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Diffeomorphisms of Manifolds with Nonsingular Poincaré Flows*

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INTRODUCTION

In this paper we study the question of which countable amenable ergodic equivalence relations can be realized as C^∞ group actions on manifolds. It has been proved that every countable equivalence relation which is amenable, nonsingular, and ergodic is orbit equivalent to an ergodic integer action on a measure space [CFW]. Katznelson showed that any ITPFI ergodic equivalence relation is orbit equivalent to some C^∞ diffeomorphism of the circle. He also proved that every ergodic C^3 diffeomorphism of the circle generates an amenable ITPFI equivalence relation [Ka].

In an unpublished preprint, R. Wong gives an explicit description of a $\mathbb{Z}$ action which has as its Poincaré flow any prescribed aperiodic ergodic flow of a measure space [W]. Earlier, C. Series had given an example of an amenable foliation in each orbit equivalence class [Se]. Hamachi and Osikawa have published examples of groups of automorphisms in all possible orbit equivalence classes [HO]. In [Ha] a method was given for constructing a diffeomorphism of a manifold whose Poincaré flow is any prescribed smooth measure-preserving flow of (another) manifold. In fact all of the above-mentioned examples are similar and are based on a theorem of Krieger which proves that two actions are orbit equivalent if and only if their Poincaré flows are isomorphic [Kr]. Recently it has been shown that every ergodic measure-preserving flow is isomorphic to a C^∞ flow on an open two-dimensional manifold [AOW].

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This solves the problem for measure-preserving and approximately transitive flows; that is, actions whose Poincaré flows fall into these categories are orbit equivalent to smooth integer actions on manifolds. It remains to determine if arbitrary amenable countable ergodic equivalence relations with non-measure-preserving Poincaré flows can be realized as diffeomorphisms of manifolds; we give a partial answer here.

We show that given any aperiodic ergodic C^∞ flow on a smooth manifold N, there exists a pair of commuting diffeomorphisms on $T^2 \times N \times \mathbb{R}$ which generates an ergodic group of diffeomorphisms with that prescribed flow as its Poincaré flow. In Section 1 we give the necessary definitions and notation. Section 2 contains the construction and some remarks about attempts to strengthen the results. In Section 3 we construct diffeomorphisms that are orbit equivalent to the $\mathbb{Z} \times \mathbb{Z}$ actions of Section 2. We also give a general method for constructing diffeomorphisms that are orbit equivalent to amenable Lie group actions.

1. Definitions and Notation

Let $(X, \mathscr{S}, \mu)$ denote a Borel space with μ a probability measure on $(X, \mathscr{S})$. We define f to be a *nonsingular ergodic transformation* of $(X, \mathscr{S}, \mu)$ if $\mu \sim f_* \mu$ (where $f_* \mu(A) = \mu(f^{-1}A)$ for every $A \in \mathscr{S}$) and if every f-invariant set $B \in \mathscr{S}$ satisfies either $\mu(B) = 1$ or $\mu(B) = 0$. We define the group $\operatorname{Aut}(X, \mathscr{S}, \mu) = \{g: (X, \mathscr{S}, \mu) \circlearrowleft$ such that g is invertible μ-a.e., bimeasurable, and nonsingular$\}$. The *full group* of $f \in \operatorname{Aut}(X, \mathscr{S}, \mu)$ is defined by $[f] = \{g \in \operatorname{Aut}(X, \mathscr{S}, \mu): g(x) = f^{n(x)}(x)$ for μ-a.e. $x \in X$, with $n: X \to \mathbb{Z}$ a measurable map$\}$.

DEFINITION 1.1. Two transformations $f, g \in \operatorname{Aut}(X, \mathscr{S}, \mu)$ are *orbit equivalent* if there exists a bimeasurable invertible (μ-a.e.) map $\psi: X \to X$ with $\psi_* \mu \sim \mu$ and such that $\psi(f\text{-orbit of } x) = g\text{-orbit of } (\psi x)$ for μ-a.e. $x \in X$.

DEFINITION 1.2. A *real-valued cocycle* for a transformation f is a Borel map $a: \mathbb{Z} \times X \to R$ satisfying:

(i) $\mu\{x: f^n x = x\} \cap \{x: a(n, x) \neq 0\} = 0$, and

(ii) $a(n+m, x) = a(n, f^m x) + a(m, x)$ for μ-a.e. $x \in X$.

The *essential range* of the cocycle a, denoted $E(a)$, is the set of all $\lambda \in \mathbb{R}$ such that for any $A \in \mathscr{S}$, $\mu(A) > 0$, and $\varepsilon > 0$, there exists $n \in \mathbb{Z}$ such that $\mu(A \cap f^{-n}A \cap \{x: |a(n, x) - \lambda| < \varepsilon\}) > 0$. The essential range of an

$\mathbb{R}$-valued cocycle forms a closed subgroup of $\mathbb{R}$ [Sc]. We say that f is of type III_1 if

$$E\left(\log \frac{d\mu f}{d\mu}\right) = \mathbb{R}.$$

DEFINITION 1.3. A one-parameter group $\{U_s\}_{s\in\mathbb{R}}$ of automorphisms of $(X, \mathscr{S}, \mu)$ is a measurable nonsingular flow if the map $(x, s) \mapsto U_s x$ from $X \times \mathbb{R}$ to X is measurable, and each U_s is nonsingular on X. If $\psi: X \to \mathbb{R}$ is measurable and satisfies $\psi(U_s x) = \psi(x)$ for μ-a.e. $x \in X$ and for every $s \in \mathbb{R}$, then ψ is called a $\{U_s\}$-invariant function. If $\{U_s\}$ admits no nonconstant invariant functions then we say that $\{U_s\}$ is an *ergodic flow*. This is equivalent to saying that the only $\{U_s\}$-invariant measurable sets have measure 1 or 0. Two flows $U_s: (X, \mathscr{S}, \mu) \circlearrowleft$ and $U'_s: (X', \mathscr{S}', \mu') \circlearrowleft$ are *isomorphic* if there exists an invertible bimeasurable map $\rho: X \to X'$ such that $\rho_* \mu \sim \mu'$ and satisfying $U'_s \rho(x) = \rho U_s(x)$ for μ-a.e. $x \in X$ and every $s \in \mathbb{R}$.

We consider the integer action of an ergodic element $f \in \mathrm{Aut}(X, \mathscr{S}, \mu)$ and we define on $X \times \mathbb{R}$ another $\mathbb{Z}$ action as

$$S_f(x, t) = \left(fx, t + \log \frac{d\mu f}{d\mu}(x)\right) \qquad \text{for every} \quad (x, y) \in X \times \mathbb{R};$$

this transformation generates the action. By $(X \times \mathbb{R}, \mathscr{S} \times \mathscr{F}, \mu \times m)$ we will denote the measure space obtained by forming the Cartesian product of $(X, \mathscr{S}, \mu)$ with $(\mathbb{R}, \mathscr{F}, m)$, where m denotes Lebesgue measure on $\mathbb{R}$, and $\mathscr{S} \times \mathscr{F}$ denotes the product σ-algebra formed in the usual way.

DEFINITION 1.4. A map π from $X \times \mathbb{R}$ onto a Lebesgue space $(Y, \mathscr{B}, \nu)$ is called a *factor map* with respect to S_f if it satisfies:

(1) $\pi^{-1}(A) \in \mathscr{S} \times \mathscr{F}$ if and only if $A \in \mathscr{B}$.

(2) $\mu(\pi^{-1}A) = 0$ if and only if $\nu(A) = 0$ for any $A \in \mathscr{B}$.

(3) $\pi \circ S_f(x, t) = \pi(x, t)$ for a.e. $(x, t) \in X \times \mathbb{R}$.

(4) If $\psi: X \times \mathbb{R}$ is an S_f-invariant function, then there is a function $\bar{\psi}: Y \to \mathbb{R}$ such that $\psi(x, t) = \bar{\psi}(\pi(x, t))$ for $\mu \times m$-a.e. $(x, t) \in X \times \mathbb{R}$.

The following lemma states that factor maps are unique up to isomorphism.

LEMMA 1.5 [HO]. *Let π_1 and π_2 be factor maps from $(X \times \mathbb{R}, \mathscr{S} \times \mathscr{F}, \mu \times m)$ onto Lebesgue spaces $(Y_1, \mathscr{B}_1, \nu_1)$ and $(Y_2, \mathscr{B}_2, \nu_2)$, respectively. Then there exists an isomorphism $\phi: (Y_1, \mathscr{B}_1, \nu_1) \to (Y_2, \mathscr{B}_2, \nu_2)$ satisfying $\phi(\pi_1(x, t)) = \pi_2(x, t)$ for a.e. $(x, t) \in X \times \mathbb{R}$.*

Let $\zeta(f)$ denote a measurable partition which generates all S_f-invariant sets, and let π_f denote the natural surjection from $X \times \mathbb{R}$ onto the measure space $X \times \mathbb{R}/\zeta(f)$. It is easy to see that π_f is a factor map with respect to S_f. We now define a flow on $X \times \mathbb{R}$ by $T_s(x, t) = (x, t+s)$ for every $(x, t) \in X \times \mathbb{R}$, $s \in \mathbb{R}$. Since S_f commutes with $\{T_s\}$, the image under π_f of $\{T_s\}$ is a flow on $(X \times \mathbb{R}/\zeta(f), \mathscr{S}_\zeta, \mu_\zeta)$ defined by $\tilde{T}_s(\pi_f(x, t)) = \pi_f(T_s(x, t))$. Kreiger proved that orbit equivalent ergodic elements of $\mathrm{Aut}(X, \mathscr{S}, \mu)$ give rise via the above construction to isomorphic flows, and every ergodic, nonsingular aperiodic flow arises in this way [Kr]. This flow has been given different names in the literature; we will use Poincaré flow.

DEFINITION 1.6. The *Poincaré flow* for an ergodic element $f \in \mathrm{Aut}(X, \mathscr{S}, \mu)$ is the isomorphism class of the flow constructed above. Furthermore, if G denotes an ergodic group of transformations of $\mathrm{Aut}(X, \mathscr{S}, \mu)$, then we obtain a G action, S_G, of $X \times \mathbb{R}$ as above. We can repeat the above process to obtain an isomorphism class of flows for the G action which we call the *Poincaré flow for the action*.

In this paper we will be considering the Poincaré flow and factor maps for $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ actions on smooth manifolds.

2. The Construction of a Group of Diffeomorphisms with a Prescribed Poincaré Flow

We begin with any C^∞ flow $\{U_s\}_{s \in \mathbb{R}}$ of a smooth connected paracompact manifold $(N, \mathscr{B}, \nu)$, where ν is a (C^∞) representative of the smooth measure class on N and $\mathscr{B}$ denotes the σ-algebra of Borel sets. We assume further that $\{U_s\}$ is aperiodic and ergodic with respect to ν. If $\{U_s\}$ preserves a measure which is equivalent to ν, then a diffeomorphism of $T^1 \times N$ which has $\{U_s\}$ as its Poincaré flow has been constructed in [Ha]. Therefore we will assume that $\{U_s\}$ does not preserve any smooth measure, or equivalently, that the cocycle $\log(d\nu U_s/d\nu)$ is not a measurable coboundary for the $\mathbb{R}$ action given by $\{U_s\}$ (cf. [HO] or [Sc] for details).

We will denote by $\delta_{U_s}(y)$ the cocycle $\log(d\nu U_s/d\nu)$ at the point $y \in N$, and when the flow $\{U_s\}$ is clearly understood we will simply write $\delta_s(y)$. Similarly, for an integer action generated by a single transformation f, we will write δ_f for the log of the Radon–Nikodym derivative.

We now consider any type III_1 diffeomorphism of $T^1 = \mathbb{R}/\mathbb{Z}$; such diffeomorphisms exist by [Ka] or [He]. By definition μ is an admissible measure for f if there exists an ergodic subgroup $H \subset [f]$ such that for every $V \in H$, $(d\mu V/d\mu)(x) = 1$ for μ-a.e. $x \in T^1$; furthermore μ is f-admissible implies that the set $\{(d\mu f^n/d\mu)(x), n \in \mathbb{Z}\}$ is dense in $\mathbb{R}^+$ for μ-a.e. $x \in T^1$.

Such measures exist by [HO]; we choose an admissible measure μ for f such that $\mu \sim m$, where m denotes Lebesgue measure on T^1. We remark that although μ may not be C^∞, it is equivalent to m so that later on in our construction we can conjugate back to a smooth representative in the measure of class of μ.

We now define a group of diffeomorphisms of $T^2 \times N \times \mathbb{R}$ which is ergodic and has $\{U_s\}$ as its Poincaré flow. The group is generated by two commuting diffeomorphisms of $T^2 \times N \times \mathbb{R}$, so gives an ergodic $\mathbb{Z} \times \mathbb{Z}$ action. Although this is not a singly generated integer action, it is clearly amenable.

We choose two type III_1 diffeomorphisms of T^1, say f and g with admissible measures for each denoted by μ_f and μ_g: then the generators for the desired action are, for each $(x_1, x_2, y, t) \in T^1 \times T^1 \times N \times \mathbb{R}$,

$$F_1(x_1, x_2, y, t) = \left(fx_1, x_2, U_{\log(d\mu_f f/d\mu_f)(x_1)}(y), t - \log \frac{dvU_{\delta_f(x_1)}}{dv}(y) \right)$$

and

$$F_2(x_1, x_2, y, t) = \left(x_1, gx_2, y, t - \log \frac{d\mu_g g}{d\mu_g}(x_2) \right).$$

Then

$$\begin{aligned} F_1 \circ F_2(x_1, x_2, y, t) = \Bigg(& fx_1, gx_2, U_{\delta_f(x_1)}(y), \\ & t - \log \frac{d\mu U_{\delta_f(x_1)}}{d\mu}(y) - \delta_g(x_2) \Bigg) \\ = {} & F_2 \circ F_1. \end{aligned}$$

This gives a $\mathbb{Z} \times \mathbb{Z}$ action: for each $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ we have $G^{(m,n)} = F_1^m \circ F_2^n$. It is clear that for each $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, $G^{(m,n)}$ either is a C^∞ diffeomorphism or is isomorphic to one (by replacing the admissible measures μ_f and μ_g by Lebesgue measure if necessary).

We first prove that the action is ergodic with respect to the smooth measure class on the manifold $T^2 \times N \times \mathbb{R}$.

LEMMA 2.1. *The $\mathbb{Z} \times \mathbb{Z}$ action defined above is ergodic with respect to $\mu_f \times \mu_g \times v \times m$ on $T^2 \times N \times \mathbb{R}$.*

Proof. We assume that $C \subset T^2 \times N \times \mathbb{R}$ is a measurable set which is invariant under $G^{(m,n)}$ for each $m, n \in \mathbb{Z} \times \mathbb{Z}$. By holding $n = 0$ and by considering an ergodic measure-preserving element of the full group of f, say $V \in [f]$, then we see that up to a set of measure 0, $C = T^1 \times C_1$ with

$C_1 \subset T^1 \times N \times \mathbb{R}$ (since C is invariant under $(x_1, x_2, y, t) \to (Vx_1, x_2, y, t)$). Now setting $m=0$, and letting n range over all the integers, since g is of type III_1 we see that F_2 has an ergodic component of the form $A \times T^1 \times \mathbb{R}$ with $A \subset T^1 \times N$, so $C_1 = T^1 \times A_1 \times \mathbb{R}$ with $A_1 \subset N$. Finally, from the fact that f is of type III_1, it follows that (f, U_{δ_f}) is itself ergodic on $T^1 \times N$ (cf. [Ha]), so $A_1 = N$, and modulo a set of measure 0, we have shown that $C = T^1 \times T^1 \times N \times \mathbb{R}$, so the action is ergodic. ∎

We now show that the Poincaré flow associated with this group of transformations is $\{U_s\}_{s \in \mathbb{R}}$ by proving that the map $\pi: T^2 \times N \times \mathbb{R}^2 \to N$ defined by $\pi(x_1, x_2, y, t_2) = U_{-(t_1+t_2)}(y)$ is a factor map. We consider the space $T^2 \times N \times \mathbb{R}^2$ with the smooth product measure $\gamma = \mu_f \times \mu_g \times v \times dt_1 \times dt_2$ and the map

$$S_G^{(m,n)}(x_1, x_2, y, t_1, t_2) = \Big(f^m x_1,\ g^n x_2,\ U_{\delta_{f^m}(x_1)}(y),\ t_1 - \log \frac{dv U_{\delta_{f^m}(x_1)}}{dv}(y) - \delta_{g^n}(x_2),\ t_2 + \log \frac{d\gamma G^{(m,n)}}{d\gamma}(x_1, x_2, y, t_1) \Big).$$

Since the Jacobian matrix for each $G^{(m,n)}$ is easy to calculate in local coordinates, it is not hard to see that

$$\log \frac{d\gamma G^{(m,n)}}{d\gamma}(x_1, x_2, y, t) = \delta_{f^m}(x_1) + \delta_{g^n}(x_2) + \delta_{U_{\delta_{f^m}(x_1)}}(y).$$

LEMMA 2.2. *The map $\pi: T^2 \times N \times \mathbb{R}^2 \to N$ given by $\pi(x_1, x_2, y, t_1, t_2) = U_{-(t_1+t_2)}(y)$ is a factor map for the $\mathbb{Z} \times \mathbb{Z}$ action S_G defined above.*

Proof. We see easily that for any measurable set in N, say $A \in \mathscr{B}$, $v(A) = 0$ if and only if $\gamma(\pi^{-1}A) = 0$. We next show that π is S_G-invariant.
Given any $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, we have

$$\begin{aligned}
\pi(S_G^{(m,n)}(x_1, x_2, y, t_1, t_2)) &= \pi\Big(f^m x_1,\ g^n x_2,\ U_{\delta_{f^m}(x_1)}(y),\\
&\quad t_1 - \log \frac{dv U_{\delta_{f^m}(x_1)}}{dv}(y) - \delta_{g^n}(x_2),\ t_2 + \delta_{f^m}(x_1) + \delta_{g^n}(x_2) + \log \frac{dv U_{\delta_{f^m}(x_1)}}{dv}(y) \Big)\\
&= U_{-(t_1+t_2)}(y)\\
&= \pi(x_1, x_2, y, t_1, t_2).
\end{aligned}$$

We suppose now that ψ is an S_G-invariant function: for every $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ and a.e. $(x_1, x_2, y, t_1, t_2) \in T^1 \times N \times \mathbb{R}^2$ we have $\psi(S_G^{(m,n)}(x_1, x_2, y, t_1, t_2)) = \psi(x_1, x_2, y, t_1, t_2)$. Since μ_f is an f-admissible measure there exists an ergodic element $V \in [f]$ such that $(d\mu_f V/d\mu_f)(x_1) = 1$ for μ_f-a.e. $x_1 \in T^1$, and we have $\psi(Vx_1, x_2, y, t_1, t_2) = \psi(x_1, x_2, y, t_1, t_2) = \psi(S_G^{(m(x_1),0)}(x_1, x_2, y, t_1, t_2))$. Therefore ψ is constant with respect to x_1; a similar argument shows that ψ is constant with respect to x_2. Therefore ψ is invariant under the maps

$$(y, t_1, t_2) \mapsto \Bigg(U_{\delta_{f^m(x_1)}}(y),$$
$$t_1 - \log \frac{dvU_{\delta_{f^m(x_1)}}}{dv} - \delta_{g^n}(x_2),\ t_2 + \delta_{f^m}(x_1) + \delta_{g^n}(x_2) + \delta_{U_{\delta_{f^m(x_1)}}}(y) \Bigg)$$

$$\text{for all}\quad m, n \in \mathbb{Z} \times \mathbb{Z},\ \text{and}\ \gamma \times m\text{-a.e.}\ (x_1, x_2, y, t_1, t_2).$$

In particular, setting $n = 0$ and allowing m to range over all of $\mathbb{Z}$, we have $\psi(y, t_1, t_2) = \psi(U_s(y), t_1 - \log(dvU_s/dv)(y), t_2 + s + \log(dvU_s/dv)(y))$; this follows from the fact that μ_f is admissible and the assumption that $\{U_s\}_{s \in \mathbb{R}}$ is a continuous flow. Also, setting $m = 0$, and letting n range over all the integers, we have $\psi(y, t_1, t_2) = (y, t_1 - r, t_2 + r)$ for all $r \in \mathbb{R}$ (by arguments similar to those used above). In particular, setting $r = -t_2$, it follows that $\psi(y, t_1, t_2) = \psi(y, t_1 + t_2, 0)$ for a.e. $(y, t_1, t_2) \in N \times \mathbb{R} \times \mathbb{R}$. Therefore ψ depends on y and $t_1 + t_2$. Now this implies that $\psi(y, t_1, t_2) = \psi(U_s(y), t_1 + t_2 + s)$ for all $s \in \mathbb{R}$; setting $s = -(t_1 + t_2)$, we obtain $\psi(y, t_1, t_2) = \psi(U_{-(t_1+t_2)}(y), 0)$, so ψ is a function of $U_{-(t_1+t_2)}$. That is, we have just shown that there exists a function $\bar{\psi}: N \to \mathbb{R}$ such that $\bar{\psi}(x_1, x_2, y, t_1, t_2) = \bar{\psi}(U_{-(t_1+t_2)}(y)) = \bar{\psi}(\pi(x_1, x_2, y, t_1, t_2))$ for $\gamma \times m$-a.e. $(x_1, x_2, y, t_1, t_2) \in T^2 \times N \times \mathbb{R}^2$. This proves that π is a factor map. We have just proved the following proposition.

PROPOSITION 2.3. *The $\mathbb{Z} \times \mathbb{Z}$ action on $T^2 \times N \times \mathbb{R}$ constructed above has $\{U_s\}_{s \in \mathbb{R}}$ as its Poincaré flow.*

COROLLARY 2.4. *The group of diffeomorphisms on $T^2 \times N \times \mathbb{R}$ given by*

$$G^{(n,k)}(x_1, x_2, y, t) = \Bigg(f^n x_1, g^k x_2, U_{\log(dmf^n/dm)(x_1)}(y),$$
$$t - \log \frac{dvU_{\delta_{f^n(x_1)}}}{dv}(y) - \log \frac{dmg^k}{dm}(x_2) \Bigg),$$

where m denotes Lebesgue measure on $T^1 = \mathbb{R}/\mathbb{Z}$ and f and g are type III$_1$ *diffeomorphisms of T^1, has $\{U_s\}_{s \in \mathbb{R}}$ as it Poincaré flow.*

Proof. Since this $\mathbb{Z}\times\mathbb{Z}$ action is obviously orbit equivalent to the action of 2.3, then the Poincaré flow is the same (isomorphic). ∎

The next corollary is related to an example of Connes described in [Se]: in that paper, it was shown that there is a foliation which gives an amenable ergodic equivalence relation in every type III orbit equivalence class. The foliation is generated by the action of a nonamenable group. Here we show the same result but we obtain continuous $\mathbb{Z}\times\mathbb{Z}$ actions of metric spaces.

COROLLARY 2.5. *Every countable ergodic amenable equivalence relation of type* III *is orbit equivalent to a continuous $\mathbb{Z}\times\mathbb{Z}$ action on a metric space. If the Poincaré flow is $\{U_s\}_{s\in\mathbb{R}}$ on the metric space (Y, ν), then an orbit equivalent $\mathbb{Z}\times\mathbb{Z}$ action exists on the metric space $T^2\times Y\times\mathbb{R}$.*

Proof. By [Ma] (cf. [Se]) we have that every ergodic aperiodic flow is isomorphic to a continuous flow on a metric space. Therefore for any countable ergodic amenable equivalence relation, we can assume that its Poincaré flow is continuous on the metric space (Y, ν). We now apply Proposition 2.3. ∎

Remarks. (1) It is known that in the case of an ergodic ITPFI relation, there exists a C^∞ diffeomorphism of the circle which is in the same orbit equivalence class [Ka]. This shows that it is not necessary that the dimension of the manifold increase when passing from the flow to the original diffeomorphism (or countable equivalence relation). In particular, in our construction we use a smooth factor map π; our map is as smooth as the flow itself. Even though this is a useful technique for producing smooth examples, it is of course not necessary.

(2) It is desirable to produce smooth representatives of orbit equivalence classes on compact manifolds. In [Ha] it was shown that if $\{U_s\}$ is a C^∞ measure-preserving ergodic flow on a compact manifold, then a C^∞ integer action having $\{U_s\}$ as its Poincaré flow can be constructed on a compact manifold.

In this paper we start with a type III smooth ergodic flow of a manifold N and produce a $\mathbb{Z}\times\mathbb{Z}$ action on $T^2\times N\times\mathbb{R}$ having the original flow on N as its Poincaré flow. Using now standard techniques of Anosov and Herman (cf. [He]) to extend this to an action on a compact manifold leads to the following interesting dilemma. We have two commuting generators of the action, say F_1 and F_2, on $T^2\times N\times\mathbb{R}$; the trick is to take their suspension flows on $T^4\times N\times\mathbb{R}$ and then extend them to flows on $T^4\times N\times T^1$. We do this by treating $\mathbb{R}$ as an open set of full measure in T^1

and multiplying the generating vector fields by appropriate C^∞ functions. Now for almost every $(t_1, t_2) \in \mathbb{R}^2$ we evaluate the new flows at t_1 and t_2, and we end up with two diffeomorphisms: G_1, which is orbit equivalent to F_1, and G_2, which is orbit equivalent to F_2. However, G_1 and G_2 will not in general be commuting diffeomorphisms of $T^5 \times N$. (Since if X and Y are vector fields of a manifold M, and f and g are differentiable functions on M, then $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$.) This has particularly bad consequences when X and Y commute and are linearly independent for most $x \in M$ as in our case. In fact, a priori, the countable equivalence relation generated by G_1 and G_2 may not be amenable. This shows that the problem of constructing orbit equivalence classes on compact manifolds using $\mathbb{Z} \times \mathbb{Z}$ actions is still open. However, in the next section, we pass from $\mathbb{Z} \times \mathbb{Z}$ to integer actions and these problems disappear.

3. Building Diffeomorphisms from Other Smooth Group Actions

In this section we show how to obtain from a smooth action of a connected amenable Lie group on a manifold a diffeomorphism which is orbit equivalent to that action. Our method gives a diffeomorphism of a higher-dimensional manifold and uses techniques of C. Series and A. Ramsay on cocycles and virtual groups. In what follows we will concentrate on our particular construction and mention more general results at the end.

In Section 2 we constructed a $\mathbb{Z} \times \mathbb{Z}$ action on a manifold $M = T^2 \times N \times \mathbb{R}$, with Poincaré flow $\{U_s\}$ on the manifold N. We obtain an orbit equivalent $\mathbb{R}^2$ action by applying the following lemma.

Lemma 3.1. *If $T_1, ..., T_n$ generate an ergodic C^∞ aperiodic $\mathbb{Z}^n$ action on a connected, locally compact manifold M, then the suspension $\mathbb{R}^n$ action on $T^n \times M$ is orbit equivalent to the original action.*

Proof. By definition, we first consider the $\mathbb{R}^n$ action on $M \times \mathbb{R}^n$ given by $F_{(t_1, t_2, ..., t_n)}(m, x_1, ..., x_n) = (m, x_1 + t_1, x_2 + t_2, ...)$; we then identify $(m, x_1, x_2, ..., x_n)$ with $(T_1^{k_1} T_2^{k_2} \cdots T_n^{k_n} m, x_1 + k_1, ..., x_n + k_n)$ for all $(k_1, ..., k_n) \in \mathbb{Z}^n$, $m \in M$, and $(x_1, ..., x_n) \in \mathbb{R}^n$. The induced action on the quotient space is clearly a C^∞ $\mathbb{R}^n$ action on $M \times T^n$ which is also ergodic. To show it has the same Poincaré flow, we look at the Radon–Nikodym derivative of this action and see easily that the flow space is the same, since the orbits of the skewed $\mathbb{R}^n$ action are in one-to-one correspondence with the orbits of the skewed $\mathbb{Z}^n$ action. Therefore the Poincaré flow of the time $(1, 1, ..., 1)$ mapping of the $\mathbb{R}^n$ action is the same as the flow of the $\mathbb{R}^n$ action itself (cf. [Se] for more details). ∎

In our construction we only use $n = 2$. We next consider an ergodic finite measure-preserving action on a manifold K which admits a cocycle which is $\mathbb{R}^2$-valued and gives an ergodic skew product extension. Equivalently, we want a smooth measure-preserving diffeomorphism and an associated C^∞ cocycle with essential range $\mathbb{R}^2$. By [He], [N] or using techniques from [HS], we can find a well-approximable irrational number $\alpha \in (0, 1)$ (a Liouville number) so that the transformation $R_\alpha(x) = x + \alpha \pmod 1$ gives a diffeomorphism of the circle, T^1, admitting a smooth $\mathbb{R}^2$-valued cocycle $\phi: \mathbb{Z} \times T^1 \to \mathbb{R}^2$ satisfying $E(\phi) = \mathbb{R}^2$. We now give the construction of the desired diffeomorphism.

A. *The New Diffeomorphism*

We begin with our original $\mathbb{Z} \times \mathbb{Z}$ action constructed in Section 2. We will let $M = T^2 \times N \times \mathbb{R}$, and we will denote by F the action described in Corollary 2.4; i.e., $F: \mathbb{Z} \times \mathbb{Z} \times M \to M$ with $F(n, k, m) = G^{(n,k)}(m)$ as defined in Corollary 2.4. Now by $F_{(s,t)}$ we denote the associated suspension $\mathbb{R}^2$ action from Lemma 3.1, an action on $M \times T^2$. Using the notation from the paragraph above, we choose $\alpha \in (0, 1)$, and a C^∞ cocycle $\phi: \mathbb{Z} \times T^1 \to \mathbb{R}^2$ for the action of R_α so that $E(\phi) = \mathbb{R}^2$.

We obtain an integer action on $T^1 \times M \times T^2$ defined by

$$G^n(x, m, y) = (R_{n\alpha} x, F_{\phi(n,x)}(m, y)) \qquad \text{for all} \quad x \in T^1, (m, y) \in M \times T^2.$$

We claim that the diffeomorphism G generates an action which is orbit equivalent to the original $\mathbb{Z} \times \mathbb{Z}$ action F.

B. *Proof That G Is Orbit Equivalent to F*

From our discussion in Section 1, it is sufficient to prove the existence of a factor map ζ from $T^1 \times M \times T^2 \times \mathbb{R}$ to N, the flow space (manifold) for the original $\mathbb{Z} \times \mathbb{Z}$ action, with respect to the map S_G (the $\mathbb{Z}$ action defined in 1.3 and 1.4; i.e., the skew product of G and the log of the Radon–Nikodym derivative of G).

By Series [Se], there is a "generalized factor map" from $T^1 \times M \times T^2 \times \mathbb{R}$ to $M \times T^2 \times \mathbb{R}$, say π_1, with the following properties:

(1) If μ_1 denotes the smooth measure on $T^1 \times M \times T^2 \times \mathbb{R}$ and μ_2 denotes smooth measure on $M \times T^2 \times \mathbb{R}$, then for μ_2 measurable sets A, $\mu_1 \pi_1^{-1}(A) = 0$ if and only if $\mu_2(A) = 0$.

(2) π_1 is constant a.e. on orbits of G, i.e., if ψ is a G-invariant real-valued function on $T^1 \times M \times T^2 \times \mathbb{R}$, then there exists a function $\tilde{\psi}: M \times T^2 \times \mathbb{R} \to \mathbb{R}$ such that $\psi(w) = \tilde{\psi}(\pi_1 w)$ for μ_1-a.e. $w \in T^1 \times M \times T^2 \times \mathbb{R}$.

(3) The $\mathbb{Z}$ action given by S_G on $T^1 \times M \times T^2 \times \mathbb{R}$ commutes with the $\mathbb{R}^2$ action given by

$$S_{F_{(s,t)}}(x, m, y, r) = \left(x, F_{(s,t)}(m, y), r + \log \frac{d\mu_2 F_{(s,t)}}{d\mu_2}(m, y)\right);$$

one easily computes that $S_G S_{F(s,t)}(x, m, y, r) = S_{F(s,t)} S_G(x, m, y, r)$, and therefore there is a factor action on $T^1 \times M \times T^2 \times R/\sim S_G$, which by [Se] is $S_{F(s,t)}$. A proof similar to our proof of Proposition 2.3 can be given.

Similarly, there is a factor map from $M \times T^2 \times \mathbb{R}$ to N, call it π_2, which gives the Poincaré flow $\{U_s\}$ on N. We claim that the desired factor map is $\zeta = \pi_2 \circ \pi_1 : T^1 \times M \times T^2 \times \mathbb{R} \to N$. This result follows from a much more general result of Ramsay [Ra, Lemma 6.9]. We sketch the proof below.

Properties (1) and (2) of Definition 1.4 are clearly satisfied.

Now $\zeta(S_G(x, m, y, r)) = \pi_2(\pi_1 \circ S_G)(x, m, y, r) = \pi_2 \circ \pi_1(x, m, y, r)$ a.e. since π_1 is S_G-invariant, so (3) is satisfied; i.e., ζ is constant on S_G orbits. To prove that (4) holds, we suppose that η is an S_G-invariant function. Then we can write it as a function of (m, y, r) alone by properties of π_1, and the function will be invariant under $(F_{\phi(n,x)}(m, y), r + \log(d\mu_2 F_{\phi(n,x)}/d\mu_2)(m, y))$ for all $n \in \mathbb{Z}$, a.e. $x \in T^1$. Since $E(\phi) = \mathbb{R}^2$, η is invariant under

$$\left(F_{(s,t)}(m, y), r + \log \frac{d\mu_2 F_{(s,t)}}{d\mu_2}(m, y)\right) \qquad \text{for all} \quad (s, t) \in \mathbb{R}^2,$$

and since π_2 is a factor map η can be written as a function of N alone with its value dependent on the $\{U_s\}$ orbit of a point in N as in Section 2. This proves that $\pi_2 \circ \pi_1$ is a factor map for G and the Poincaré flow is $\{U_s\}$.

We have just proved the following theorem.

THEOREM 3.2. *Given any finite collection of k commuting diffeomorphisms of a manifold M, there exists a single diffeomorphism of $T^k \times M \times \mathbb{R}$ which is orbit equivalent to the original $\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ action.*

The same proof gives the corollary.

COROLLARY 3.3. *Let H be a connected amenable Lie group which acts ergodically with respect to a smooth measure on a manifold. Then there exists an ergodic diffeomorphism of a (higher-dimensional) manifold which is orbit equivalent to the H action.*

Proof. We just apply [He, IV.5] plus the proof of Theorem 3.2. ∎

COROLLARY 3.4. *Given an aperiodic C^∞ ergodic flow $\{U_s\}$ on a manifold N, there exists a diffeomorphism of a (higher-dimensional) manifold which has $\{U_s\}$ as its Poincaré flow.*

Remarks. (1) If the original flow $\{U_s\}$ is an ergodic aperiodic $\mathbb{R}$ action of a compact manifold N, then using techniques discussed in Section 2 (cf. [Ha]) one can construct the diffeomorphism G on a compact manifold as well.

(2) These results show that any measure-preserving flow and any C^∞ flow which is aperiodic and ergodic occurs as the Poincaré flow of a diffeomorphism of some manifold. It remains to treat the case of non-smoothable, non-measure-preserving, and non-AT flows, if such flows exist.

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