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Ratio Sets of Endomorphisms which Preserve a Probability Measure[1]

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Introduction. In this paper we compare the concept of weak orbit equivalence of finite measure preserving ergodic automorphisms, which is well-known to be virtually trivial, with the corresponding notion for n-to-one endomorphisms, which we show to be highly nontrivial. Two nonsingular ergodic automorphisms T and S of standard Borel spaces X and X' respectively, are said to be orbit equivalent if there exists a nonsingular invertible a.e. map from X onto X' which maps almost every T-orbit onto an S-orbit; this definition was introduced by H. Dye **[D]**. He subsequently proved that all finite measure-preserving ergodic automorphisms are orbit equivalent, so the concept has mainly been studied in the context of non-measure-preserving transformations, and a complete classification of nonsingular automorphisms up to orbit equivalence has been given **[K1]**, **[K3]**.

We would like to extend Dye's definition to the case of nonsingular endomorphisms, in particular to n-to-one mappings with $n \geq 2$. We assume throughout this paper that all measure spaces are standard probability spaces endowed with the σ-algebra of Borel sets. If we say two endomorphisms T and S are orbit equivalent if there exists a nonsingular invertible map taking almost every T-orbit onto an S-orbit, we are still left with defining an "orbit" of an endomorphism. There are several natural candidates. For $n = 1$, we always define for a nonsingular endomorphism $T, \mathrm{Orb}_T(x) = \{T^n x\}_{n \in \mathbf{Z}}$. In **[HS]** the authors defined for $n \geq 2, \mathrm{Orb}_T^+(x) = \{T^n x\}_{n \in \mathbf{N}}$; using these forward orbits, the definition of orbit equivalence was shown to be the same as (nonsingular) isomorphism, and the classification problem has not been simplified. There is a natural weakening of this notion of orbit equivalence which occurs by looking at larger orbits; we give that here and show it is still much more rigid for noninvertible endomorphisms than for invertible ones.

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We say two points x *and* y *are in the same orbit of* T if $T^m x = T^n y$ for some $m, n \geq 1$; that is, we allow the orbit of a point x to include in addition to its forward orbit, all points whose forward orbits eventually coincide with that of x. This gives an orbit, which for $n > 1$, is strictly larger than $\mathrm{Orb}^+(x)$, and in the case $n = 1$, *gives precisely the classical notion of orbit and orbit equivalence as described above.* The relation "being in the same orbit of T" is an equivalence relation on $X \times X$ (which is not the case for the definition of **[HS]**), which we define as R_T; we write $(x, y) \in R_T$ (or just R when T is understood) when x and y are in the same orbit. There is a subrelation of R, call it R'_T (or just R'), which is given by $(x, y) \in R'$ if $T^n x = T^n y$ for some $n \geq 1$; x and y are related in this way if they are in the same "lateral" orbit of T. If T is ergodic and nonsingular, then both of these equivalence relations are countable amenable nonsingular equivalence relations **[V]**, which is to say they are generated by a single automorphism of the space.

It is now natural to define two endomorphisms T and S of X and X' respectively, to be *orbit equivalent* if there exists a nonsingular map ψ from X onto X' taking almost every T-orbit onto an S-orbit; that is $\psi R_T(x) = \psi(\{y \in X : (x, y) \in R_T\}) = R_S(\psi x)$ for μ-a.e. $x \in X$. (Equivalently, T and S are orbit equivalent if R_T and R_S are isomorphic.) Since countable amenable equivalence relations have been completely classified up to isomorphism **[K3]**, **[CFW]**, we can now ask the question: are all finite measure-preserving ergodic n-to-one endomorphisms orbit equivalent? We know for $n = 1$, the answer is yes. For $n \geq 2$, we will see there are uncountably many orbit equivalence classes. We show this by computing the ratio set of R and R' associated to some well-studied endomorphisms such as one-sided Bernoulli and Markov shifts.

Starting with a nonsingular endomorphism T, we know there is a natural ergodic automorphism which generates the relation R_T up to isomorphism (an odometer); by computing which ratio set is associated to a given relation R_T we actually know in many cases which odometer has the orbit structure of R_T. More precisely, if the ratio set of R_T is $\{\lambda^n\}_{n \in \mathbb{Z}}$ for some $\lambda \in (0, 1)$, then the equivalence relation is uniquely defined up to isomorphism, the associated generating odometer is uniquely defined up to orbit equivalence, and the endomorphism is uniquely determined up to orbit equivalence. (See below for definitions.) We first compute the ratio sets of the relations R and R' associated to ergodic one-sided measure-preserving Bernoulli and Markov shifts. We prove that the relation R for these endomorphisms is never type II_1, a seemingly paradoxical situation. (In the case of ergodic automorphisms, the ones giving type II_1 relations R are characterized by being ergodic finite measure-preserving **[D]**.) We show that for some shifts the subrelation R' is II_1 however; in fact for each k there is exactly one Bernoulli k-shift with this property. We show that any $\lambda \in (0, 1)$ can occur as an element of the ratio set of a finite measure preserving Bernoulli shift on two or more states, and type III_1 relations also occur for R and R'. To obtain the (unique) type III_1

relation R we need either a Markov shift or three states. In the case of one-sided Markov shifts, the ratio sets of R and R' turn out to be the closure in $\mathbf{R}^+$ of some invariants of finitary isomorphism (using an amenable equivalence relation between two-sided shifts other than orbit equivalence) introduced by Krieger **[K2]** and developed further by W. Parry and K. Schmidt **[PS]**.

We also show that if we drop the condition that the measure of the shift be preserved, then many more (but not all) equivalence relations can be realized by a finite state shift with a product measure. We conclude by showing that every odometer has a representative as the relation R' for the two-shift with respect to some nonsingular measure. The author would like to thank Klaus Schmidt for helpful discussions and for bringing **[PS]** to her attention.

We begin with some introductory definitions and terminology about ratio sets of amenable countable ergodic equivalence relations. Ratio sets were introduced by Araki and Woods **[AW]** for odometers in the context of von Neumann algebras and generalized to arbitrary ergodic nonsingular transformations by Krieger **[K1]**. For a complete account of ideas mentioned here, see **[CFW]**. We assume that $(X, \mathscr{B}, \mu)$ is a standard Borel space with a countable measurable relation $R \subset X \times X$. For any amenable countable ergodic equivalence relation R, we define a partial automorphism on R to be an invertible map $\phi\colon A \to B$, with $A, B \subset X$ and $\operatorname{graph}(\phi) \subset R \subset X \times X$. We then define a nonnegative real number λ to be in the ratio set of R, and write $\lambda \in r(R)$, if for every $\varepsilon > 0$ and every set $C \in \mathscr{B}$ of positive measure, there exist sets $A, B \in \mathscr{B}$ of positive measure, with $A, B \subset C$, and a partial isomorphism $\phi\colon A \to B$ satisfying $|d\mu * \phi / d\mu(x) - \lambda| < \varepsilon$ for all $x \in B$, where $\mu * \phi(D) = \mu(\phi^{-1}D)$ for all measurable sets $D \subset B$. One can show that the ratio set is independent of the measure chosen in the measure class of μ, it is a closed subgroup of $\mathbf{R}^+\backslash\{0\}$, and could contain the limiting value 0 **[K1]**. We say that the relation R is of type:

II if $r(R) = \{1\}$; it is II_1 if the generating automorphism preserves a finite equivalent measure, and II_∞ if it preserves an infinite equivalent measure;

III_λ if $r(R) = \{\lambda^n\}_{n \in \mathbf{Z}}$ for some $\lambda \in (0,1)$ (which implies that the generating automorphism preserves no equivalent measure);

III_1 if $r(R) = [0, \infty)$;

III_0 if $r(R) = \{0, 1\}$.

Since the ratio sets obtained in this way are invariant under orbit equivalence of the generating automorphism, isomorphism of R_T, and orbit equivalence of T, we see that the ratio set is also invariant under the usual isomorphism of endomorphisms. Of course, the ratio set described in this paper is by no means a complete isomorphism invariant; for example, it does not distinguish the cardinality of the endomorphism, but we will see that for some examples such as one-sided Bernoulli and Markov shifts, it is a complete invariant under orbit equivalence.

Furthermore, it bears noting that the ratio set of an endomorphism computed in this paper is not the same as the ratio set of **[HS]** (different notation is also used); in **[HS]** the ratio set computed is the essential range of the Radon-Nikodym derivative cocycle of the endomorphism, (considered as a semigroup action), and is equal to 1 when the measure is preserved by the map (but is not necessarily 1 if an equivalent measure is preserved **[ES]**). Here we look at partial automorphisms between sets related by the orbits of an endomorphism and determine the Radon-Nikodym derivative of these (invertible) maps; this is the same ratio set computed for one-sided Markov shifts in **[B]**. This has also been computed in the context of two-sided Markov shifts in **[K2]**, **[PS]**, with one important difference: the relations R and R' associated to a two-sided shift in **[PS]** do not arise from the orbits of the shift as we have defined them above, but from a different amenable equivalence relation defined on the two-sided shift space. If T is a two-sided shift, R_T is always the trivial ratio set, which agrees with the original definition of Krieger **[K1]**.

§**1. Bernoulli shifts.** We start by considering the one-sided Bernoulli k-shift. Let $X = \prod_{i=0}^{\infty}\{0, 1, \dots, k-1\}_i$, $\mathscr{B}$ = the σ-algebra generated by cylinder sets of the form $[c_0, \dots, c_r] = \{x \in X: x_0 = c_0, \dots, x_r = c_r\}$ for $r \geq 0$ and $c_i \in \{0, 1, \dots, k-1\}$. Consider the product measure on $\{0, \dots, k-1\}_i$ determined by a vector $p_i = p = \{\lambda_1/(\lambda_1 + \cdots + \lambda_k), \dots, \lambda_k/(\lambda_1 + \cdots + \lambda_k)\}$ for $\lambda_j \in \mathbf{R}^+$ so that $p_i\{j\} = \lambda_j/(\lambda_1 + \cdots + \lambda_k)$ for each $i \in \mathbf{N}$ and each $j = 0, \dots, k-1$. The product measure $\mu = \prod_{i=0}^{\infty} p_i$ gives an invariant ergodic probability measure for the one-sided Bernoulli k-shift on X; the shift is defined by $(Tx)_i = x_{i+1}$ for all $i \geq 0$. (The only product measure preserved by the shift T with $p = \{\lambda_1/(\lambda_1+\cdots+\lambda_k), \dots, \lambda_k/(\lambda_1+\cdots+\lambda_k)\}$ as one factor has p as each of its factors, cf. **[J**, p. 161].) The first result we mention is the following.

PROPOSITION 1.1. *Let T be the one-sided Bernoulli k-shift on*

$$X = \prod_{i=0}^{\infty}\{0, 1, \dots, k-1\}_i$$

with measure $p = \{\alpha_1, \dots, \alpha_k\}$ on each factor, with $\alpha_j > 0$ and $\sum_{j=1}^{k} \alpha_j = 1$. Then the ratio sets of both R and R' contain all elements of the form α_i/α_j, for $i, j = 1, \dots, k$.

PROOF. The cylinder sets of the form $C = [c_0, \dots, c_r] = \{x \in X: x_0 = c_0, \dots, x_r = c_r\}$ form a countable dense subset of the measure algebra $\mathscr{B}_0$; then by Lemma 2.2 in **[CHP]** it suffices to show that for any $\beta \in (0, 1]$ which is a quotient of the form α_i/α_j, and any set $C = [c_0, \dots, c_r]$, there exist sets $A, B \subset C$, with $\mu(B) > k\mu(C)$, where k is a constant which only depends on p, (and not on C), and a partial automorphism $\phi: A \to B$ satisfying graph$(\phi) \subset R$, and $d\mu * \phi/d\mu(x) = \beta$ for all $x \in B$. Let $A \subset C$ be defined by $A = [c_0, \dots, c_r, a]$ and $B = [c_0, \dots, c_r, b]$ where $a, b \in \{0, \dots, k-1\}$ are such that $p(\{b\}) = \alpha_i$ and $p(\{a\}) = \alpha_j$. Then we map A to B by $[\phi(x)]_i = x_i$ if

$i \neq r+1$, and $[\phi(x)]_{r+1} = b$. Since $T^{r+2}x = T^{r+2}(\phi x)$ for each $x \in A$, we have graph$(\phi) \subset R' \subset R$. Furthermore for each $x \in B, d\mu * \phi/d\mu(x) = \alpha_i/\alpha_j = \beta$ and $\mu(B) = \alpha_i\mu(C) \geq k\mu(C)$, by choosing $k = \min\{\alpha_i\}$. □

In fact, the following stronger proposition can be proved.

PROPOSITION 1.2. *Let T be the one-sided Bernoulli k-shift on*

$$X = \prod_{i=0}^{\infty}\{0, 1, \ldots, k-1\}_i$$

with measure $p = \{\alpha_1, \ldots, \alpha_k\}$ on each factor, with $\alpha_j > 0$ and $\sum_{j=1}^{k}\alpha_j = 1$. Then $r(R)$ is the closure of the subgroup generated by α_i^m/α_j^n with $m, n \in \mathbf{N}$; $r(R')$ is the closure of the subgroup generated by α_i^n/α_j^n with $n \in \mathbf{N}$; (i and j are not necessarily distinct).

PROOF. We fix any α_i^m/α_j^n as above. Letting $C = [c_0, \ldots, c_r]$, assume that $p(\{a\}) = \alpha_j$ and $p(\{b\}) = \alpha_i$. Then define $A = [c_0, \ldots, c_r, a, a, \ldots, a]$ (exactly n a's) and $B = [c_0, \ldots, c_r, b, \ldots, b]$ (exactly m b's). The set B fills up a fixed proportion of C, independent of C as required. We define $\phi\colon A \to B$ as follows.

$$\begin{aligned} &\phi(x)_i = x_i \text{ if } i \leq r; \\ &\phi(x)_i = b \text{ if } r+1 \leq i \leq m; \text{ and} \\ &\phi(x)_i = x_{i+(m-n)} \text{ if } i > m. \end{aligned}$$

In this way we define a one-to-one map from A onto B. Also the definition of ϕ is independent of the relation between m and n. We now observe that $T^{r+n+1}x = T^{r+m+1}\phi x$ for all $x \in A$, so graph$(\phi) \subset R$. Finally we see that $d\mu * \phi/d\mu(x) = \mu(B)/\mu(A) = \alpha_i^m/\alpha_i^n$.

It now remains to show that nothing else is in the ratio set. Suppose $\beta \in r(R)$ and β is not in the closure of the above group, so β has distance at least $\delta > 0$ from any one of the elements. By choosing $C = X$ and $\varepsilon = \delta$ and noting that every partial automorphism between subsets of X must be of the form described above in order that graph$(\phi) \subset R$ we see that $|d\mu * \phi/d\mu(x) - \beta| > \varepsilon$ everywhere on X for allowable ϕ. This proves that β is not in $r(R)$ or $r(R')$. The ratio set of R' is the obvious restriction to equal powers of α_i and α_j^{-1}. □

We obtain the following corollaries.

COROLLARY 1.3. *The relation R associated to the Bernoulli k-shift is of type* $\mathrm{III}_{1/k}$ *with respect to the measure $\mu = \prod_{i=0}^{\infty} p_i, p_i = \{1/k, \ldots, 1/k\}$. The relation R' is of type* II_1 *in this case.*

COROLLARY 1.4. *There are uncountably many orbit equivalence classes of finite ergodic measure-preserving n-to-1 Bernoulli shifts if $n > 1$.*

REMARKS.

1. From the above results, we see that using Bernoulli 2-shifts, we can only obtain a type III_λ relation R with $\lambda \in (0, 1)$, but every such λ can occur

by choosing $p = \{\lambda/1+\lambda, 1/1+\lambda\}$. It will follow from Theorem 2.3 of this paper and **[GS]** that there is a product measure on the 2-state space which gives the type III_1 relation R for the shift endomorphism; this measure will not be equivalent to any probability measure which is preserved by the shift.

2. To obtain a type III_1 finite measure-preserving Bernoulli shift, we use a three-state space X and choose the measure p on each factor to be $p = \{\lambda/(1+\lambda+\beta), \beta/(1+\lambda+\beta), 1/(1+\lambda+\beta)\}$ with λ and β rationally independent numbers in $(0,1)$ (so that they generate a dense subgroup of $\mathbf{R}^+$). From Proposition 1.1, it follows that $r(R)$ and also $r(R')$ are of type III_1.

3. None of the partial isomorphisms given in the proofs of the above propositions would be allowable under the relation R obtained by a two-sided (invertible) Bernoulli shift. This, of course, must be the case since all finite measure-preserving Bernoulli shifts give the same type II_1 relation R. In the case of the two-sided shift, the relation R' is just the trivial relation on X; i.e., $x \sim x$ only. This is also a II_1 relation.

4. Many examples of type III_λ one-sided finite measure-preserving ergodic Bernoulli shifts on $k+2$ states for any $\lambda \in (0,1)$ can be obtained by defining each factor measure to be of the form

$$p = (1/(1+\lambda+\lambda^{r_1}+\cdots+\lambda^{r_k}), \lambda/(1+\lambda+\lambda^{r_1}+\cdots+\lambda^{r_k}),$$
$$\lambda^{r_1}/(1+\lambda+\lambda^{r_1}+\cdots+\lambda^{r_k}), \ldots, \lambda^{r_k}/(1+\lambda+\lambda^{r_1}+\cdots+\lambda^{r_k}))$$

or a permutation of the entries of this vector, where $r_1, \ldots, r_k$ is any sequence of integers, not necessarily distinct. This follows form an easy computation based on the conditions mentioned above. Even though the ratio set in the case of type III_λ ergodic equivalence relations is a complete invariant under isomorphism of the relation **[K1]**, it is not a complete invariant under measure theoretic isomorphism of ergodic endomorphisms (even within a fixed cardinality class). We can obtain two Bernoulli shifts which are of type III_λ and which have different entropies. For example, a three-shift with measure $p = \{1/7, 2/7, 4/7\}$ is of type $\mathrm{III}_{1/2}$ as is the three-shift with measure $q = \{1/11, 2/11, 8/11\}$. In each case both the relation R and the subrelation R' are of type $\mathrm{III}_{1/2}$. However their entropies are different so they are not isomorphic.

5. We mention briefly the case of countable state Bernoulli shifts. We let $X_i = \{0, 1, 2, \ldots\}$, and $X = \prod_{i=0}^{\infty} X_i$. We define a probability vector on X_i by choosing a sequence of positive numbers a_k whose sum is 1, and then define $p_i(\{k\}) = a_k$. Letting $p_i = p_j$ for every i, j, we obtain a measure on X (the product of the p_i's) which is preserved by the one-sided Bernoulli shift. We will not obtain any new ratio sets in this way; in fact it seems clear that R and R' will usually be the III_1 relation, unless each a_k is a term of some normalized geometric series. (In that case, type III_λ can occur.) The relation R' could also be II_∞ if $p_i = \{1, 1, \ldots\}$; this endomorphism preserves an infinite measure, not a finite one; one can never obtain a type II_1 relation.

§2. Markov shifts.

In this section we consider one-sided Markov shifts which preserve a probability measure. Our results are similar to those for Bernoulli shifts. One new result is that of type III_1 relation can be obtained from a Markov shift on two symbols. We also show that if the relation R' is of type II_1 for a full Markov shift, then in fact it is a Bernoulli shift; that is, the ergodic measure for the shift is just a Bernoulli product measure. (We deduce this directly, but it also follows from Example 5.4.2 in **[PS]**.) The ratio sets associated to one-sided Markov shifts have been studied before in **[B]**, **[S]**, **[K2]** and **[PS]**, but in different contexts.

We recall the basic description of an ergodic measure preserving Markov shift. We begin with a finite collection of states, $\{0, 1, \ldots, k-1\}$ and a $k \times k$ matrix A of 0's and 1's such that every row and column contains at least one 1. We consider also a consistent Markov matrix for the system, that is, a nonnegative $k \times k$ matrix $P = (p_{ij})$ with $\sum_j p_{ij} = 1$ for each i; in addition $p_{ij} > 0$ precisely when $a_{ij} = 1$. If A is assumed to be irreducible (for i, j with $1 \le i, j \le k$ there is a positive integer n such that $a_{ij}^{(n)} > 0$ where $a_{ij}^{(n)} = ij^{th}$ entry of A^n), then there exists a positive row vector $p = (p_1, p_2, \ldots, p_k)$ with $\sum_i p_i = 1$ such that $pP = p$. Using this, we define an ergodic probability measure invariant under the shift on $X_A = \{x \in \prod_{k=0}^{\infty}\{0, 1, \ldots, k-1\} \colon a_{x_i x_{i+1}} > 0$ for all $i\}$. This measure is determined by its values on cylinders and is defined by $\mu(C) = p_{i_0} p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{r-1} i_r}$ where $C = [i_0, i_1, \ldots, i_r]$ is the cylinder defined as in §1.

We remark that in the case of the full shift, A is just the $k \times k$ matrix consisting of 1's and all the entries of P are positive with each row sum equal to 1. When the rows of P are all identical, the shift is in fact a Bernoulli shift.

The method for computing ratio sets of R and R' for a Markov shift is different from the Bernoulli shift due to the fact that the Radon-Nikodym derivative of a partial automorphism is not necessarily constant on arbitrary cylinders; the method of proof used in Proposition 1.2 has to be applied more carefully. For example, it was pointed out to the author by K. Schmidt that certain ratios of entries of the matrix may fail to appear in the ratio set of R or R'; an example of this phenomenon is given below. In particular the ratio set of R' is exactly the topological closure of the group Δ_P defined by Parry and Schmidt **[PS]**, and $r(R)$ is the closure of the group they call Γ_p. By definition, Δ_P is the multiplicative subgroup of $\mathbf{R}^+$ consisting of all possible ratios with numerator of the form $p_{i_0 i_1} \cdots p_{i_{n-1} i_0}$ and denominator of the form $p_{i_0 j_1} \cdots p_{j_{n-1} i_0}$ with $n \ge 1$, and $i_r, j_r \in \{0, 1, \ldots, k-1\}$, and involving only nonzero entries of P. The group $\Gamma_P \supset \Delta_P$ is defined similarly; it is the multiplicative subgroup of $\mathbf{R}^+$ consisting of all possible (nonzero) ratios with numerator of the form $p_{i_0 i_1} \cdots p_{i_{n-1} i_0}$ and denominator of the form $p_{i_0 j_1} \cdots p_{j_{m-1} i_0}$ with $n, m \ge 1$. Using techniques similar to those in §1, we prove the following.

PROPOSITION 2.1. *Let T denote an ergodic one-sided Markov shift with stochastic matrix $P = p_{ij}$. Then $r(R')$ = the closure in $\mathbf{R}^+$ of Γ_P, and $r(R)$ = the closure in $\mathbf{R}^+$ of Δ_P.*

PROOF. The proof is virtually the same as the proof of Proposition 1.2; we work with cylinder sets and show that $\Gamma_P \subset r(R')$; the discussion of **[PS]** (or an easy computation) shows that no real number outside the closure of Γ_P will be in the ratio set. Then an approximation argument gives the rest. □

COROLLARY 2.2. *If T is a full Markov shift on k states and $r(R'_T) = \{1\}$, then T is the Bernoulli shift on k states with $p = \{1/k, \dots, 1/k\}$.*

PROOF. Clearly if P is the matrix all of whose entries are $1/k$, then $r(R') = \{1\}$. We show this is the only possibility. From Proposition 2.1 we see that all entries along the main diagonal must be equal to some $\alpha > 0$ (since $p_{ii}/p_{ij}p_{ji} = 1 = p_{jj}/p_{ij}p_{ji}$). All products of the form $p_{ij}p_{ji}$ for any i, j satisfy $p_{ij}p_{ji} = \alpha^2$. If $k = 2$, then the fact that P is stochastic gives the result immediately. We assume now that $k \geq 3$ and that $p_{12} > \alpha$. From this it follows that $p_{23}p_{31} < \alpha^2$ (since $p_{12}p_{23}p_{31} = \alpha^3$), $p_{21} < \alpha$, and $p_{13}p_{32} > \alpha^2$. This implies that $p_{23}p_{31} < p_{13}p_{32}$; writing $p_{23} = \alpha^2/p_{32}$ and $p_{31} = \alpha^2/p_{13}$, we have that $\alpha^4 < p_{13}^2 p_{32}^2$ which is a contradiction. Since our choice of p_{12} was completely arbitrary, the result follows, since only $\alpha = 1/k$ will work. □

The following stronger assertion was made in **[PS]**.

COROLLARY 2.3 **[PS]**. *If T is an ergodic Markov shift and $r(R') = \{1\}$, then P is the matrix (consistent with A) giving the Markov measure of maximal entropy.*

This situation is much more rigid than what is suggested by Dye's Theorem **[D]** where knowing that the orbit relation of an automorphism (such as the two-sided Bernoulli shift) is type II_1 tells you nothing about the transformation other than that it is finite measure-preserving and ergodic.

EXAMPLES.

1. We give an example of a Markov shift on two states which is of type III_1; this is the only new relation obtained by passing from Bernoulli to Markov measures, and is only new in the case of two states. Since the ratio set is an invariant under isomorphism of R, and therefore of isomorphism of the shift itself (as is cardinality in the case of a one-sided shift), this one-sided Markov shift is not isomorphic to any one-sided Bernoulli shift. However, it is well-known that its two-sided counterpart is isomorphic to a Bernoulli shift.

The shift is given by the matrix $P = \begin{bmatrix} \alpha(1+\alpha)^{-1} & (1+\alpha)^{-1} \\ \beta(1+\beta)^{-1} & (1+\beta)^{-1} \end{bmatrix}$, which then has as its eigenvector for eigenvalue 1, the vector

$$p = (\alpha(1+\beta)/1 + 2\alpha + \alpha\beta, 1 + \alpha/1 + 2\alpha + \alpha\beta).$$

As long as α and β are rationally independent, the ratio set is all of $\mathbf{R}^+$, so the relation is of type III_1. Any Markov shift on two states which is of type III_1 will have to be the full shift (that is, have no zero entries in the associated stochastic matrix). However, one can show that entropy will not be an invariant under isomorphism of R, since many different entropies can occur for type III_1 shifts within each cardinality class by varying the choice of α and β.

2. Type III_λ Markov shifts occur if and only if the ratios which arise from all possible quotients of the form $(p_{i_0 i_1} \cdots p_{i_{n-1} i_0})/(p_{i_0 j_1} \cdots p_{j_{m-1} i_0})$ involving only non-zero entries of the stochastic matrix are equal to λ^q for some integer q. Of course the ratio of precisely λ needs to occur, or the shift is not of type III_λ.

3. We see that although the matrix $P = \begin{bmatrix} \alpha(1+\alpha)^{-1} & (1+\alpha)^{-1} \\ \alpha(1+\alpha)^{-1} & (1+\alpha)^{-1} \end{bmatrix}$ defines a one-sided Markov (Bernoulli) shift of type III_α (for R and R'), the matrix $Q = \begin{bmatrix} \alpha(1+\alpha)^{-1} & (1+\alpha)^{-1} \\ (1+\alpha)^{-1} & \alpha(1+\alpha)^{-1} \end{bmatrix}$ gives a shift T_Q such that $\alpha \notin r(R')$; in particular, R' is of type III_{α^2}. (This example is given in **[PS]**.) We conclude that T_P and T_Q are not orbit equivalent even though they have the same entropy, which is a strengthening of the nonisomorphism result of Furstenberg **[F]**.

We remark that we have shown that for finite measure preserving Bernoulli shifts and Markov shifts, the ratio set provides a complete invariant for the isomorphism of R. That is to say, only those equivalence classes completely characterized by the ratio set occur (type III_0 never occurs) and all such classes occur. Furthermore, even though in the type II case the triviality of the ratio set leaves some ambiguity as to which type R' could be, in this case either the shift is of finite cardinality and a trivial ratio set means the relation is of type II_1 or it is countable to one, in which case II_1 can never occur.

If we drop the condition that the measure for the shift should be preserved and do not insist that the measure on the k-state space be a product measure, then we can realize all orbit equivalence classes as 2 to 1 shifts. This will follow from a well-known result about odometers and the following natural connection between orbits of odometers and relations R' for one-sided shifts.

We recall that the odometer transformation S on

$$X = \prod_{i=0}^{\infty} \{0, 1, \ldots, k_i - 1\}$$

is defined to be addition of 1 on the left (in $\mathbf{Z}_{k_i}$) with carrying to the right if necessary. Unlike the shift, k can vary with i for an odometer.

THEOREM 2.3. *Let S be the odometer on the space*

$$X = \prod_{i=0}^{\infty} \{0, 1, \ldots, k - 1\}_i,$$

with $\mathscr{B}$ the σ-algebra of Borel sets and μ some ergodic nonsingular measure for S. Then the orbit relation $R_S = \{(x, y) \colon y = S^n x \text{ for some } n \in \mathbf{Z}\}$ is isomorphic to R'_T for the one-sided shift T on $(X, \mathscr{B}, \mu)$.

PROOF. The isomorphism is implemented by the identity mapping defined on a set of full measure in X; that is, we just show that $(x, y) \in R_S$ if and only if $(x, y) \in R'_T$ for the shift map on the same space with the same measure. This just follows from the fact that $y = S^n x$ if and only if y and x agrees in all but finitely many coordinates; say for all $i \geq r, x_i = y_i$. This implies that if T denotes the shift on $X, T^r x = T^r y$, so $(x, y) \in R'$. Similarly if $T^r x = T^r y$, then x and y differ in only finitely many places, so one can easily map x to y via a power of S (since it acts transitively on finite products of factors in X). This proves the orbit equivalence. □

Starting with an ergodic nonsingular odometer ensures that R'_T is nonsingular and ergodic, but T itself may not be. We consider the following example of a type II_∞ odometer.

EXAMPLE 2.4 Let $X = \prod_{k=0}^{\infty}\{0, 1\}$, and we define the measure $\mu = \prod_{k=0}^{\infty} \mu_k$ as follows: for each $k = 2q, q = 0, 1, \ldots$, we define $\mu_k(\{0\}) = \mu_k(\{1\}) = 1/2$. For each $k = 2q - 1, q = 1, 2, \ldots$, we define $\mu_k(\{0\}) = 1 - 1/k^2, \mu_k(\{1\}) = 1/k^2$. The odometer transformation with this measure is of type II_∞ **[CHP]** and gives rise via Theorem 2.3 to a shift T on two states with R'_T of type II_∞. However, one can check that the endomorphism T is singular with respect to the measure μ. Let $C = \{x \in X \colon x_{2q-1} = 0 \text{ for } q \in \mathbf{N}\}$; we have that $\mu(C) = \prod_{q=1}^{\infty}(1 - 1/(2q - 1)^2) > 0$. However $T^{-1}C = \{x \in X \colon X_{2q} = 0 \text{ for } q \in \mathbf{N}\}$ which satisfies $\mu(T^{-1}C) = 0$.

In fact one can easily verify that the shift T obtained in Theorem 2.3 is nonsingular and ergodic if and only if the corresponding R_T is nonsingular and ergodic. (In the case of finite measure preserving shifts the nonsingularity and ergodicity of R and R' are the same.)

It is not possible to realize every odometer as a product odometer (an odometer with product measure μ), and not every product odometer can be realized a shift on k elements for finite k. However Giordano and Skandalis proved that any bounded odometer (k is bounded over all $i \in \mathbf{N}$) with a product measure is orbit equivalent to some binary odometer with a product measure **[GS]**. This gives us the corollary that every relation R'_T which can be realized by a k-to-1 product shift is isomorphic to a relation R'_T for a 2-to-1 shift, with some nonsingular ergodic product measure. Finally we obtain as a corollary the result that every orbit equivalence class of odometers can be realized as a shift orbit relation R' on a two-state space with some nonsingular ergodic measure, because of the fact that all ergodic nonsingular odometers are orbit equivalent to some binary one **[D]**, **[K1]**, **[K3]**.

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