -4$, and giving $e_1 = -i$ and $e_2 = i$. For the standard lattice any square forming a period parallelogram has side length $\gamma \approx 2.62206$, and we denote the standard lattice by $\Gamma = [b + bi, b - bi]$, with $\gamma/\sqrt{2} = b \approx 1.85407$ (see for example the tables in [17]).

2.2.1 Quarter lattice values of $\wp_\Lambda$

There are formulas for the quarter period values of the Weierstrass elliptic function. Define for each $i, j, k = 1, 2, 3$

$$d_i^2 = (e_i - e_j)(e_i - e_k) = 3e_i^2 - g_2/4, \tag{6}$$

and choose the square root so that we obtain the quarter lattice values:

$$\wp_\Lambda(\lambda_i/4) = e_i + d_i, \tag{7}$$

with $\Lambda = [\lambda_1, \lambda_2]$, and $\lambda_3 = \lambda_1 + \lambda_2$.

We recall the result from [7].

Lemma 2.10. *If $\Lambda = [a + ai, a - ai]$, $a > 0$ is a real rhombic square lattice, then $\wp_\Lambda(a/2) = d_3 = \sqrt{-g_2/4} = |e_2|$, where g_2 is the invariant associated to the lattice Λ. In particular, for the standard lattice Γ, $\wp_\Gamma(b/2) = 1$ and $\wp'_\Gamma(b/2) = -2\sqrt{2}$.*

Proof. By Equation 7 we have that $\wp_\Gamma(b/2) = \sqrt{-g_2/4} = 1$ since $e_3 = 0$. Moreover, by Equation 1 we have that

$$(\wp'_\Gamma)^2 = 4 - g_2 = 8,$$

and since the function is decreasing at $b/2$, the result follows. □

The only other possibility for Λ to be a real square lattice is if $g_2 > 0$. In this case the preceding discussion gives $\Lambda = [\lambda, \lambda i]$ for some $\lambda > 0$, and $\wp_\Lambda(\lambda/2) = e_1 = -e_2 = \dfrac{\sqrt{g_2}}{2} > 0$. Additionally we have that $\wp_\Lambda(\lambda/4) = \dfrac{(1+\sqrt{2})\sqrt{g_2}}{2} > 0$.

2.3 Fatou and Julia sets for elliptic functions

We review the basic dynamical definitions and properties for meromorphic functions which appear in [2], [3], [4] and [5]. As above, let $f \colon \mathbb{C} \to \mathbb{C}_\infty$ be a meromorphic function; by $f^{\circ n}$, $n \in \mathbb{N}$, we denote the composition of f with itself n times (on points where all the iterates are defined). The notation $f^{-\circ n}(z)$ is used for the points $w \in \mathbb{C}_\infty$ such that $f^{\circ n}(w) = z$; if $n = 1$ we just use f and f^{-1}. The *Fatou set* $F(f)$ is the set of points $z \in \mathbb{C}_\infty$ such that $\{f^{\circ n} \colon n \in \mathbb{N}\}$ is defined and normal in some neighborhood of z. The *Julia set* is the complement of the Fatou set on the sphere, $J(f) = \mathbb{C}_\infty \backslash F(f)$. We note that $\mathbb{C}_\infty \backslash \overline{\bigcup_{n \geq 0} f^{-\circ n}(\infty)}$ is the largest open set where all iterates are defined. Since $f(\mathbb{C}_\infty \backslash \overline{\bigcup_{n \geq 0} f^{-\circ n}(\infty)}) \subset \mathbb{C}_\infty \backslash \overline{\bigcup_{n \geq 0} f^{-\circ n}(\infty)}$, Montel's theorem implies that

$$J(f) = \overline{\bigcup_{n \geq 0} f^{-\circ n}(\infty)}.$$

Let $Crit(f)$ denote the set of critical points of f, *i.e.*,

$$Crit(f) = \{z \colon f'(z) = 0\}.$$

If z_0 is a critical point then $f(z_0)$ is a *critical value.* For each lattice, $\wp_\Lambda$ has three critical values and no asymptotic values. The *singular set* $Sing(f)$ of f is the set of critical and finite asymptotic values of f and their limit points. A function is called *Class S* if f has only finitely many critical and asymptotic values; for each lattice Λ, every elliptic function with period lattice Λ is of Class S. The *postcritical set* of $\wp_\Lambda$ is:

$$P(\wp_\Lambda) = \overline{\bigcup_{n \geq 0} \wp_\Lambda^{\circ n}(e_1 \cup e_2 \cup e_3)}.$$

Definition 2.11. For a meromorphic function f, a point z_0 is *periodic* of period p if there exists a $p \geq 1$ such that $f^{\circ p}(z_0) = z_0$. We also call the set $\{z_0, f(z_0), \ldots, f^{\circ p-1}(z_0)\}$ a *p-cycle.* The *multiplier* of a point z_0 of period p is the derivative $(f^{\circ p})'(z_0)$. A periodic point z_0 is called *attracting, repelling,* or *neutral* if $|(f^{\circ p})'(z_0)|$ is less than, greater than, or equal to 1 respectively. If $|(f^{\circ p})'(z_0)| = 0$ then z_0 is called a *superattracting* periodic point. As in the case of rational maps, the Julia set is the closure of the repelling periodic points [2].

Suppose U is a connected component of the Fatou set. We say that U is *preperiodic* if there exists $n > m \geq 0$ such that $f^{\circ n}(U) = f^{\circ m}(U)$, and the minimum of $n - m = p$ for all such n, m is the *period* of the cycle.

PROPOSITION 2.12. *If p is an attracting fixed point or a rationally neutral fixed point for $\wp_\Lambda$, then the local coordinate chart for the point is completely contained in one fundamental period of $\wp_\Lambda$ (in fact in one half of one fundamental period).*

Proof. This is due to the periodicity of $\wp_\Lambda$; in each case the local form is invertible. If the local coordinate chart spills into another (half) fundamental period or region, then injectivity fails. □

The Julia set of $\wp_\Lambda$ does not depend on the shape of the lattice Λ as much as on its invariants g_2 and g_3 (cf. [9], [10], [11], [8]), or equivalently on the side length and angle that a fundamental period makes with the positive real axis. For example, the following result shows that every lattice shape gives rise to a similar lattice with a superattracting fixed point, and a similar result in [12] (Lemma 2.3) shows that a small variation in the choice of k results in critical orbits terminating in poles.

PROPOSITION 2.13. *[7] Let $\Lambda = [1, \tau]$ be a lattice such that the critical value $\wp_\Lambda(1/2) = \epsilon \neq 0$. If m is any odd integer and $k = \sqrt[3]{2\epsilon/m}$ (taking any root) then the lattice $\Gamma = k\Lambda$ has a superattracting fixed point at $mk/2$.*

Julia sets for square lattices exhibit additional symmetry. A special case of the following was proved in [9].

THEOREM 2.14. *If Λ is square, then $e^{\pi i/2}J(\wp_\Lambda) = J(\wp_\Lambda)$ and $e^{\pi i/2}F(\wp_\Lambda) = F(\wp_\Lambda)$.*

Proof. Let $z \in F(\wp_\Lambda)$; then by definition, $\wp_\Lambda^{\circ n}(z)$ exists and is normal for all n. By Equation 2, $\wp_\Lambda(iz) = -\wp_\Lambda(z)$ and since $\wp_\Lambda$ is even we know that $\wp_\Lambda^{\circ n}(iz) = \wp_\Lambda^{\circ n}(z)$ for all $n \geq 2$. The result follows from this. □

2.3.1 Summary of properties of $\wp_\Lambda$ when Λ is a real rhombic square lattice.

Much of the focus of this paper is on $\wp_\Lambda$ when Λ is a real rhombic square lattice. If Λ is real rhombic square then all of the following properties hold:

(1) $\Lambda = [\lambda e^{\pi i/4}, \lambda e^{-\pi i/4}]$ for some $\lambda > 0$.
(2) $i\Lambda = \Lambda$, $iJ(\wp_\Lambda) = J(\wp_\Lambda)$, and $iF(\wp_\Lambda) = F(\wp_\Lambda)$.
(3) $g_3 = 0$ and $g_2 < 0$.
(4) For $k > 0$, $\Lambda' = k\Lambda$ if and only if $g_2(\Lambda') = k^{-4}g_2(\Lambda)$
(5) $e_3 = 0$, and $e_1 = -e_2$ are purely imaginary and satisfy

$$-4e_1e_2 = g_2,$$

or equivalently,

$$e_1 = -\sqrt{g_2}/2.$$

(6) The standard lattice corresponds to $g_2 = -4$, and thus $e_1 = -i$ and $e_2 = i$.
(7) For the standard lattice we have $\lambda = \gamma \approx 2.62206$, and we denote it by $\Gamma = [b + bi, b - bi]$, with $b \approx 1.85407$; $\gamma = \sqrt{2}b$.
(8) For any Λ, $\wp_\Lambda(z) \geq 0$ if and only if $z \in \mathbb{R}$. Equality holds if and only if z is a real critical point, that is, a real half lattice point.

Parameters	$\{e_1, e_2, e_3\}$	$\{g_2, g_3\}$	Side length
Standard	$\{-i, i, 0\}$	$\{-4, 0\}$	γ
e_1	$\{-ci, ci, 0\}$	$\{-4c^2, 0\}$	$\dfrac{\gamma}{\sqrt{c}}$
g_2	$\left\{\dfrac{-\sqrt{-c}}{2}, \dfrac{\sqrt{-c}}{2}, 0\right\}$	$\{-c, 0\}$	$\left(\dfrac{4}{c}\right)^{1/4}\gamma$
side length	$\left\{\dfrac{-\gamma^2}{c^2}i, \dfrac{\gamma^2}{c^2}i, 0\right\}$	$\left\{\dfrac{-4\gamma^4}{c^4}, 0\right\}$	c
side length$\cdot\gamma$	$\left\{\dfrac{-i}{c^2}, \dfrac{i}{c^2}, 0\right\}$	$\left\{\dfrac{-4}{c^4}, 0\right\}$	$c\gamma$

Table 1. The relationships among parameters of rhombic square lattices for $c > 0$

We summarize some relationships among the various invariants for rhombic square lattices and the associated Weierstrass $\wp$ function in Table 1. The left hand column tells which invariant is being prescribed.

3. The Schwarzian derivative for $\wp$ for real lattices

The Schwarzian derivative is a classical object in complex analysis used to measure the nonlinearity of an analytic map locally (see eg., [1]); it was brought into the forefront of interval dynamics in 1987 [19] and used in the study of the dynamics of interval maps. Elliptic functions based on real lattices have some connection with both of these studies. We recall the definition.

Definition 3.1. Suppose that f is meromorphic in $\mathbb{C}$. We define:

$$S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 \tag{8}$$

to be the *Schwarzian derivative* of f at z.

It is clear that S_f is not defined at poles or critical points of f. We give some properties of S_f for elliptic functions.

Proposition 3.2.

(1) If $\wp_\Lambda$ denotes the Weierstrass elliptic function on a lattice Λ, then $S_{\wp_\Lambda}$ is an even elliptic function.
(2) If Λ is real, then $S_{\wp_\Lambda}$ is a real-valued meromorphic function when restricted to $\mathbb{R}$.

Proof. We fix a lattice Λ throughout and write $S_\wp$ for the Schwarzian derivative of $\wp_\Lambda$.

We apply several standard identities. The first is Equation (1), the second is

$$\wp''(z) = 6\wp(z)^2 - \frac{1}{2}g_2, \tag{9}$$

and the chain rule gives:

$$\wp'''(z) = 12\wp\wp'. \tag{10}$$

These allow us to replace Equation (8) with

$$S_\wp = 12\wp - \frac{36\wp^2 - 3g_2}{4(4\wp^3 - g_2\wp - g_3)}, \tag{11}$$

which is an even elliptic function since it is a rational expression of $\wp$. It also follows from the real coefficients of (11) that if the period lattice for $\wp$ is real, then $S_\wp(z) \in \mathbb{R}$ if $z \in \mathbb{R}$ is not a pole. From Equation (8) it is clear there are poles at all lattice and half lattice points of $\wp$ (the poles and critical points of $\wp$ respectively) for the function $S_\wp$. Since $\wp$ is doubly periodic, so is $S_\wp$, and the result is proved. □

Proposition 3.3. *If Λ is a real square lattice, then $S_\wp(z) \le 0$, for all $z \in \mathbb{R}$ for which it is defined, and $S_\wp(z) = 0$ if and only if $g_2 < 0$ and $z = \pm\frac{\sqrt{2}\lambda}{4} + \Lambda$ (real quarter lattice points).*

Proof. The hypothesis is equivalent to $g_3 = 0$, so Equation (8) can be rewritten using (9) and (10) as:

$$S_\wp(z) = -\frac{3}{8}\frac{(g_2 + 4\wp^2)^2}{(\wp')^2}(z) < 0 \tag{12}$$

for $z \in \mathbb{R}$ unless z is a pole of $S_\wp$ or unless $\wp(z)^2 = \frac{-g_2}{4}$. By Lemma 2.10 the second possibility occurs for some $z_0 \in \mathbb{R}$ precisely when $g_2 < 0$ and z_0 is a real quarter lattice point for $\wp$.

□

We apply the Schwarzian derivative to this setting in the same way in which it is used for interval maps, namely to establish that there must be a real critical point in the basin of any nonrepelling fixed point of $\wp^{\circ n}$. Using the chain rule, on every point $z \in \mathbb{R}$ for which $\wp^{\circ n}$ is defined, we have:

$$S\wp^{\circ n}(z) = \sum_{i=0}^{n-1} S_\wp(\wp^{\circ i} z) \cdot |\wp'(z)|^2.$$

Therefore unless z_0 is a real quarter lattice point, and we have a lattice Λ such that $\wp_\Lambda(z_0) = z_0$, we have that $S\wp^{\circ n}(z) < 0$ for all n and z. This exceptional situation can occur as was shown in ([7], Theorem 3.1), and in this case it was proved that the fixed point is repelling and $J(\wp_\Lambda) = \mathbb{C}_\infty$. We will assume then from now on that we are not working with this exceptional lattice.

Lemma 3.4. **(Minimum Principle)** *Assume that Λ is a real square lattice. Suppose we have a closed interval $I = [l, r] \subset \mathbb{R}$, not containing any poles or critical points of $\wp$. Then*

$$|\wp'(z)| > \min\{|\wp'(l)|, |\wp'(r)|\}, \forall z \in (l, r). \tag{13}$$

Proof. The minimum of $|\wp'(z)|$ on I occurs either at an interior point of I or at an endpoint. If the minimum occurs at $z_0 \in (l, r)$, then z_0 is critical point of $|\wp'(z)|$; i.e., $\wp''(z_0) = 0$. If z_0 is a quarter lattice point, then Proposition 3.3 implies that $S_\wp(z_0) = 0$ and $\wp''(z_0) = 0$, so by Equations (8) and (10), we have that $\wp(z_0) = 0$. But this is impossible for a quarter lattice point. By our hypotheses $\wp(z) > 0$

on I, so $S_\wp(z_0) = 12\wp(z_0) > 0$ which contradicts Proposition 3.3. Therefore the minimum must occur at an endpoint of I as claimed. $\square$

4. Singer's Theorem for $\wp_\Lambda$ for real square lattices

We turn to the setting of maps from $\mathbb{R}$ to $\mathbb{R}_\infty \equiv \mathbb{R} \cup \{\infty\}$. For a real lattice Λ, $\wp_\Lambda(\mathbb{R}) \subset \mathbb{R}_\infty$, and for any p-cycle

$$S = \{z_0, \wp_\Lambda(z_0), \ldots, \wp_\Lambda^{\circ p-1}(z_0)\} \subset \mathbb{R},$$

we associate to it a set:

$$B(S) = \{x \in \mathbb{R} : \wp_\Lambda^{\circ k}(x) \to S \text{ as } k \to \infty\}.$$

We say S is *topologically attracting* if $B(S)$ contains an open interval; in this case we call $B(S)$ the *real attracting basin of* S. By $B_0(S)$ we denote the union of components of $B(S)$ in $\mathbb{R}$ containing points from S; $B_0(S)$ is the *real immediate (attracting) basin* of S. For Λ real, if S is an attracting p-cycle for $\wp_\Lambda$ in the sense of Definition 2.11, then $S \subset [0, \infty)$ and $B(S) \neq \emptyset$. Therefore the p-cycle S is topologically attracting on $\mathbb{R}$ [9]. We extend a theorem of Singer on interval maps to this setting. Our proof follows the proof for C^3 interval maps given in [16] with necessary modifications for our setting.

THEOREM 4.1 . *If Λ is a real square lattice, then:*

1. *the real immediate basin of a topologically attracting periodic orbit of $\wp_\Lambda$ contains a real critical point.*
2. *If $y \in \mathbb{R}$ is in a rationally neutral p-cycle for $\wp_\Lambda$ then it is topologically attracting; i.e., there exists an open interval V such that for every $x \in V$, $\lim_{n\to\infty} \wp_\Lambda^{\circ np}(x) = y$.*

Proof . (1): We suppose first that we have a topologically attracting periodic point y with no real critical point in its immediate attracting basin. Let p be the period of y, and denoting by F_0 the component of the Fatou set containing y and hence its immediate basin of attraction in $\mathbb{C}$, we write $U \subset F_0 \cap \mathbb{R}$ for the component of $F_0 \cap \mathbb{R}$ containing y. It follows that U is an open interval not containing any poles or prepoles so U and $\overline{U}$ are completely contained in the interior of one fundamental period of $\wp|_\mathbb{R}$; in addition, we have that $\wp_\Lambda^{\circ k}(x)$ is defined for all $k \in \mathbb{N}$ and for all $x \in U$. Then $\wp_\Lambda^{\circ p}(U) \subset U$, and since $\partial U \subset J(\wp_\Lambda)$, $\wp_\Lambda^{\circ p}(\partial U) \subset \partial U$. We note that $y \in \partial U$ is a possibility, but if y is an attracting period point in the sense of Definition 2.11, then y is topologically attracting and $y \notin \partial U$.

Assume first that $(\wp_\Lambda^{\circ p})'(x) = 0$ for some $x \in U$; then for some integer $k = 0, \ldots, p-1$, $\wp_\Lambda^{\circ k}(x)$ is a critical point in $\wp_\Lambda^{\circ k}(U)$ and therefore is in the immediate basin of $\wp_\Lambda^{\circ k}(y)$ and (1) holds. Therefore we assume by contradiction that for all $x \in U$, $(\wp_\Lambda^{\circ p})'(x) \neq 0$. We assume that $(\wp_\Lambda^{\circ p})'(x) > 0$ for $x \in U$; otherwise use $\wp_\Lambda^{\circ 2p}$ in what follows. Since U is an interval, this implies that $\wp_\Lambda^{\circ p}(U) = U$ and $\wp_\Lambda^{\circ p}(w_j) = w_j$ for $w_j \in \partial U$, $j = 1, 2$. Then $(\wp_\Lambda^{\circ p})'(w_j) \geq 1$, otherwise w_j is itself a topologically attracting fixed point under $\wp_\Lambda^{\circ p}$, which cannot be the case since it lies on the boundary of U, the attracting basin of y. Therefore by the Minimum Principle, $(\wp_\Lambda^{\circ p})'(x) > 1$ for all $x \in U$, a contradiction since $\wp_\Lambda^{\circ p}(U) = U$. This proves (1).

To prove (2), suppose $S = \{y, \ldots, \wp_\Lambda^{\circ 2p-1}(y)\}$, $y \in \mathbb{R}$, and y is a neutral p-cycle. Then $(\wp_\Lambda^{\circ 2p})'(y) = 1$; if every neighborhood of y in $\mathbb{R}$ satisfies $(\wp_\Lambda^{\circ 2p})'(x) \geq 1$, this contradicts the Minimum Principle 3.4, so this cannot occur. Therefore given any neighborhood, it must contain an interval V on which $(\wp_\Lambda^{\circ 2p})'(x) < 1$, for all $x \in V$ from which one can conclude that $V \subset B(S)$, so y is topologically attracting.

$\square$

Recalling the notation from Definition 2.2, for $\Lambda = [a + ai, a - ai] = [\lambda_1, \lambda_2]$, we label the critical point cosets as $c_j + \Lambda = \lambda_j/2 + \Lambda$, and $e_j = \wp_\Lambda(c_j + \Lambda)$.

THEOREM 4.2. *If Λ is a rhombic square lattice, then $J(\wp_\Lambda) = \mathbb{C}_\infty$,*

Proof. For a rhombic square lattice, the critical values are of the form: $e_1 = -ci$, $e_2 = ci$, and $e_3 = 0$, with $c > 0$ (see Table 1, Row 3). Every real critical point is mapped to 0; since 0 is a pole, $e_3 \in J(\wp_\Lambda)$. Since $\wp_\Lambda|_\mathbb{R}$ maps $\mathbb{R}$ onto $\mathbb{R}$, if there is a nonrepelling cycle for $\wp_\Lambda$ the cycle must lie on the positive real axis and contain a critical point, hence a critical value. However this is impossible, so there is no such cycle. Herman rings and Siegel disks have already been eliminated in [9], so the result follows.

$\square$

It was shown directly in [10] that among the real rhombic square lattice there are infinitely many parameters for which all critical values are prepoles. In this case, for each corresponding map $\wp$ we have that $J(\wp) = \mathbb{C}_\infty$, and from [14] it follows that $\wp$ preserves a σ-finite measure μ equivalent to m, where m denotes two-dimensional Lebesgue measure, and that $\wp$ is ergodic with respect to μ. We discuss possible critical orbits in more detail in the next section.

5. Types of critical orbits possible

We have shown that all Weierstrass elliptic $\wp$ functions with associated invariants $(g_2, g_3) = (-c, 0)$, $c > 0$ satisfy $J(\wp_\Lambda) = \mathbb{C}_\infty$ and correspond to rhombic square lattices. We turn to a discussion of the substructure of the critical orbits; the conditions imposed on (g_2, g_3) under our assumptions reduce the number of free critical orbits from three to one. This is because we always have that $\wp(c_3+\Lambda) = 0$, so we are guaranteed that one critical orbit ends at ∞. In addition, it is always that case that the other two critical orbits will coincide in exactly two iterations, and for the third iteration, assuming we have not hit a pole, we have that $\wp^{\circ 3}(c_1) = \wp^{\circ 3}(c_2) > 0$. Hence except for at most three points, the postcritical set $P(\wp) \subset (0, \infty]$ so we always have $m(P(\wp)) \leq m(\mathbb{R}) = 0$, where m is two-dimensional Lebesgue measure.

We run through some possibilities for critical orbits after developing a simple tool to understand the critical behavior.

5.1 The role of the real quarter lattice point

The real quarter lattice points play an important role in understanding the critical orbits of a rhombic square lattice. This is because the quarter lattice point has a forward orbit that coincides with the forward orbit of the free critical value e_1 after two iterations.

We begin with a lemma from [7]. Recall from Proposition 2.3 that if $\Lambda = [a + ai, a - ai]$, $a > 0$, then real quarter lattice points occur at $a/2 + 2ma$, $m \in \mathbb{Z}$.

LEMMA 5.1. *Let $\Lambda = [a + ai, a - ai]$, $a > 0$. Then the following hold:*

(1) $g_2 = -4$, and we have the standard lattice $\Lambda(-4, 0) = \Gamma = [b+bi, b-bi]$ with

square side length γ if and only if $b = \gamma/\sqrt{2}$, $\wp_\Lambda(b/2) = 1$ and $\wp'_\Lambda(b/2) = -2\sqrt{2}$.

(2) *For an arbitrary negative real g_2 the real quarter lattice value of the corresponding lattice is $\wp_\Lambda(a/2) = \sqrt{-g_2}/2 = |e_1| = |e_2|$; moreover if the lattice is $k > 0$ times the standard lattice, $\wp_\Lambda(a/2) = 1/k^2$, and $\wp'_\Lambda(a/2) = (-1/2)(-g_2)^{3/4} = -2\sqrt{2}k^{-3}$.*

Proof. These statements follow from the classical identities given by Equations (2) and (4). □

Corollary 5.2. *Let $\Lambda = [a + ai, a - ai]$, $a > 0$. Then for the corresponding map $\wp_\Lambda$ we have:*

$$a/2 \mapsto |e_1| \mapsto |\wp_\Lambda^{\circ 2} 2(c_j + \Lambda)| \mapsto \wp_\Lambda^{\circ 3}(c_j + \Lambda),$$

for $j = 1, 2$.

For the next result, we note that the real lattice values of $\Lambda = [a + ai, a - ai]$ are integer multiples of $2a$, so we obtain a formula for quarter points to be mapped by $\wp_\Lambda$ to specified fractional values of real lattice points. These next results are extensions of earlier work published in [9], [10], and [12].

Corollary 5.3. *Starting with the standard lattice $\Gamma = [b + bi, b - bi]$, if $\Lambda = [a+ai, a-ai]$, $a > 0$ is k times the standard lattice, $k > 0$, then for any nonnegative integer m, and $c \in (0, 1]$ we have that*

$$\wp_\Lambda(a/2) = (c + m)2a \tag{14}$$

if and only if

$$k = \frac{1}{\sqrt[3]{(c + m)2b}} = \frac{1}{\sqrt[6]{2}\,\sqrt[3]{(c + m)\gamma}} \tag{15}$$

Proof. We first note that γ is the length of the side of a standard square lattice, so $b = \gamma/\sqrt{2}$. Then we just compute:

$$\wp_\Lambda(a/2) = \frac{1}{k^2}$$

$$\frac{1}{k^2} = (c + m)2a = (c + m)2kb = (c + m)k\gamma\sqrt{2}$$

if and only if

$$\frac{1}{k^3} = (c + m)\gamma\sqrt{2}.$$

Taking inverses and cube roots gives the result.

□

5.1.1 Parameters with finite postcritical sets

Special cases of the next result are discussed in [9], [10], and [12].

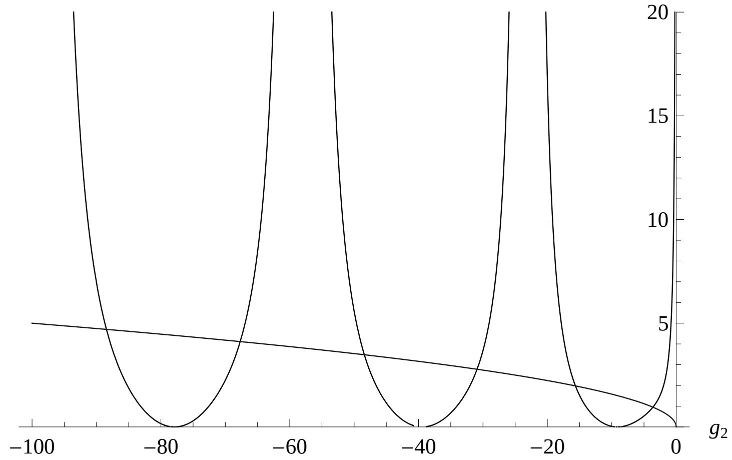

Figure 3. Graphs of $|e_1(g_2)|$ and $\wp_\Lambda(|e_1(g_2)|)$ (with poles) for $\Lambda = \Lambda(g_2, 0)$ real rhombic square

PROPOSITION 5.4. *There are infinitely many values of $g_2 < 0$ for which the critical orbits of $\wp_{\Lambda(g_2,0)}$ satisfy:*

(1) e_1 and e_2 have finite orbits terminating in the same repelling fixed point.
(2) e_1 and e_2 are both prepoles.

Proof. (1) We set $c = \frac{1}{4}$ or $c = \frac{3}{4}$ and pick an integer $m \geq 0$; applying Corollary 5.3 to choose the appropriate value of k we obtain a new lattice $\Lambda = k\Gamma$. When $c = \frac{1}{4}$ this results in $g_2(\Lambda) = -4/k^4$ such that the quarter lattice point

$$\frac{a}{2} = \frac{k\gamma}{2\sqrt{2}}$$

(and hence every point of the form $z = \frac{a}{2} + \lambda$, $\lambda \in \Lambda$) gets mapped under $\wp_\Lambda$ to the quarter lattice point

$$\frac{a}{2} + mk\gamma\sqrt{2} = \frac{a}{2} + m2a.$$

Therefore the quarter lattice point $q_m := \frac{a}{2} + m2a$ is fixed. Since $k < 1$ for each nonnegative integer m, using Lemma 5.1 (2) we have $\wp'_\Lambda(q_m) = -2\sqrt{2}k^{-3} > 1$; the proof for $c = \frac{3}{4}$ is similar since $\wp_\Lambda(a/2) = \wp_\Lambda(3a/2)$.

(2) The proof is similar to (1) using $c = 1$ or $c = \frac{1}{2}$. In the first case e_1 is mapped to a lattice point, which is a pole. In the second e_1 is mapped to a real half lattice lattice point, which in turn gets mapped to the pole at 0.

□

In Figure 3 we show the graphs of $|e_1| = |e_1(g_2)| = \sqrt{-g_2}/2$ (the continuous curve) and $\wp_{\Lambda(g_2,0)}(|e_1|)$ (the graph with poles). The g_2 coordinates where the graphs cross show the values in (1) of Proposition 5.4. The poles give the values where g_2 has a quarter or three quarter lattice point landing on a lattice point, and the zeros are where e_1 gets mapped to 0. This is part (2) of Proposition 5.4.

It is also the case, though we do not prove it here, that there are values of $g_2 < 0$ for which setting $\Lambda = \Lambda(g_2, 0)$, we have $\wp_\Lambda^{\circ j}(a/2) = a/2 + m2a$, $j \in \mathbb{N}$, but not for any $n < j$. In this way we find that we have many negative g_2 values giving rise to finite postcritical sets with terminating repelling periodic orbits of all periods. The period two examples can be seen in Figure 4 at parameters where the graphs of $|e_1| = |e_1(g2)| = \sqrt{-g_2}/2$ and $\wp^{\circ 2}_{\Lambda(g_2,0)}(|e_1|)$ cross; ignoring the intersections

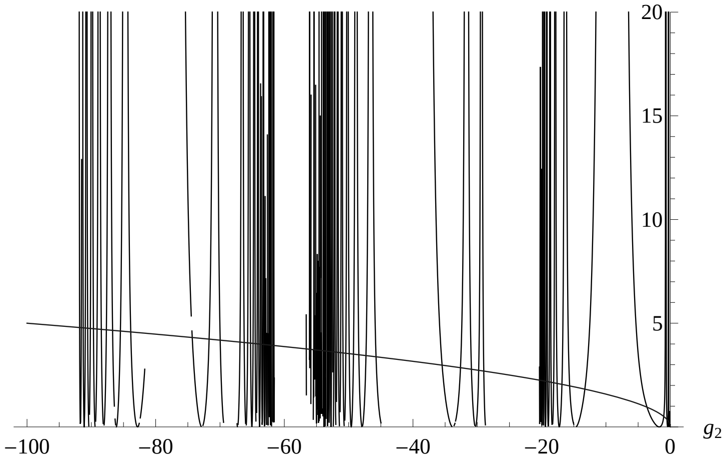

Figure 4. The graphs of $|e_1(g_2)|$ and $\wp_\Lambda^{\circ 2}(|e_1(g_2)|)$ (with poles) for $\Lambda = \Lambda(g_2, 0)$ real rhombic square

which also appear in Figure 3, there are evidently infinitely many new parameters appearing with the property. This phenomenon is the topic of ongoing work on the g_2 parameter space with M. McClure and is loosely related to work done in [12].

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