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## SMOOTH TYPE III Diffeomorphisms of Manifolds

BY

JANE HAWKINS

**ABSTRACT.** In this paper we prove that every smooth paracompact connected manifold of dimension  $\geq 3$  admits a smooth type  $\text{III}_\lambda$  diffeomorphism for every  $0 \leq \lambda \leq 1$ . (Herman proved the result for  $\lambda = 1$  in [7].) The result follows from a theorem which gives sufficient conditions for the existence of smooth ergodic real line extensions of diffeomorphisms of manifolds.

**1. Introduction.** Let  $f$  be a nonsingular ergodic automorphism of a Lebesgue space  $(X, \mathcal{S}, \mu)$ . The problem of describing the conditions under which  $\mu$  is equivalent to an  $f$ -invariant measure has been the subject of much study [2, 3, 4, 6, 11, 13]. Herman and Katznelson have constructed examples of diffeomorphisms of the circle which do not admit any invariant measure equivalent to Lebesgue measure [8, 11]. In [6], some relationships between the rotation number of a diffeomorphism of  $T^1$  and its measure theoretic properties are studied further. In [8] Herman proves the existence of smooth type  $\text{III}_1$  diffeomorphisms of every paracompact connected manifold of dimension  $m \geq 3$ .

In this paper we study the more pathological types of non-measure-preserving diffeomorphisms of manifolds. In particular we construct examples of smooth type  $\text{III}_0$  and  $\text{III}_\lambda$ ,  $0 < \lambda < 1$ , diffeomorphisms of  $C^\infty$  manifolds. The main theorem of the paper is the following.

**THEOREM 5.5.** *Every smooth paracompact connected manifold of dimension  $\geq 3$  admits a type  $\text{III}_\lambda$  diffeomorphism for every  $0 \leq \lambda \leq 1$ .*

We prove the theorem in a sequence of steps, following a general method introduced by Anosov in [5] and used by Herman to construct type  $\text{III}_1$  diffeomorphisms. In order to obtain a type  $\text{III}_0$  diffeomorphism on an  $m$ -dimensional manifold  $M$ , we construct a flow with the desired property on  $T^2 \times \mathbf{R}^{m-2}$  and extend it to a globally defined flow with the same property on  $M$ .

§2 supplies some necessary definitions and notation. In §3, we prove that starting with a type  $\text{III}_0$  diffeomorphism of the circle (which we know exists by [11]) we can obtain a type  $\text{III}_0$  flow and diffeomorphisms on higher dimensional tori. We prove the following slightly more general assertion: if  $X$  is any smooth manifold which admits a type  $\text{III}_0$  diffeomorphism  $f$ , then the diffeomorphism of  $X \times T^1$  given by  $(x, y) \mapsto (fx, y + t(\text{mod } 1))$ ,  $x \in X, y \in T^1$ , is of type  $\text{III}_0$  for  $m$ -a.e.  $t \in (0, 1)$ .

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§4 offers a proof that we can obtain real line extensions of type  $\text{III}_0$  diffeomorphisms which preserve the ratio set (cf. §2). This result is not always true in the measure-preserving case. In fact it is well known that for most irrational numbers  $\alpha$ , the diffeomorphism of the circle given by rotation through  $\alpha$  admits no ergodic real line extensions of the form:  $(y, z) \mapsto (y + \alpha, z + \psi y)$ , where  $\psi: T^1 \rightarrow \mathbf{R}$  is smooth. (Using Fourier coefficients the equation  $\psi(y) = \varphi(y) - \varphi(y + \alpha)$  has a  $C^\infty$  solution for  $m$ -a.e.  $\alpha$ , hence the extension by  $\psi$  is not even ergodic.) However, Theorem 4.13 states that every type  $\text{III}_0$  diffeomorphism of a paracompact manifold (and type  $\text{III}_\lambda$ ,  $0 < \lambda < 1$ , as well by a small modification) admits weakly equivalent real line extensions.

The last section, §5, then applies the methods of Anosov and Herman to extend the type  $\text{III}_\lambda$  flows we have constructed on  $T^2 \times \mathbf{R}^{m-2}$  to arbitrary paracompact connected manifolds of dimension  $m$ .

The contents of this paper form a part of the author’s Ph.D. thesis. Dr. Klaus Schmidt, of Warwick University, is gratefully acknowledged for supervising this work.

**2. Ergodic theory preliminaries.** Let  $(X, \mathfrak{S}, \mu)$  denote a Borel space where  $\mu$  is a probability measure on  $(X, \mathfrak{S})$ . Let  $f$  denote a nonsingular ergodic transformation of  $(X, \mathfrak{S}, \mu)$ , i.e. every  $f$ -invariant set  $B \in \mathfrak{S}$  satisfies either  $\mu(B) = 0$  or  $\mu(B) = 1$ . We define the set  $\text{Aut}(X, \mathfrak{S}, \mu) = \{T: (X, \mathfrak{S}) \leftarrow \text{such that } T \text{ is a nonsingular Borel automorphism of } (X, \mathfrak{S})\}$ , and let

$$O_f(x) = \{f^n x: n \in \mathbf{Z}\}.$$

The *full group* of  $f$  is defined by

$$[f] = \{V \in \text{Aut}(X, \mathfrak{S}, \mu): Vx \in O_f(x) \text{ for } \mu\text{-a.e. } x \in X\}.$$

**DEFINITION 2.1.** Two transformations  $f, g \in \text{Aut}(X, \mathfrak{S}, \mu)$  are *weakly equivalent* if there exists a measurable invertible map  $\psi: X \rightarrow X$  with  $\psi_*^{-1}\mu \sim \mu$  and  $\psi(O_f(x)) = O_g(\psi x)$  for  $\mu$ -a.e.  $x \in X$ .

We now introduce an invariant of weak equivalence.

**DEFINITION 2.2.** Let  $f \in \text{Aut}(X, \mathfrak{S}, \mu)$  be an invertible, ergodic transformation. A nonnegative real number  $t$  is said to lie in the *ratio set* of  $f$ ,  $r^*(f)$ , if for every Borel set  $B \in \mathfrak{S}$  with  $\mu(B) > 0$ , and for every  $\varepsilon > 0$ ,

$$\mu \left( \bigcup_{n \in \mathbf{Z}} \left( B \cap f^n B \cap \left\{ x \in X: \left| \frac{d\mu f^{-n}}{d\mu}(x) - t \right| < \varepsilon \right\} \right) \right) > 0.$$

Here  $d\mu f^{-n}/d\mu$  denotes the Radon-Nikodym derivative of  $f_*^{-n}\mu$  with respect to  $\mu$ . We set  $r(f) = r^*(f) \setminus \{0\}$ . One can show that  $r(f)$  is a closed subgroup of the multiplicative group of positive real numbers  $\mathbf{R}^+$ , and that  $f$  admits a  $\sigma$ -finite invariant measure if and only if  $r^*(f) = \{1\}$  [13]. If  $f$  has no  $\sigma$ -finite invariant measure equivalent to  $\mu$ , there are three possibilities:

- (1)  $r^*(f) = \{t \in \mathbf{R}: t \geq 0\}$ , in which case  $f$  is said to be of type  $\text{III}_1$ ;
- (2)  $r^*(f) = \{0\} \cup \{\lambda^n: n \in \mathbf{Z}\}$  for  $0 < \lambda < 1$ ; in this case  $f$  is said to be of type  $\text{III}_\lambda$ ; or,
- (3)  $r^*(f) = \{0, 1\}$ ; then  $f$  is of type  $\text{III}_0$ .

The ratio set is actually an example of the set of essential values for a particular cocycle for  $f$ . We shall briefly introduce these more general concepts from the study of nonsingular group actions on measure spaces. For the purposes of this paper, we will give the definitions in the differentiable context; for the most general definitions, we refer the reader to [16].

Let  $(X, \mathfrak{S}, \mu)$  denote a compact Riemannian manifold with  $\mathfrak{S}$  the  $\sigma$ -algebra of Borel sets, and let  $\mu$  denote a smooth probability measure. By  $\text{Diff}^\infty(X)$  we will denote the set of  $C^\infty$  diffeomorphisms of  $X$ . We define the metric

$$d_\infty(f, g) = \sum_{r=1}^\infty 2^{-r} \frac{d_r(f, g)}{1 + d_r(f, g)} \quad \text{for every } f, g \in \text{Diff}^\infty(X),$$

where  $d_r$  denotes the usual  $C^r$  metric. Clearly  $\text{Diff}^\infty(X)$  is a Baire space.

Let  $f \in \text{Diff}^\infty(X)$  be  $\mu$ -ergodic and let  $H$  be a locally compact second countable abelian group. The action  $(n, x) \mapsto f^n(x)$  of  $\mathbf{Z}$  on  $X$  is nonsingular since for every  $n \in \mathbf{Z}$ ,  $x \mapsto f^n x$  is a Borel automorphism of  $X$  which leaves  $\mu$  quasi-invariant.

DEFINITION 2.3. A Borel map  $a: \mathbf{Z} \times X \rightarrow H$  is called a *cocycle* for  $\mathbf{Z}$  if the following condition holds:

For every  $n, m \in \mathbf{Z}$  and for every  $x \in X$ , we have

$$a(n, f^m x) - a(n + m, x) + a(m, x) = 0.$$

A cocycle  $a: \mathbf{Z} \times X \rightarrow H$  is called a *coboundary* if there exists a Borel map  $b: X \rightarrow H$  with  $a(n, x) = b(f^n x) - b(x)$ ,  $n \in \mathbf{Z}$ , for  $\mu$ -a.e.  $x \in X$ . Two cocycles  $a_1$  and  $a_2$  are said to be *cohomologous* if their difference is a coboundary.

The following defines a cohomology invariant which generalises the concept of the ratio set.

DEFINITION 2.4. Let  $(X, \mathfrak{S}, \mu)$  be as above,  $G$  a countable group which acts nonsingularly and ergodically on  $(X, \mathfrak{S}, \mu)$  and let  $a: G \times X \rightarrow H$  be a cocycle for  $G$ . An element  $\alpha \in \overline{H} = H \cup \{\infty\}$  is called an *essential value* of  $a$  if, for every Borel set  $B \in \mathfrak{S}$  with  $\mu(B) > 0$  and for every neighbourhood  $N(\alpha)$  of  $\alpha$  in  $\overline{H}$ ,

$$\mu(B \cap g^{-1}B \cap \{x: a(g, x) \in N(\alpha)\}) > 0$$

for some  $g \in G$ . The set of essential values is denoted by  $\overline{E}(a)$ , and we put  $E(a) = \overline{E}(a) \cap H$ . We will state a few well-known properties of  $\overline{E}(a)$ .

- (1)  $\overline{E}(a)$  is a nonempty closed subset of  $\overline{H}$ ;
- (2)  $E(a)$  is a closed subgroup of  $H$ ;
- (3)  $\overline{E}(a) = \{0\}$  if and only if  $a$  is a coboundary;
- (4)  $\overline{E}(a_1) = \overline{E}(a_2)$  whenever  $a_1$  and  $a_2$  are cohomologous.

DEFINITION 2.5. Let  $(X, \mathfrak{S}, \mu)$  be as above,  $G$  a countable group acting nonsingularly and ergodically on  $X$ , and  $a: G \times X \rightarrow H$  a cocycle for  $G$ . The cocycle  $a$  is *recurrent* if, for every  $B \in \mathfrak{S}$  with  $\mu(B) > 0$ , and for every neighbourhood  $N(0)$  in  $H$ ,

$$\mu\left(\bigcup_{g \in G} (B \cap g^{-1}B \cap \{x: a(g, x) \in N(0)\}) \cap \{x: gx \neq x\}\right) > 0.$$

We will now recall some definitions relating to flows, or  $\mathbf{R}$ -actions, on manifolds.

DEFINITION 2.6. A  $C^r$  flow on a manifold  $X$  is a  $C^r$  map  $f: X \times \mathbf{R} \rightarrow X$  such that if we denote  $f_t(x) = f(x, t) \forall x \in X, t \in \mathbf{R}$ , then

- (i)  $f_{t+s}(x) = f_t f_s(x) \forall t, s \in \mathbf{R}, \forall x \in X$ ;
- (ii)  $f_0 = \text{Id}_X$ .

Then it follows that

- (iii)  $f_{-t} = f_t^{-1}$ .

A flow on  $(X, \mathfrak{S}, \mu)$  is  $\mu$ -ergodic if  $f_t(A) = A (A \in \mathfrak{S}) \forall t \in \mathbf{R}$  implies  $\mu(A) = 0$  or  $\mu(A^c) = 0$ .

DEFINITION 2.7. A nonsingular flow  $f_t$  on  $(X, \mathfrak{S}, \mu)$  is of type III if it admits no  $\sigma$ -finite invariant measure equivalent to  $\mu$ .

If  $f$  is any orientation-preserving diffeomorphism of a smooth manifold of dimension  $n \geq 1$ , there exists a canonical method for obtaining a flow on an  $(n + 1)$ -dimensional manifold, from  $f$ , called the suspension flow of  $f$ . We first define a flow on  $X \times \mathbf{R}$  by  $G_t(x, y) = (x, y + t) \forall x \in X, y \in \mathbf{R}, t \in \mathbf{R}$ . We next consider the equivalence relation on  $X$  given by  $(x, y) \sim (f^n x, y + n) \forall n \in \mathbf{Z}$  and we see that  $G_t$  induces a flow,  $F_t$ , on  $X \times \mathbf{R} / \sim \cong X \times T^1$ . We call  $F_t$  the suspension flow of  $f$ .

The following lemma appears in [7 and 14].

LEMMA 2.8. Let  $(Y, \mathfrak{B}, \nu)$  be a Lebesgue space,  $\nu$  a positive  $\sigma$ -finite measure and  $G_t$  a flow on  $Y$  preserving  $\nu$  and  $\nu$ -ergodic. If  $G_t$  has no orbit of full  $\nu$ -measure, then there exists a set  $B \subset [0, 1], m(B) = 1$ , such that for all  $t_0 \in B, G_{t_0} \in \text{Aut}(Y, \nu)$  is  $\nu$ -ergodic.

Finally, we define type  $\text{III}_\lambda$  flows for  $0 \leq \lambda \leq 1$ , and give a definition for weak equivalence of flows.

DEFINITION 2.9. A smooth flow  $G_t$  on a  $C^\infty$  manifold  $(X, \mathfrak{S}, \mu)$  is of type  $\text{III}_\lambda (0 \leq \lambda \leq 1)$  if for  $m$ -a.e.  $t_0 \in \mathbf{R}, G_{t_0} \in \text{Aut}(X, \mu)$  is of type  $\text{III}_\lambda$ .

Two flows  $g_t$  and  $G_t$  are weakly equivalent if there exists a measurable invertible map  $\psi: X \rightarrow X$  with  $\psi_*^{-1}\mu \sim \mu$  and  $\psi(g_t(x)) = G_t(\psi x)$  for  $\mu$ -a.e.  $x \in X$ , and some  $s, t \in \mathbf{R}$ .

**3. Type  $\text{III}_0$  flows and diffeomorphisms of  $T^n$ .** This section contains a crucial step in the process of obtaining a type  $\text{III}_0$  diffeomorphism on an arbitrary manifold; we prove the existence of a flow with the desired property. In order to obtain a type  $\text{III}_0$  flow on  $T^2$ , we start with a type  $\text{III}_0$  diffeomorphism,  $f$ , of  $T^1$  and then take the suspension flow. We need to check that for  $m$ -a.e.  $t \in (0, 1)$ , the diffeomorphism given by  $(x, w) \mapsto (fx, w + t)$  is also of type  $\text{III}_0$ . This involves proving that, usually, the nontrivial ergodic decomposition of the skew product given by

$$(x, z) \mapsto \left( fx, z + \log \frac{d\mu f^{-1}}{d\mu}(x) \right) \text{ on } X \times \mathbf{R}$$

is preserved when we pass to one higher dimension.

In Theorem 3.2 we prove that a type  $\text{III}_0$  diffeomorphism of an arbitrary smooth manifold  $X$  can be extended to a type  $\text{III}_0$  flow on  $X \times T^1$ . As a corollary we obtain type  $\text{III}_0$  diffeomorphisms of  $T^n$ , for  $n \geq 2$ .

**PROPOSITION 3.1.** *Let  $(X, \mathcal{S}, \mu)$  be a smooth manifold with  $\mu$  a  $C^\infty$   $\sigma$ -finite measure on  $X$ . Suppose  $f \in \text{Aut}(X, \mu)$  is a  $\mu$ -ergodic diffeomorphism of type  $\text{III}_\lambda$  ( $0 \leq \lambda \leq 1$ ), and there exists a  $\beta \in [0, 1]$  satisfying:*

(1)  $F_\beta: X \times T^1 \rightarrow X \times T^1$  defined by  $(x, w) \mapsto (fx, w + \beta)$  is  $\mu \otimes m$ -ergodic;

(2)  $(Y, \mathcal{Y}, \rho)$  and  $\{q_y: y \in Y\}$  are the Borel space and family of ergodic measures, respectively, corresponding to the ergodic decomposition of  $S_f: X \times \mathbf{R} \rightarrow X \times \mathbf{R}$  given by

$$(x, z) \mapsto \left( fx, z + \log \frac{d\mu f^{-1}}{d\mu}(x) \right)$$

(see [16]), and the ergodic decomposition of  $S_{F_\beta} = S_\beta: X \times T^1 \times \mathbf{R} \rightarrow X \times T^1 \times \mathbf{R}$  defined by

$$(x, w, z) \mapsto \left( fx, w + \beta, z + \log \frac{d\mu \otimes m F_\beta^{-1}}{d\mu \otimes m}(x, w) \right)$$

has Borel space  $(\tilde{Y}, \tilde{\mathcal{Y}}, \tilde{\rho})$  and ergodic measures  $\{q_{\tilde{y}}: \tilde{y} \in \tilde{Y}\}$  such that  $q_{\tilde{y}} = q_y \otimes m$  for  $\tilde{\rho}$ -a.e.  $\tilde{y} \in \tilde{Y}$  and  $\rho$ -a.e.  $y \in Y$ .

Then  $F_\beta$  is also of type  $\text{III}_\lambda$ .

**PROOF.** We prove the theorem by showing that  $r^*(F_\beta) = r^*(f)$  which is equivalent to proving that  $\bar{E}(a_f) = \bar{E}(a_\beta)$  where  $a_f$  and  $a_\beta$  are the following cocycles:

$$a_f: \mathbf{Z} \otimes X \rightarrow \mathbf{R}, \quad a_f(n, x) = \log \frac{d\mu f^{-n}}{d\mu}(x) \quad \text{and}$$

$$a_\beta: \mathbf{Z} \times X \times T^1 \rightarrow \mathbf{R}, \quad a_\beta(n, x, w) = \log \frac{d\mu \otimes m F_\beta^{-n}}{d\mu \otimes m}(x, w) = \log \frac{d\mu f^{-n}}{d\mu}(x)$$

for every  $n \in \mathbf{Z}, x \in X, w \in T^1$ .

From [16, §5] we recall that  $I(a_f) = \{\lambda \in \mathbf{R}: R_\lambda B = B(\mu \text{ mod } 0) \text{ for every } B \in \mathcal{S} \times \mathcal{C} \subset X \times \mathbf{R} \text{ which is } S_f\text{-invariant}\}$ , and  $I(a_\beta)$  is defined analogously, where  $R_\lambda(x, z) = (x, z + \lambda)$  for all  $x \in X, z \in \mathbf{R}, \lambda \in \mathbf{R}$ , and  $R_\lambda^\beta(x, w, z) = (x, w, z + \lambda)$  for every  $x \in X, w \in T^1, z \in \mathbf{R}, \lambda \in \mathbf{R}$ . Now by Theorem 5.2 in [16] we have that  $E(a) = I(a)$  for all cocycles  $a: \mathbf{Z} \times X \rightarrow \mathbf{R}$  (and for all  $a: \mathbf{Z} \times X \times T^1 \rightarrow \mathbf{R}$ ) so the proposition is true for  $\lambda > 0$  if we show that  $I(a_f) = I(a_\beta)$ , and if  $\lambda = 0, I(a_\beta) \subset I(a_f)$  plus the uniqueness of ergodic decomposition suffice to prove the result. (This will be discussed further later.)

Let  $\pi: X \times T^1 \times \mathbf{R} \rightarrow X \times \mathbf{R}$  denote the projection mapping defined by

$$(x, w, z) \mapsto (x, z) \quad \text{for every } x \in X, w \in T^1, z \in \mathbf{R}.$$

Clearly,  $S_f \circ \pi = \pi \circ S_\beta$  (as maps from  $X \times T^1 \times \mathbf{R}$  to  $X \times \mathbf{R}$ ), and also for every  $\lambda \in \mathbf{R}, R_\lambda \circ \pi = \pi \circ R_\lambda^\beta$ .

We now prove that  $\pi$  defines an isomorphism (mod sets of  $\mu \otimes m$ -measure zero) from  $S_f$ -invariant sets of  $X \times \mathbf{R}$  to  $S_\beta$ -invariant sets of  $X \times T^1 \times \mathbf{R}$ . Suppose that  $B \in \mathcal{S} \times \mathcal{C} \subset X \times \mathbf{R}$  is  $S_f$ -invariant. Then  $S_f B = B$  implies that  $B = S_f \circ \pi(\pi^{-1}B) = \pi \circ S_\beta(\pi^{-1}B)$ , hence  $\pi^{-1}B = S_\beta(\pi^{-1}B)$  so  $\pi^{-1}B$  is  $S_\beta$ -invariant. Now let  $D \in \mathcal{S} \times \mathcal{Y} \times \mathcal{C} \subset X \times T^1 \times \mathbf{R}$  satisfy  $S_\beta D = D$ . Then  $S_f \pi D = \pi \circ S_\beta D = \pi D$ , so  $\pi D$  is  $S_f$ -invariant.

Now  $\lambda \in I(a_f)$  implies for every  $S_\beta$ -invariant  $D$ ,  $R_\lambda(\pi D) = \pi D$ , which implies that  $\pi^{-1}R_\lambda\pi D = \pi^{-1}\pi D$ . Equivalently,  $\pi^{-1}\pi R_\lambda^\beta D = D$ , so  $\lambda \in I(a_\beta)$ .

If  $\lambda \in I(a_\beta)$ , then for every  $S_f$ -invariant  $B$ ,  $R_\lambda^\beta\pi^{-1}B = \pi^{-1}B$ . Then  $R_\lambda B = R_\lambda\pi(\pi^{-1}B) = \pi R_\lambda^\beta\pi^{-1}B = \pi\pi^{-1}B = B$ , so  $\lambda \in I(a_f)$ .

In the case  $\lambda = 0$ , we have  $\bar{E}(a_f) = \{0, \infty\}$ . Since  $\tilde{Y} \cong Y$  by hypothesis, and by the one-to-one correspondence between  $S_f$  and  $S_\beta$  invariant sets,  $F_\beta$  cannot be of type II (i.e. have  $r^*(F_\beta) = \{1\}$  which would imply that  $S_\beta$  is conjugate to  $(F_\beta, \text{Id})$ ), so it must be of type III<sub>0</sub>.

Using Proposition 3.1, it is now an easy consequence to prove the following theorem.

**THEOREM 3.2.** *If  $X$  is any smooth manifold which admits a type III<sub>0</sub> diffeomorphism, then the suspension flow of that diffeomorphism is of type III<sub>0</sub> on  $X \times T^1$  (i.e., for  $m$ -a.e.  $t \in (0, 1)$ , the diffeomorphism obtained by fixing the flow at  $t$  is of type III<sub>0</sub>).*

**PROOF.** Assume that  $f$  is a type III<sub>0</sub> diffeomorphism of  $X$ . We consider the ergodic decomposition of the skew product defined by

$$F: X \times \mathbf{R} \rightarrow X \times \mathbf{R},$$

$$(x, z) \mapsto \left( fx, z + \log \frac{d\mu f^{-1}}{d\mu}(x) \right), \quad x \in X, z \in \mathbf{R},$$

which preserves the measure  $\nu = e^{-z} d\mu \otimes dz$ . By [16] there exists a Borel probability space  $(Y, \mathfrak{T}, \rho)$  and  $\sigma$ -finite measures  $q_y$  on  $(T^k \times \mathbf{R}, \mathfrak{S}_k, \nu)$  such that:

- (i)  $y \mapsto q_y(B)$  is Borel for every  $B \in \mathfrak{S}_k$ .
- (ii)  $\nu(B) \sim \int_{T^k \times \mathbf{R}} q_y(B) d\rho(y)$ .
- (iii) Every  $q_y, y \in Y$ , is invariant and ergodic under  $F$ , and  $q_y$  and  $q_{y'}$  are mutually singular when  $y \neq y'$ .
- (iv) Let  $\mathfrak{L} = \{B \in \mathfrak{S}_k: F(B) = B\}$ . For every  $B \in \mathfrak{L}$ , put  $B_y = \{y \in Y: q_y(B) > 0\}$ . Then  $\mathfrak{L}_Y = \{B_Y: B \in \mathfrak{L}\}$  is equal to  $\mathfrak{T}$  mod sets of  $\rho$ -measure zero.

For each  $y \in Y$ , the map  $F_t: X \times \mathbf{R} \times T^1 \rightarrow X \times \mathbf{R} \times T^1$  defined by

$$(x, z, w) \rightarrow \left( fx, z + \log \frac{d\mu f^{-1}}{d\mu}(x), w + t \right)$$

is  $q_y \otimes m$ -ergodic for  $m$ -a.e.  $t \in [0, 1]$ . This can be shown by taking the suspension flow of  $F$  and applying Lemma 2.8 for each  $y \in Y$ . If we can prove that the set  $Q = \{(y, t) \in (Y \times I, \mathfrak{T} \times \mathfrak{I}, \rho \times m): F_t \text{ is } q_y \otimes m\text{-ergodic}\}$  is  $\rho \otimes m$ -measurable, then by Fubini's Theorem, since  $\rho \otimes m(Q) = 1$  there exists a set  $C \subset [0, 1]$ ,  $m(C) = 1$  such that if  $t \in C$  for  $\rho$ -a.e.  $y \in Y$ ,  $F_t$  is  $q_y \otimes m$ -ergodic. By the uniqueness of ergodic decomposition, this implies that  $F_t$  is of type III<sub>0</sub>.

We conclude the proof with the following lemma.

**LEMMA 3.3.** *The set*

$$Q = \{(y, t) \in (Y \times I, \mathfrak{T} \times \mathfrak{I}, \rho \otimes m): F_t \text{ is } q_y \otimes m\text{-ergodic}\}$$

*is measurable.*

PROOF. Let  $\tilde{X} = X \times \mathbf{R}$ , and denote by  $X_n$  the manifold defined by  $X_n = X \times (-n, n)$ . The skew product  $F$  defined in 3.1 is always conservative, so we can induce on  $X_n$ . Let  $F_n$  denote the induced transformation on  $X_n$ , and we write  $F_{nt}$  for the map

$$F_{nt}: X_n \times T^1 \rightarrow X_n \times T^1, \\ (x, z, w) \mapsto (F_n(x, z), w + t) \quad \text{for every } x \in X, z \in (-n, n), w \in T^1.$$

We define, for every  $y \in Y$  and  $n \geq 1$ , a normalised measure on  $X_n$  equivalent to the induced measure obtained from  $q_y$  restricted to  $X_n$ . Call these measures  $p_y^{(n)}$ ; then  $p_y^{(n)}(X_n) = 1$  for  $n \geq 1$ . Clearly we have  $F_t$  is  $q_y \otimes m$ -ergodic if and only if  $F_{nt}$  is  $p_y^{(n)} \otimes m$ -ergodic for every  $n \geq 1$ . To show  $Q$  is measurable, we show that  $Q_n = \{(y, t) \in Y \times I \mid F_{nt} \text{ is } p_y^{(n)} \otimes m\text{-ergodic}\}$  is  $\rho \otimes m$ -measurable for each  $n \in \mathbf{N}$ .

To show  $Q_n$  is measurable, we use the following claim, a different version of which is proved in [7].

Claim. With the above notation,  $F_{nt}$  is  $p_y^{(n)} \otimes m$ -ergodic if and only if

$$\inf_{m \geq 1} \left| \frac{1}{m} \sum_{j=0}^{m-1} h_i \circ F_{nt}^j - \int_{X_n \times I} h_i dp_y^{(n)} \otimes m \right|_{L^2(X_n \times I, p_y^{(n)} \otimes m)} = 0,$$

where  $\|\cdot\|_{L^2(X_n \times I, p_y^{(n)} \otimes m)}$  denotes the relevant  $L^2$  norm and  $h_i \in \{h_k\}_{k \in \mathbf{N}}$  and  $\{h_k\}_{k \in \mathbf{N}}$  is a countable dense sequence of Borel ( $L^2$ ) functions on  $X_n \times I$  (hence measurable for  $p_y^{(n)} \otimes m$ , for every  $y \in Y$ ).

Since the infimum of measurable functions is measurable, and since the countable intersection of measurable sets is measurable, it suffices to show that for each fixed  $m, i$ , and  $n \in \mathbf{N}$ , the map

$$(x, z, w, y, t) \rightarrow \left| \frac{1}{m} \sum_{j=0}^{m-1} h_i \circ F_{nt}^j(x, z, w) - \int_{X_n \times I} h_i dp_y^{(n)} \otimes m \right|_{L^2(X_n \times I, p_y^{(n)} \otimes m)}$$

is measurable, for every  $x \in X_n, z \in (-n, n), w \in T^1, y \in Y$ , and  $t \in [0, 1]$ .

Using the definition of Lebesgue integral and elementary facts about measurable functions, it is not difficult to see that the map  $\Phi_{(m,i,n)}: X \times (-n, n) \times T^1 \times Y \times I \rightarrow \mathbf{R}$  given by

$$(x, z, w, y, t) \mapsto \left| \frac{1}{m} \sum_{j=0}^{m-1} h_i \circ F_{nt}^j(x, z, w) - \int_{X_n \times T^1} h_i dp_y^{(n)} \otimes m \right|_{L^2(X_n \times I)}$$

is Borel for each fixed triplet  $(m, i, n)$  and hence the infimum map, denoted  $\Phi_{i,n}$ , is Borel also.

Thus the set  $\bar{Q} = \bigcap_{n \in \mathbf{N}} \bigcap_{i \in \mathbf{N}} \Phi_{i,n}^{-1}(0)$  is a measurable set in  $X \times \mathbf{R} \times T^1 \times Y \times I$ , and by Fubini's Theorem we have that the projection on  $Y \times I$  of  $\bar{Q}$  is measurable in  $Y \times I$ . Now we conclude by observing that

$$\pi_{Y \times I}(\bar{Q}) = \{(y, t) \in Y \times I \mid F_t \text{ is } \rho \otimes m\text{-ergodic}\} = Q,$$

so the lemma is proved.

COROLLARY 3.4. For every  $n \geq 1$ , there exist type III<sub>0</sub> diffeomorphisms of  $T^n$ .

PROOF. We use induction. For  $n = 1$ , the theorem is true by [11]. At the  $k$ th step we take the suspension flow and apply Theorem 3.2.

**4. Type  $III_0$  diffeomorphisms of  $T^1 \times \mathbf{R}$ .** In this section we prove the existence of smooth, ergodic real line extensions for the  $\mathbf{Z}$ -action of a type  $III_\lambda$  diffeomorphism on a smooth, paracompact manifold  $X$ , for any  $0 \leq \lambda \leq 1$ . It is not difficult to show that in the type  $III_0$  case, most of these ergodic extensions give type  $III_0$  diffeomorphisms of  $X \times \mathbf{R}$ . Our goal is to obtain type  $III_0$  diffeomorphisms of  $T^1 \times \mathbf{R}$ , so we can complete our construction on arbitrary manifolds in §5. Special thanks are due to Klaus Schmidt and Ralf Spatzier for helpful discussions on the construction given in Theorem 4.8.

We begin with a theorem proved in [16].

**THEOREM 4.1.** *Let  $(X, \mathcal{S}, \mu)$  denote a  $C^\infty$  manifold with smooth probability measure  $\mu$ , and let  $f \in \text{Diff}^\infty(X)$  be  $\mu$ -ergodic. If  $H$  denotes a locally compact second countable abelian group with Haar measure  $\lambda_H$ , and if  $a: \mathbf{Z} \times X \rightarrow H$  is a cocycle for the  $\mathbf{Z}$ -action of  $f$  on  $X$ , then the map  $S_a: X \times H \rightarrow X \times H$  given by  $(x, h) \mapsto (fx, h \cdot a(1, x))$  is  $\mu \otimes \lambda_H$ -ergodic if and only if  $E(a) = H$ .*

Any real-valued function  $\phi$  on a smooth manifold  $(X, \mathcal{S}, \mu)$  gives rise to a cocycle for the  $\mathbf{Z}$ -action of any ergodic diffeomorphism,  $f$ , of  $X$  in the following way.

Using additive notation we have

$$a_\phi(n, x) = \begin{cases} \sum_{k=0}^{n-1} \phi(f^k x) & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \\ -a(-n, f^n x) & \text{if } n \leq -1 \end{cases}$$

for  $\mu$ -a.e.  $x \in X$ .

From the above definition one can check that  $a_\phi$  is a coboundary if and only if  $\phi$  can be written as  $\phi(x) = \eta - \eta \circ f(x)$  for  $\mu$ -a.e.  $x \in X$  and for some Borel map  $\eta: X \rightarrow \mathbf{R}$ . We say that  $\phi$  is a coboundary if  $a_\phi$  is a coboundary and we will write  $\bar{E}(a_\phi) = \bar{E}(\phi)$  unless confusion arises.

If  $V \in [f]$ , then recall that  $V(x) = f^{n(x)}(x)$  for  $\mu$ -a.e.  $x \in X$ . Using the notation of [16] we let  $a_\phi(V, x) = a_\phi(n(x), x)$  if  $V(x) = f^{n(x)}(x)$ .

We prove a result which is motivated by a result of Jones and Parry [10] for continuous and compact extensions.

**PROPOSITION 4.2.** *Let  $(X, \mathcal{S}, \mu)$  be a smooth connected compact manifold with  $\mu$  a  $C^\infty$  probability measure on  $X$ . Let  $f \in \text{Diff}^\infty(X)$  be an ergodic diffeomorphism. Suppose there exists an element which is not a coboundary in the set*

$$\mathcal{C} = \overline{\{ \phi \in C^\infty(X, \mathbf{R}) \mid \phi = \eta - \eta \circ f \text{ for some Borel map } \eta: X \rightarrow \mathbf{R} \}}$$

(where the closure is taken with respect to the  $C^\infty$  topology). Then the set of coboundaries is meagre in  $\mathcal{C}$ .

PROOF. The proof is similar to that of [10]. We consider  $\mathcal{C}$  as a complete topological group under pointwise addition and with respect to the  $C^\infty$  topology. If

we let

$$E = \{ \phi: X \rightarrow \mathbf{R} \mid \phi \text{ is a Borel map and } \phi - \phi \circ f = h \text{ a.e. for some } h \in \mathcal{C} \},$$

and identify two functions in  $E$  if and only if they are equal  $\mu$ -a.e., then we see that  $E$  is a group under pointwise addition.

We define the map  $L: E \rightarrow \mathcal{C}$  by setting  $L(\phi) = \phi - \phi \circ f$ . We see that  $L$  is a group homomorphism and  $\ker L = \text{constant maps} \cong \mathbf{R}$ . We now define a metric on  $E$  by

$$\delta_{\mathbf{R}}(\phi_1, \phi_2) = \int_X \frac{|\phi_1 - \phi_2|}{1 + |\phi_1 - \phi_2|} d\mu + \|L\phi_1 - L\phi_2\|_{\infty} \quad \text{for all } \phi_1, \phi_2 \in E;$$

then we see that  $E$  is complete and separable with respect to  $\delta_{\mathbf{R}}$ , and that  $L$  is a continuous group homomorphism.

Using the Open Mapping Theorem, and the assumption that there exists  $\psi \in \mathcal{C}$  such that  $\psi \notin \text{image } L$ , the proposition is proved.

REMARK 4.3. Under the assumption that there exists at least one element which is not a coboundary in  $\mathcal{C}$ , we will prove that there is in fact a dense  $G_{\delta}$  of elements in  $\mathcal{C}$  which contain all of  $\mathbf{R}$  in their essential range; i.e. there is a dense  $G_{\delta}$  in  $\mathcal{C}$ , call it  $\mathcal{E}_{\mathcal{C}}$ , such that if  $\phi \in \mathcal{E}_{\mathcal{C}}$ , then the skew product given by  $F: X \times \mathbf{R} \rightarrow X \times \mathbf{R}$ , where  $(x, y) \mapsto (fx, y + \phi x)$  is  $\mu \otimes m$ -ergodic. We begin with an easy lemma, whose proof is similar to [16, 9.6].

LEMMA 4.4. *Let  $(X, \mathcal{S}, \mu)$  and  $f$  be as in 4.2. The set  $\mathcal{E}_{\mathcal{C}} = \{ \phi \in C^{\infty}(X, \mathbf{R}) \mid F: X \times \mathbf{R} \text{ defined in Remark 4.3 is } \mu \otimes m\text{-ergodic} \}$  is a  $G_{\delta}$ .*

PROOF. By Theorem 4.1,  $\mathcal{E}_{\mathcal{C}} = \{ \phi \in C^{\infty}(X, \mathbf{R}) \mid E(\phi) = \mathbf{R} \}$ . Let  $\mathcal{S}_0$  denote a countable, dense subalgebra for  $X$ . Let  $\{ \lambda_i \}_{i \in \mathbf{N}}$  be a dense sequence in  $\mathbf{R}$ . Fix any element  $B \in \mathcal{S}_0$  and any number  $\beta \in \{ \lambda_i \}_{i \in \mathbf{N}}$ . We claim that for every fixed  $\delta > 0$  and  $\varepsilon > 0$  the set

$$U(B, \beta, \varepsilon, \delta) = \left\{ \phi \in C^{\infty}(X, \mathbf{R}) \mid \sup_{V \in \{f\}} \mu(B \cap V^{-1}B \cap \{x: a_{\phi}(V, x) \in (\beta - \varepsilon, \beta + \varepsilon)\}) > \delta \right\}$$

is open in  $C^{\infty}(X, \mathbf{R})$  with the  $C^{\infty}$  topology.

(PROOF OF CLAIM. Clearly, for fixed  $n$ , the map  $\phi \mapsto \sum_{i=0}^{n-1} \phi \circ f^i$  is continuous in  $C^{\infty}(X, \mathbf{R})$  with respect to the  $C^{\infty}$  topology. We recall that  $Vx = f^{n(x)}x$  for some  $n \in \mathbf{Z}$  and for  $\mu$ -a.e.  $x \in X$ . By [16, 2.6],  $a_{\phi}(V, x) = a_{\phi}(n(x), x) = \sum_{i=0}^{n(x)-1} \phi \circ f^i(x)$ . Therefore by continuity, we can find  $\delta > 0$  small enough s.t.  $\|\phi - \tilde{\phi}\|_{\infty} < \delta$  implies that  $\mu\{x: |a_{\phi}(V, x)| < \varepsilon\} = \mu\{x: |a_{\tilde{\phi}}(V, x)| < \varepsilon\}$ .) Then

$$\bigcap_{B \in \mathcal{S}_0} \bigcap_m U\left(B, \beta, \frac{1}{m}, \frac{\mu(B)}{4}\right) = \{ \phi \in C^{\infty}(X, \mathbf{R}) \mid \beta \in E(\phi) \},$$

and finally we have

$$= \bigcap_i \bigcap_B \bigcap_m U\left(B, \lambda_i, \frac{1}{m}, \frac{\mu(B)}{4}\right) = \{ \phi \in C^{\infty}(X, \mathbf{R}) \mid E(\phi) = \mathbf{R} \},$$

which proves the proposition.

REMARK 4.5. Since  $\mathcal{C} \subset C^\infty(X, \mathbf{R})$  is a closed subgroup, the set  $\mathcal{E}_\mathcal{C} = \{\phi \in C^\infty(X, \mathbf{R}) \mid E(\phi) = \mathbf{R}\}$  is also a  $G_\delta$ . It is easy to see that the set  $\mathcal{E}_\mathcal{C}$  is dense in  $\mathcal{C}$  if it is not empty. For if there exists an element  $\psi \in \mathcal{E}_\mathcal{C}$ , then by adding a suitable coboundary to  $\psi$  we can find another element of  $\mathcal{E}_\mathcal{C}$  arbitrarily close in the  $C^\infty$  topology to any  $\phi \in \mathcal{C}$ .

In the next lemma we will prove that if there exists an element  $\psi \in \mathcal{C}$  such that  $E(\psi) = \{n\lambda\}_{n \in \mathbf{Z}}$  for some  $\lambda > 0$ , then there exists an element  $\tilde{\psi} \in \mathcal{E}_\mathcal{C}$ .

LEMMA 4.6. *Let  $(X, \mathcal{S}, \mu)$  and  $f$  be as in 4.2. Suppose there exists an element  $\psi \in \mathcal{C}$  such that  $E(\psi) = \{n\lambda\}_{n \in \mathbf{Z}}$  for some  $\lambda > 0$ . Then  $\mathcal{E}_\mathcal{C}$  is a dense  $G_\delta$  in  $\mathcal{C}$ .*

PROOF. Since  $\lambda \in E(\psi)$ , we have for all  $B \in \mathcal{S}$ ,  $\mu(B) > 0$  and all  $\varepsilon > 0$ ,

$$\sup_{V \in [f]} \mu\left(B \cap V^{-1}B \cap \left\{x: |a_\psi(V, x) - \lambda| < \varepsilon\right\}\right) > \frac{\mu(B)}{2},$$

as proved in [16].

We can choose an irrational scalar  $c \in \mathbf{R}$ ,  $0 < c < 1$ , such that  $\beta = c\lambda$ , and  $\lambda$  and  $\beta$  are rationally independent. Then for all  $x \in \{x: |a_\psi(V, x) - \lambda| < \varepsilon\}$ , we have

$$a_{c\psi}(V, x) = \sum_{i=0}^{n(x)-1} c\psi \circ f^i(x) = c \sum_{i=0}^{n(x)-1} \psi \circ f^i(x),$$

for some  $n(x) \in \mathbf{Z}$ , so  $|a_{c\psi}(V, x) - \beta| < \varepsilon$  as well. Let  $\tilde{\psi} = c\psi$ . Thus  $\beta \in E(\tilde{\psi})$ . By adding suitable coboundaries to  $\tilde{\psi}$ , we see that the set  $\{\phi \in \mathcal{C} \mid E(\phi) \ni \{n\beta\}_{n \in \mathbf{Z}}\}$  is dense in  $\mathcal{C}$ , and the proof of Lemma 4.4 shows us that it is in fact a  $G_\delta$ . Similarly, we have that  $U_\lambda = \{\phi \in \mathcal{C} \mid E(\phi) \ni \{n\lambda\}_{n \in \mathbf{Z}}\}$  is a dense  $G_\delta$ . Then  $U_\lambda \cap U_\beta$  is also a dense  $G_\delta$  in  $\mathcal{C}$ , and since the set of essential values for  $\phi \in \mathcal{C}$  forms a closed additive subgroup of  $\mathbf{R}$ ,  $\lambda \in E(\phi)$  and  $\beta \in E(\phi)$  imply that  $E(\phi) = \mathbf{R}$  since  $\lambda$  and  $\beta$  are rationally independent. Therefore the proposition is proved.

We have verified the existence of a nonempty set  $\mathcal{E}_\mathcal{C}$  if there exists  $\psi \in \mathcal{C}$  such that  $\bar{E}(\psi) = \mathbf{R}$  or  $\{n\lambda\}_{n \in \mathbf{Z}}$ . The only remaining possibility is that every element  $\phi \in \mathcal{C}$  which is not a coboundary satisfies  $\bar{E}(\phi) = \{0, \infty\}$ . In Theorem 4.8 we will show that even under this assumption we can construct elements in  $\mathcal{E}_\mathcal{C}$ . This theorem is sufficient to ensure that type III<sub>0</sub> diffeomorphisms of compact manifolds have ergodic real line extensions, which is what is needed to extend type III<sub>0</sub> diffeomorphisms to arbitrary manifolds.

We first prove a proposition which is necessary for the construction in Theorem 4.8.

PROPOSITION 4.7. *Let  $(X, \mathcal{S}, \mu)$  be a smooth paracompact connected manifold with  $\mu$  a smooth  $\sigma$ -finite measure on  $X$ . Let  $f \in \text{Diff}^\infty(X)$  be a  $\mu$ -ergodic diffeomorphism. We denote by  $\mathcal{S}_0$  a countable dense subalgebra of  $\mathcal{S}$ , and  $\mathcal{C}$  is as in 4.2. Suppose there exists  $\phi \in \mathcal{C}$  such that  $\phi$  is recurrent and  $\bar{E}(\phi) = \{0, \infty\}$ . Then there exists a set  $\Psi \subset \mathcal{C}$ , such that  $\Psi$  is a dense  $G_\delta$  in  $\mathcal{C}$  with respect to the  $C^\infty$  topology and every element  $\psi \in \Psi$  satisfies the following condition:*

*For every  $\varepsilon > 0$ , for every  $M \in \mathbf{R}^+$ , and for every  $B \in \mathcal{S}_0$ , there exists  $\phi \in \mathcal{C}$  with  $\|\phi\|_\infty \leq 1$  and  $\bar{E}(\phi) = \{0, \infty\}$ , and  $V \in [f]$  such that*

$$\mu\left(B \cap V^{-1}B \cap \left\{x: |a_\psi(V, x)| < \varepsilon\right\} \cap \left\{x: |a_\phi(V, x)| > M\right\}\right) \geq \mu(B)/2.$$

PROOF. We choose a countable dense set  $\{\phi_i\}_{i \in \mathbb{N}}$  in the unit ball of  $\mathcal{C}$ , where for every  $i \in \mathbb{N}$ ,  $\bar{E}(\phi_i) = \{0, \infty\}$  and  $\phi_i$  is recurrent. We choose a countable dense set in the full group of  $f$ , denoted  $\{V_k\}_{k \in \mathbb{N}}$ . Let  $M \in \mathbb{N}$  denote a positive integer. We define the set

$$\begin{aligned} \Lambda(B, M, \varepsilon, l, j, k, i) &= \left\{ \psi \in \mathcal{C} \mid \mu \left( B \cap V_k^{-1}B \cap \left\{ x : |a_\psi(V_k, x)| < \varepsilon \right\} \cap \left\{ x \mid |a_{\phi_i/j}(V_k, x)| > M \right\} \right) \right. \\ &> \left. \left( 1 - \frac{1}{l} \right) \mu(B) \cdot 2^{-1} \right\} \end{aligned}$$

By the continuity of  $\psi$ , and using techniques from 4.4 we can show that for each fixed  $(B, M, \varepsilon, l, j, k, i)$  the set  $\Lambda(B, M, \varepsilon, l, j, k, i)$  is open in  $\mathcal{C}$  with respect to the  $C^\infty$  topology.

We also claim that  $\Gamma(B, M, \varepsilon, l, j) = \bigcup_k \bigcup_i \Lambda(B, M, \varepsilon, l, j, k, i)$  is open and dense in  $\mathcal{C}$ . Clearly it is open, and it is dense because each  $\Gamma(B, M, \varepsilon, l, j)$  contains the coboundaries. To show this, fix  $\varepsilon_0, M_0, B_0, j_0$  and  $l_0$ . Suppose  $\psi \in \mathcal{C}$  is a coboundary. Then choose any  $\phi_0 \in \mathcal{C}$  which satisfies  $\bar{E}(\phi_0) = \{0, \infty\}$  and  $\|\phi_0\|_\infty/j_0 \leq 1$ . Since  $\psi$  is a coboundary we write  $\psi = \eta - \eta \circ f$  where  $\eta$  is a Borel function on  $X$ , and we find a set  $D_0 \subset B_0$  such that  $|\eta(x) - \eta(y)| < \varepsilon_0/4$  for all  $x, y \in D_0$ . Since  $\infty \in \bar{E}(\phi_0)$ , we can find an integer  $p$  such that

$$\mu \left( D_0 \cap f^{-p}D_0 \cap \left\{ x : |a_{\phi_0/j_0}(p, x)| > M_0 \right\} \right) > 0.$$

Using the exhaustion argument method of [15, 9.4], we can find an element  $V_k \in [f]$  such that

$$\begin{aligned} \mu \left( B \cap V_k^{-1}B \cap \left\{ x : |a_\psi(V_k, x)| < \varepsilon_0 \right\} \cap \left\{ x : |a_{\phi_0/j_0}(V_k, x)| > M_0 \right\} \right) \\ > \left( 1 - \frac{1}{l_0} \right) \mu(B_0) 2^{-1}. \end{aligned}$$

This proves that  $\psi \in \Gamma(B_0, M_0, \varepsilon_0, l_0, j_0)$ .

We now define  $\Psi = \bigcap_{B \in \mathfrak{S}_0} \bigcap_M \bigcap_\varepsilon \bigcap_l \bigcap_j \Gamma(B, M, \varepsilon, l, j)$  (where  $\varepsilon \in \{\varepsilon_r\}_{r \in \mathbb{N}}$  is a countable set such that  $\varepsilon_r \leq 1/r$ ).

Clearly  $\Psi$  is a dense  $G_\delta$ , and it remains to show that  $\psi \in \Psi$  satisfies the hypotheses of the proposition. If  $\psi \in \Psi$ , then for every  $\varepsilon_r > 0$ , for every  $M \in \mathbb{N}$ , for every  $B \in \mathfrak{S}_0$ , and for every  $j, l \in \mathbb{N}$  there exists  $\phi_i \in \mathcal{C}$ ,  $\bar{E}(\phi_i) = \{0, \infty\}$ ,  $\phi_i$  is recurrent, and there exists  $V_k \in [f]$  satisfying

$$\begin{aligned} \mu \left( B \cap V_k^{-1}B \cap \left\{ x : |a_\psi(V_k, x)| < \varepsilon \right\} \cap \left\{ x : |a_{\phi_i/j}(V_k, x)| > M \right\} \right) \\ > \left( 1 - \frac{1}{l} \right) \mu(B) \cdot 2^{-1}. \end{aligned}$$

This concludes the proof.

We make  $\Psi$  into a complete metric space by defining a metric on  $\Psi$  given by

$$D_\infty(\phi_1, \phi_2) = \|\phi_1 - \phi_2\|_\infty + d_\infty(\phi_1, \phi_2),$$

where  $d_\infty(\phi_1, \phi_2)$  is defined in the following way.

Let  $d(\phi, A) = \inf_{\psi \in A} \|\phi - \psi\|_\infty$  for any set  $A \subset \mathcal{C}$ . We index the countable set of sets  $\Gamma(B, \varepsilon, M, j, k)$  by  $s \in \mathbb{N}$ , say. Then we define

$$d_s(\phi_1, \phi_2) = \left| \frac{1}{d(\phi_1, \Gamma_s^c)} - \frac{1}{d(\phi_2, \Gamma_s^c)} \right|,$$

where  $\Gamma_s^c$  denotes the complement of the open set  $\Gamma_s = \Gamma_s(B, M, \varepsilon, l, j)$ .

Finally we let  $d_\infty(\phi_1, \phi_2) = \sum_{s=1}^\infty 2^{-s} d_s(\phi_1, \phi_2) / (1 + d_s(\phi_1, \phi_2))$ . An easy calculation shows that  $D_\infty$  is a metric on  $\Psi$ , and that  $\Psi$  is complete with respect to  $D_\infty$ .

We are now ready for the main theorem of this section.

**THEOREM 4.8.** *Let  $(X, \mathcal{S}, \mu)$  and  $f \in \text{Diff}^\infty(X)$  be as in 4.7. Suppose there exists an element  $\phi \in \mathcal{C}$  which is recurrent and not a coboundary. Then  $\mathfrak{E}_\mathcal{C}$  is a dense  $G_\delta$  in  $\mathcal{C}$  with the  $C^\infty$  topology.*

**PROOF.** By 4.4–4.6 it suffices to assume that every element of  $\mathcal{C}$  which is not a coboundary satisfies  $\bar{E}(\phi) = \{0, \infty\}$ .

Let  $\mathcal{S}_0$  denote a countable, dense subalgebra of  $X$ . We fix an element  $B \in \mathcal{S}_0$ ,  $\mu(B) > 0$ , and we choose and fix any  $\varepsilon > 0$ .

We will construct  $\psi \in \mathcal{C}$  and  $V \in [f]$  such that

$$\mu(B \cap V^{-1}B \cap \{x: |a_\psi(V, x) - 1| < \varepsilon\}) \geq \mu(B)/2.$$

Then, using the notation and methods of Lemmas 4.4–4.6 we see that  $U(B, 1, \varepsilon, \mu(B)/2)$  is open, dense, and nonempty in  $\mathcal{C}$  (in the  $C^\infty$  topology) for each  $B \in \mathcal{S}_0$  and  $\varepsilon > 0$ , and therefore the theorem is proved.

**A. Setting up the construction.** Let  $\Psi$  be defined as in 4.7. We start the induction process by defining  $\phi_0 = 0$ ,  $\tilde{B}_0 = B$ ,  $M_0 = 1$ , and  $\varepsilon_0 = \varepsilon/2$ . Since  $\phi_0 \in \Psi$  we apply 4.7 to obtain  $p_1$  and  $\phi_1$  satisfying

$$(4.1) \quad \mu(\tilde{B}_0 \cap f^{-p_1}\tilde{B}_0 \cap \{x: |a_{\phi_0}(p_1, x)| < \varepsilon_0\} \cap \{x: |a_{\phi_1}(p_1, x)| > M_0\}) > 0.$$

Since the set  $\Psi$  is dense in  $\mathcal{C}$ , we can perturb  $\phi_1$  slightly if necessary so that  $\phi_1 \in \Psi$ , and (4.1) still holds. We choose

$$B_1 \subseteq \tilde{B}_0 \cap f^{-p_1}\tilde{B}_0 \cap \{x: |a_{\phi_0}(p_1, x)| < \varepsilon_0\} \cap \{x: |a_{\phi_1}(p_1, x)| > M_0\}$$

such that  $B_1 \cap f^{p_1}B_1 = \emptyset$ . Choose  $c_1 \leq 1$  satisfying

$$\mu(\tilde{B}_0 \cap f^{-p_1}\tilde{B}_0 \cap \{x: |a_{\phi_0}(p_1, x)| < \varepsilon_0\} \cap \{x: |a_{c_1\phi_1}(p_1, x) - 1| < \varepsilon_0\}) > 0.$$

We define  $V_1 \in [f]$  by

$$V_1(x) = \begin{cases} f^{p_1}x & \text{if } x \in B_1, \\ f^{-p_1}x & \text{if } x \in f^{p_1}B_1 \\ x & \text{otherwise,} \end{cases}$$

and let  $\tilde{B}_1 = \tilde{B}_0 \setminus (B_1 \cup f^{p_1}B_1)$ . We define  $\zeta_1 = c_1\phi_1$ .

**B. The  $j$ th stage.** We will define inductively,  $\phi_j \in \mathcal{C}$ ,  $\|\phi_j\|_\infty \leq 1$ ,  $c_j \in \mathbb{R}^+$ ,  $s_j \in \mathbb{N}$ ,  $M_j \in \mathbb{R}^+$ ,  $\varepsilon_j > 0$ ,  $B_j \subset B$ ,  $\tilde{B}_j \subset B$ ,  $p_j \in \mathbb{N}$ ,  $\zeta_j \in \Psi$  and  $V_j \in [f]$  satisfying:

$$(1)_j \quad \zeta_j = \sum_{l=1}^{s_j} c_l \phi_l \in \Psi,$$

- (2)<sub>j</sub>  $\mu(\tilde{B}_{j-1} \cap f^{-p_j} \tilde{B}_{j-1} \cap \{x: |a_{c_j \phi_j}(p_j, x) - 1| < \epsilon_j\} \cap \{x: |a_{\zeta_{j-1}}(p_j, x)| < \epsilon_j\}) > 0,$
- (3)<sub>j</sub>  $D_\infty(\zeta_{j-1}, \zeta_j) < \epsilon_j,$
- (4)<sub>j</sub>  $c_j \|\sum_{l=0}^{p_l-1} \phi_l\|_\infty < \epsilon(2^{j+2l})^{-1}$  for  $0 \leq l \leq j - 1,$
- (5)<sub>j</sub>  $p_j > p_{j-1}, \epsilon_j = \epsilon(2^{j+1})^{-1}, M_j \geq M_{j-1},$
- (6)<sub>j</sub>  $B_j \subset \tilde{B}_{j-1} \cap f^{-p_j} \tilde{B}_{j-1} \cap \{x: |a_{c_j \phi_j}(p_j, x) - 1| < \epsilon_j\}, \mu(B_j) > 0,$  and  $B_j \cap f^{p_j} B_j = \emptyset.$  We define  $\tilde{B}_j = \tilde{B}_{j-1} \setminus (B_j \cup f^{p_j} B_j).$
- (7)<sub>j</sub>

$$V_j(x) = \begin{cases} V_k(x) & \text{if } x \in B_k, k \leq j - 1, \\ f^{p_j}(x) & \text{if } x \in B_j, \\ f^{-p_j}(x) & \text{if } x \in f^{p_j} B_j, \\ x & \text{otherwise.} \end{cases}$$

C. *The induction step.* Assume we are at the  $j$ th stage. First we choose  $s_{j+1} \in \mathbb{N}$  large enough so that  $\sum_{s=s_{j+1}}^\infty 2^{-s} < \epsilon(2^{j+3})^{-1}$ . Then we define  $\gamma_{s_{j+1}} = \gamma_{j+1} = \min_{s \leq s_{j+1}} d(\zeta_j, \Gamma_s^c)$ . Since  $\zeta_j \in \Psi, \gamma_{j+1} > 0$ .

Now we choose  $M_{j+1} \geq \max(M_j, \epsilon_j^{-1} \gamma_{j+1}^{-2} s_{j+1} 2^4 p_j)$ . Using (1)<sub>j</sub> and Proposition 4.7 we can find  $p_{j+1} > p_j$  and  $\phi \in \mathcal{C}$  such that  $\|\phi\|_\infty \leq 1$ , and for  $\epsilon_{j+1} = \epsilon_j \cdot 2^{-1}$  we have (4.2)

$$\mu(\tilde{B}_j \cap f^{-p_{j+1}} \tilde{B}_j \cap \{x: |a_{\zeta_j}(p_{j+1}, x)| < \epsilon_{j+1}\} \cap \{x: |a_\phi(p_{j+1}, x)| > M_{j+1}\}) > 0.$$

Let  $\phi_{j+1} = \phi$ . We choose  $c_{j+1} < 1/M_{j+1}$  such that

$$(4.3) \quad \mu(\tilde{B}_j \cap f^{-p_{j+1}} \tilde{B}_j \cap \{x: |a_{\zeta_j}(p_{j+1}, x)| < \epsilon_{j+1}\} \cap \{x: |a_{c_{j+1} \phi_{j+1}}(p_{j+1}, x) - 1| < \epsilon_{j+1}\}) > 0.$$

We define the set

$$B_{j+1} \subseteq \tilde{B}_j \cap f^{-p_{j+1}} \tilde{B}_j \cap \{x: |a_{\zeta_j}(p_{j+1}, x)| < \epsilon_{j+1}\} \cap \{x: |a_{c_{j+1} \phi_{j+1}}(p_{j+1}, x) - 1| < \epsilon_{j+1}\}$$

such that  $\mu(B_{j+1}) > 0$  and  $B_{j+1} \cap f^{p_{j+1}} B_{j+1} = \emptyset$ .

We can assume that  $\zeta_j + c_{j+1} \phi_{j+1} \in \Psi$ , because if not we could have chosen  $\tilde{c}_{j+1}$  or  $\tilde{\phi}_{j+1}$  arbitrarily close by such that (4.2) still holds and  $\zeta_j + \tilde{c}_{j+1} \tilde{\phi}_{j+1} \in \Psi$  (since  $\Psi$  is dense in  $\mathcal{C}$ ). Define  $\zeta_{j+1} = \zeta_j + c_{j+1} \phi_{j+1} = \sum_{i=1}^{j+1} c_i \phi_i$ .

We must now check to see if (1)<sub>j+1</sub> through (7)<sub>j+1</sub> hold.

For (7)<sub>j+1</sub>, we just define

$$V_{j+1}(x) = \begin{cases} V_k(x) & \text{if } x \in B_k, k \leq j, \\ f^{p_{j+1}}(x) & \text{if } x \in B_{j+1}, \\ f^{-p_{j+1}}(x) & \text{if } x \in f^{p_{j+1}} B_{j+1}, \\ x & \text{otherwise.} \end{cases}$$

By our construction, (1)<sub>j+1</sub>, (2)<sub>j+1</sub>, (5)<sub>j+1</sub>, and (6)<sub>j+1</sub> obviously are satisfied.

To check  $(3)_{j+1}$  we proceed as follows:

$$\begin{aligned}
 D_\infty(\xi_j, \xi_{j+1}) &= \|\xi_j - \xi_{j+1}\|_\infty + \sum_{s=1}^\infty 2^{-s} \frac{d_s(\xi_j, \xi_{j+1})}{1 + d_s(\xi_j, \xi_{j+1})} \\
 &= \|\xi_j - \xi_{j+1}\|_\infty + \sum_{s=1}^{s_{j+1}} 2^{-s} \frac{d_s(\xi_j, \xi_{j+1})}{1 + d_s(\xi_j, \xi_{j+1})} + \sum_{s=s_{j+1}+1}^\infty 2^{-s} \frac{d_s(\xi_j, \xi_{j+1})}{1 + d_s(\xi_j, \xi_{j+1})} \\
 &\leq c_{j+1} \|\phi_{j+1}\|_\infty + \sum_{s=1}^{s_{j+1}} 2^{-s} \frac{d_s(\xi_j, \xi_{j+1})}{1 + d_s(\xi_j, \xi_{j+1})} + \sum_{s=s_{j+1}+1}^\infty 2^{-s} \\
 &\leq c_{j+1} \|\phi_{j+1}\|_\infty + \sum_{s=1}^{s_{j+1}} 2^{-s} \frac{d_s(\xi_j, \xi_{j+1})}{1 + d_s(\xi_j, \xi_{j+1})} + \epsilon(2^{j+3})^{-1},
 \end{aligned}$$

by our choice of  $s_{j+1}$ .

We have that

$$\begin{aligned}
 d_s(\xi_j, \xi_{j+1}) &= \left| \frac{1}{d(\xi_j, \Gamma_s^c)} - \frac{1}{d(\xi_{j+1}, \Gamma_s^c)} \right| \\
 &\leq \left| \frac{1}{d(\xi_j, \Gamma_s^c)} - \frac{1}{d(\xi_j, \Gamma_s^c) + \|\xi_j - \xi_{j+1}\|_\infty} \right|,
 \end{aligned}$$

since  $|d(\xi_j, \Gamma_s^c) - d(\xi_{j+1}, \Gamma_s^c)| \leq \|\xi_j - \xi_{j+1}\|_\infty$  for each  $s$ , so the denominators can vary by at most  $\|\xi_j - \xi_{j+1}\|_\infty$ , and now we have

$$\begin{aligned}
 \|\xi_j - \xi_{j+1}\|_\infty &\leq c_{j+1} \|\phi_{j+1}\|_\infty \leq c_{j+1} \cdot 1 \\
 &\leq \epsilon_j \cdot \gamma_{j+1}^2 \cdot 2^{-4j} \cdot p_j^{-1} \cdot s_{j+1}^{-1} \leq \epsilon \cdot \gamma_{j+1}^2 \cdot 2^{-5j} \cdot p_j^{-1} \cdot s_{j+1}^{-1}
 \end{aligned}$$

and recalling our choice of  $\gamma_{j+1} > 0$ , we have

$$\begin{aligned}
 D_\infty(\xi_j, \xi_{j+1}) &\leq \epsilon \cdot 2^{-5j} + \sum_{s=1}^{s_{j+1}} \frac{\gamma_{j+1}^2 \cdot \epsilon \cdot 2^{-(j+6)} s_{j+1}^{-1}}{\gamma_{j+1}^2} \Big/ 1 + \frac{\gamma_{j+1}^2 \cdot \epsilon \cdot 2^{-(j+6)} s_{j+1}^{-1}}{\gamma_{j+1}^2} \\
 &\quad + \epsilon \cdot 2^{-(j+3)} \\
 &\leq \epsilon \cdot 2^{-5j} + s_{j+1} (\epsilon \cdot 2^{-(j+6)} s_{j+1}^{-1}) + \epsilon \cdot 2^{-(j+3)} \\
 &\leq \epsilon \cdot 2^{-5j} + \epsilon \cdot 2^{-(j+6)} + \epsilon \cdot 2^{-(j+3)} \\
 &< \epsilon \cdot 2^{-(j+3)} = \epsilon_{j+1}.
 \end{aligned}$$

We check  $(4)_{j+1}$  as follows. Clearly

$$\begin{aligned}
 c_{j+1} \left\| \sum_{i=0}^{p_l-1} \phi_{j+1} \circ f^i \right\|_\infty &\leq c_{j+1} \cdot p_j \cdot \|\phi_{j+1}\|_\infty \quad \text{since } 0 \leq l \leq j, \\
 &\leq \epsilon_j \cdot \gamma_{j+1}^2 \cdot s_{j+1}^{-1} \cdot p_j^{-1} \cdot 2^{-4j} \cdot p_j \cdot \|\phi_{j+1}\|_\infty \\
 &\leq \epsilon \cdot 2^{-(j+1)} \cdot 2^{-(4j)} \leq \epsilon \cdot 2^{-(j+1+2l)}.
 \end{aligned}$$

D. *Taking the limit.* We let  $\psi = \sum_{i=1}^\infty c_i \phi_i$ . Since  $(3)_j$  holds for all  $j \geq 1$ ,  $\psi$  is the limit of a Cauchy sequence in  $\Psi$ . More precisely, given any  $\delta > 0$  (we might as well

assume  $0 < \delta < \varepsilon$ , we choose  $j_0 \in \mathbb{N}$  such that  $\sum_{j=j_0}^\infty 2^{-j} < \delta/\varepsilon$ . Then for any  $m > n \geq j_0$  we have

$$D_\infty(\zeta_n, \zeta_m) \leq \sum_{k=0}^{m-n-1} D(\zeta_{n+k}, \zeta_{n+k+1}) \leq \sum_{k=0}^{m-n-1} \varepsilon_{n+k+1}, \text{ by (3)}_j.$$

Now

$$\sum_{k=0}^{m-n-1} \varepsilon_{n+k+1} \leq \varepsilon \sum_{k=0}^{m-n-1} 2^{-(n+k+1)} \leq \varepsilon \sum_{k=j_0}^\infty 2^{-k} < \delta,$$

so  $D_\infty(\zeta_n, \zeta_m) < \delta$ .

Therefore  $\psi \in \Psi$ , since  $\Psi$  is complete. We can apply 4.7 to  $\psi$  and continue the induction process. Then, using an exhaustion argument, we obtain a sequence of sets  $B_i$  such that  $B_i \cap B_j = B_i \cap f^{p_j} B_j = B_j \cap f^{p_i} B_i = B_i \cap f^{p_i} B_i = B_j \cap f^{p_j} B_j = \emptyset$ , for all  $i \neq j$ , and such that  $\mu(\cup_{i \in \mathbb{N}} (B_i \cup f^{p_i} B_i)) > \mu(B)/2$ . We also obtain  $V \in [f]$  such that

$$(4.4) \quad V(x) = \begin{cases} V_k(x) & \text{if } x \in B_k \cup f^{p_k} B_k, \\ x & \text{otherwise.} \end{cases}$$

For all  $x \in B_j$ , we have  $V(x) = f^{p_j}(x)$ , and this implies for  $x \in B_j$ ,

$$\begin{aligned} |a_\psi(V, x)| &= |a_\psi(p_j, x)| \\ &\leq |a_{\zeta_{j-1}}(p_j, x)| + |a_{c_j \phi_j}(p_j, x)| + \left| \sum_{k=0}^{p_j-1} \left( \sum_{i=j+1}^\infty c_i \phi_i \right) \circ f^k(x) \right| \\ &\leq \varepsilon_j + 1 + \varepsilon_j + \left| \sum_{i=j+1}^\infty c_i \left( \sum_{k=0}^{p_j-1} \phi_i \circ f^k(x) \right) \right| \end{aligned}$$

by (2)<sub>j</sub>, and now by (4)<sub>j</sub>,

$$\begin{aligned} &\leq \varepsilon_j + 1 + \varepsilon_j + \sum_{i=j+1}^\infty \varepsilon(2^{i+2(i-1)})^{-1} \\ &\leq \varepsilon_j + 1 + \varepsilon_j + \varepsilon/8 \leq 1 + (5/8)\varepsilon. \end{aligned}$$

From the above and (4.4), an easy calculation shows that

$$\mu(B \cap V^{-1}B \cap \{x : |a_\psi(V, x) - 1| < \varepsilon\}) \geq \mu(B)/2,$$

and we are done.

We should point out that 4.4 is true for noncompact  $X$ , and the hypotheses on  $(X, \mathfrak{S}, \mu)$  in 4.7 are sufficient for 4.8 to be true. We have proved the existence of a dense  $G_\delta$  of  $\mathcal{C}$  whose elements give ergodic extensions for  $f$ ; we now need to see which of these skew products have the same ratio sets as  $f$ . We will give a necessary and sufficient condition, but first we will recall some easily proved facts.

LEMMA 4.9. Let  $(X, \mathcal{S}, \mu)$  be as in 4.7 and let  $f \in \text{Diff}^\infty(X)$  be any ergodic diffeomorphism of  $X$ . If we consider the skew product  $F_\phi$  defined by

$$F_\phi: X \times \mathbf{R} \rightarrow X \times \mathbf{R}, \quad (x, y) \mapsto (fx, y + \phi x) \quad \text{where } \phi \in \mathcal{C},$$

then  $r^*(F_\phi) \subset r^*(f)$ .

PROOF. Let  $F = F_\phi$ . By definition,  $r^*(F) = \bar{E}(\log(d\mu \otimes mF^{-1}/d\mu \otimes m))$  where  $(d\mu \otimes mF^{-1}/d\mu \otimes m)(x, y)$  denotes the Radon-Nikodym derivative of the measure  $\mu \otimes m(F^{-1})$  with respect to  $\mu \otimes m$  at the point  $(x, y) \in X \times \mathbf{R}$ . This implies

$$\begin{aligned} \frac{d\mu \otimes mF^{-1}}{d\mu \otimes m}(x, y) &= \det DF(x, y) = \det \begin{vmatrix} df(x) & 0 \\ d\phi(x) & 1 \end{vmatrix} \\ &= df(x) = \frac{d\mu f^{-1}}{d\mu}(x). \end{aligned}$$

From this we see that  $\lambda \in r^*(F) \Rightarrow \lambda \in r^*(f)$ .

REMARK 4.10. Given two cocycles on  $X$ ,  $\phi_1, \phi_2 \in C^\infty(\mathbf{Z} \times X, \mathbf{R})$  we can define a cocycle  $(\phi_1, \phi_2): \mathbf{Z} \times X \rightarrow \mathbf{R}^2$  by  $(\phi_1, \phi_2)(n, x) = (\phi_1(n, x), \phi_2(n, x)) \forall n \in \mathbf{Z}, x \in X$ . We compactify  $\mathbf{R}^2$  by adding lines of the form  $(\alpha, \infty), (\alpha, -\infty), (\infty, \alpha), (-\infty, \alpha)$  for all  $\alpha \in \mathbf{R}$ , plus four points at  $(-\infty, \infty), (\infty, \infty), (\infty, -\infty), (-\infty, -\infty)$ . Then  $(\lambda, \beta) \in \bar{E}(\phi_1, \phi_2)$  means that for every  $B \in \mathcal{S}, \mu(B) > 0$  and for every  $\varepsilon > 0$ , there exists  $n \in \mathbf{Z}$  such that  $\mu(B \cap f^{-n}B \cap \{x: |(\phi_1(n, x), \phi_2(n, x)) - (\lambda, \beta)| < \varepsilon\}) > 0$ , or equivalently,  $\mu(B \cap f^{-n}B \cap \{x: |\phi_1(n, x) - \lambda| < \varepsilon\} \cap \{x: |\phi_2(n, x) - \beta| < \varepsilon\}) > 0$ .

It is clear that  $(\lambda, \beta) \in \bar{E}(\phi_1, \phi_2)$  implies that  $\lambda \in \bar{E}(\phi_1)$  and  $\beta \in \bar{E}(\phi_2)$ , but the converse is not necessarily true. We give an example of the usefulness of considering two cocycles together in the next proposition.

PROPOSITION 4.11. With  $(X, \mathcal{S}, \mu)$  an  $m$ -dimensional manifold and  $f$  as in 4.7, we assume further that  $f$  is of type  $\text{III}_0$  and that the map  $F_\phi$  defined in 4.9 is  $\mu \otimes m$ -ergodic. Then  $(0, \infty) \in \bar{E}(\phi, \log(d\mu f^{-1}/d\mu))$  if and only if  $F_\phi$  is of type  $\text{III}_0$ .

PROOF. ( $\Rightarrow$ ) Assume that  $(0, \infty) \in \bar{E}(\phi, \log(d\mu f^{-1}/d\mu))$ . By 4.9 it suffices to show that  $\infty \in r^*(F_\phi)$ . Let  $C \in \mathcal{S} \times \mathcal{G} \subset X \times \mathbf{R}$  be such that  $\mu \otimes m(C) > 0$ . Choose  $t_0$  to be a point of density of  $C$ . Then there exists an  $m + 1$ -dimensional cube  $R \subset X \times I$  of volume  $\delta > 0$ , centered at  $t_0 = (t_1, t_2)$  such that  $\mu \otimes m(R \cap C) > .99\delta$ . By setting  $B = \Pi_X(R \cap C)$ , we see that  $\mu(B) > 0$ . Since  $(0, \infty) \in \bar{E}(\phi, \log(d\mu f^{-1}/d\mu))$ , there exists  $n \in \mathbf{Z}$  such that

$$\mu \left( B \cap f^{-n}B \cap \left\{ x: \left| \sum_{i=0}^{n-1} \phi \circ f^i(x) \right| < \delta^{1/(m+1)} \right\} \cap \left\{ x: \left| \log \frac{d\mu f^n}{d\mu}(x) \right| > M \right\} \right) > 0.$$

This implies that

$$\begin{aligned} \mu \otimes m \left( (R \cap C) \cap \left\{ (x, y) \in R \cap C: \left( f^{-n}x, y - \sum_{i=0}^{n-1} \phi \circ f^i(x) \right) \in R \cap C \right\} \right. \\ \left. \cap \left\{ (x, y) \left| \log \frac{d\mu f^n}{d\mu}(x) \right| > M \right\} \right) > 0. \end{aligned}$$

Therefore  $\infty \in r^*(F)$ .

( $\Leftarrow$ ) Suppose that  $F$  is of type  $\text{III}_0$ , i.e.  $r^*(F) = \{0, \infty\}$ . Then for every set  $C \subset \mathfrak{S} \times \mathfrak{Y}$ ,  $\mu \otimes m(C) > 0$ , and for every  $M \in \mathbf{R}^+$ , we can find an integer  $n$  such that

$$\mu \otimes m \left( C \cap F^{-n}C \cap \left\{ (x, y) : \left| \log \frac{d\mu \otimes mF^n}{d\mu \otimes m} (x, y) \right| > M \right\} \right) > 0.$$

Since  $\log(d\mu \otimes mF^n/d\mu \otimes m)(x, y) = \log(d\mu f^n/d\mu)(x)$ , and since  $F^{-n}C = \{(f^{-n}x, y - \sum_{i=0}^{n-1} \phi \circ f^i(x)), (x, y) \in C\}$ , then for any  $B \in \mathfrak{S}$ ,  $\mu(B) > 0$ , we just choose  $C = B \times (-\varepsilon/2, \varepsilon/2)$ .

Then clearly there exists  $n \in \mathbf{Z}$  such that

$$\mu \left( B \cap f^{-n}B \cap \left\{ x : \left| \sum_{i=0}^{n-1} \phi \circ f^i(x) \right| < \varepsilon \right\} \cap \left\{ x : \left| \log \frac{d\mu f^{-n}}{d\mu} (x) \right| > M \right\} \right) > 0.$$

This implies that  $(0, \infty) \in \bar{E}(\phi, \log(d\mu f^{-1}/d\mu))$ .

Finally we prove the existence of a dense  $G_\delta$  of elements in  $\mathcal{C}$  which satisfy the hypotheses of 4.11. We assume  $X$  and  $f$  are as in 4.7.

**PROPOSITION 4.12.** *Let  $\psi: X \rightarrow \mathbf{R}$  be a fixed ( $C^\infty$ ) cocycle for  $f$  with  $\bar{E}(\phi) = \{0, \infty\}$ . Then the set  $\mathfrak{A} = \{\phi \in \mathcal{C} \mid (0, \infty) \in \bar{E}(\phi, \psi)\}$  is a dense  $G_\delta$  in  $\mathcal{C}$ .*

**PROOF.** Let  $\mathfrak{S}_0$  be a countable dense subalgebra for  $X$ . Choose any  $B \in \mathfrak{S}_0$ , and fix  $M \in \mathbf{R}^+$ . Since  $\infty \in \bar{E}(\phi)$ , there exists  $V \in [f]$  such that

$$\mu(B \cap V^{-1}B \cap \{x : |a_\psi(V, x)| > M\}) > \mu(B)/2.$$

If we define the set

$$\mathcal{Q}(B, M, \varepsilon) = \left\{ \phi \in \mathcal{C} \mid \sup_{V \in [f]} \mu(B \cap V^{-1}B \cap \{x : |a_\psi(V, x)| > M\}) \cap \{x : |a_\phi(V, x)| < \varepsilon\} > \mu(B)/4 \right\},$$

using the same argument as in Lemma 4.4 we see that it is open for fixed  $B, M$ , and  $\varepsilon$ . Now

$$\mathfrak{A} = \bigcap_{B \in \mathfrak{S}_0} \bigcap_{m \in \mathbf{N}} \bigcap_{M \in \mathbf{N}} \mathcal{Q}\left(B, M, \frac{1}{m}\right) = \{\phi \in \mathcal{C} \mid (0, \infty) \in \bar{E}(\phi, \psi)\}.$$

Clearly this set is a  $G_\delta$ . To show that it is dense, we observe that the coboundaries are dense in  $\mathcal{C}$  and obviously lie in  $\mathfrak{A}$ .

**THEOREM 4.13.** *With  $X$  and  $f$  as in 4.7, suppose further that  $f$  is of type  $\text{III}_0$ . Then the set  $\mathcal{C}_0 = \{\phi \in \mathcal{C} \mid (x, y) \mapsto (fx, y + \phi x) \text{ is of type } \text{III}_0\}$  is a dense  $G_\delta$  in  $\mathcal{C}$ .*

**PROOF.** By 4.8, we have that  $\mathfrak{E}_\mathcal{C}$  is a dense  $G_\delta$ . By 4.11 and 4.12, we have  $\mathfrak{A}$  is a dense  $G_\delta$ . Then  $\mathfrak{E}_\mathcal{C} \cap \mathfrak{A}$  is a dense  $G_\delta$  of  $\mathcal{C}$  and  $\mathcal{C}_0 = \mathfrak{E}_\mathcal{C} \cap \mathfrak{A}$ .

**COROLLARY 4.14.** *There are uncountably many  $C^\infty$  type  $\text{III}_0$  diffeomorphisms on  $T^n \times \mathbf{R}^p$ , for every  $n \geq 1, p \geq 0$ .*

PROOF. For  $n = 1, p = 0$ , we use Katznelson’s construction. By 4.2, the result is true for all  $n \geq 1$  when  $p = 0$ . By repeated applications of 4.4–4.13 and an induction argument, the corollary is proved.

REMARK 4.15. All of the results from this section hold true for diffeomorphisms of type  $\text{III}_\lambda$ , with  $0 < \lambda \leq 1$ . In some sense, type  $\text{III}_\lambda$  transformations are better behaved than type  $\text{III}_0$ ; there is only one type  $\text{III}_\lambda$  ergodic transformation, up to weak equivalence, for each  $0 < \lambda \leq 1$  which is not true in the  $\text{III}_0$  case. In particular, using obvious modifications of the preceding theorems, we can prove the following.

THEOREM 4.16. *For every  $\lambda, 0 < \lambda \leq 1$ , there exists a  $C^\infty$  type  $\text{III}_\lambda$  diffeomorphism of  $T^n \times \mathbf{R}^p$ , for every  $n \geq 1, p \geq 0$ .*

5. Type  $\text{III}_\lambda, 0 \leq \lambda \leq 1$  diffeomorphisms of arbitrary manifolds. Herman proved in [7] that every connected paracompact manifold of dimension  $\geq 3$  has a  $C^\infty$  type  $\text{III}_1$  diffeomorphism on it. He used a nice method for extending maps on  $T^2 \times \mathbf{R}^{m-2}$  to any connected paracompact  $m$ -dimensional manifold for  $m \geq 3$ ; this procedure was introduced by Anosov in 1974 [5]. We will outline the method here without proof, also including some modifications for our particular circumstances.

LEMMA 5.1. *Let  $X$  be an  $m$ -dimensional  $C^\infty$  paracompact connected manifold and  $\mu$  a  $C^\infty$  measure on  $X$ . Then there exists an open set  $V \subset X$ , diffeomorphic to  $\mathbf{R}^m$  and satisfying  $\mu(X - V) = 0$ .*

LEMMA 5.2. *If  $m \geq 3$ , there exists an open set  $U$  of  $\mathbf{R}^m$  diffeomorphic to  $T^2 \times \mathbf{R}^{m-2}$  such that  $\mu(\mathbf{R}^m - U) = 0$ .*

LEMMA 5.3. *There exists a  $C^\infty$  type  $\text{III}_0$  ( $\text{III}_\lambda, 0 < \lambda \leq 1$ ) flow on  $T^2 \times \mathbf{R}^p$  for every  $p \in \mathbf{N}$ .*

PROOF. We apply Corollary 4.14 (Theorem 4.16) to obtain a  $C^\infty$  type  $\text{III}_0$  ( $\text{III}_\lambda, 0 < \lambda \leq 1$ ) diffeomorphism of  $T^1 \times \mathbf{R}^p$ , then take the suspension flow.

LEMMA 5.4. *Let  $U$  be an open set of  $\mathbf{R}^m$ , and let  $f_t$  be a  $C^\infty$  flow of type  $\text{III}_\lambda, 0 \leq \lambda \leq 1$ , on  $U$ . Let  $\chi$  be the infinitesimal generator of  $f_t$ , i.e.,  $\chi$  is defined by*

$$\left. \frac{\partial f_t}{\partial t}(x) \right|_{t=0} = \chi \circ f_t(x).$$

*Let  $\phi \in C^\infty(U, \mathbf{R}), \phi > 0$ , be defined such that the vector field  $\phi\chi$  is globally integrable and defines a flow  $g_t$ . Then the flow  $g_t$  is weakly equivalent to  $f_t$ .*

LEMMA 5.5. *There exists a  $C^\infty$  type  $\text{III}_0$  ( $\text{III}_\lambda, 0 < \lambda \leq 1$ ) flow on every paracompact, connected manifold  $X$  of dimension  $m \geq 3$ .*

PROOF. By 5.1 and 5.2 we have an open set  $U \subset X$  of full  $\mu$ -measure and such that  $U$  is diffeomorphic to  $T^2 \times \mathbf{R}^{m-2}$ . Let  $f_t$  be a type  $\text{III}_0$  ( $\text{III}_\lambda, 0 < \lambda \leq 1$ ) flow on  $U$  with infinitesimal generator  $\chi$ ; such a flow exists by 5.3. Let  $\phi \in C^\infty(X, \mathbf{R})$  be such that  $\phi > 0$  on  $U, \phi = 0$  on  $X - U$ , and such that the vector field

$$Y(x) = \begin{cases} \phi(x)\chi(x), & \text{if } x \in U, \\ 0, & \text{if } x \in X - U \end{cases}$$

is  $C^\infty$  on  $X$  and globally integrable, thus defining a flow,  $\tilde{f}_t$ , on  $X$ . The flow  $\tilde{f}_t$  is of type  $\text{III}_0$  ( $\text{III}_\lambda$ ) by 5.4.

**COROLLARY 5.6.** *There exists a  $C^\infty$  type  $\text{III}_0$  ( $\text{III}_\lambda$ ,  $0 < \lambda \leq 1$ ) diffeomorphism on every connected paracompact manifold of dimension  $\geq 3$ .*

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