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SMOOTH T^n-VALUED COCYCLES FOR ERGODIC DIFFEOMORPHISMS

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ABSTRACT. We prove that if f is any conservative, ergodic diffeomorphism of a smooth, connected, paracompact manifold, then in the set of smooth T^n-valued cocycles on X for the $\mathbf{Z}$-action determined by f, there is a residual set which give ergodic skew product extensions.

Introduction. In this paper we prove that if f is any conservative, ergodic diffeomorphism of a smooth, connected, paracompact manifold, then in the set of smooth T^n-valued cocycles on X for the $\mathbf{Z}$-action determined by f, there is a residual set which give ergodic skew product extensions. In other words, we show that the set $\{\psi \in C^\infty(X, T^n) | (f, \psi)\colon X \times T^n \to X \times T^n$ defined by $(f,\psi)(x, y) = (fx, y \cdot \psi x)$ is ergodic$\}$ is residual in the Baire space $C^\infty(X, T^n)$ for each ergodic diffeomorphism f. As a corollary to this theorem we have that if f is of type III_1 (respectively $\mathrm{II}_1, \mathrm{II}_\infty$) then the set of $\psi \in C^\infty(X, T^n)$ giving type III_1 ($\mathrm{II}_1, \mathrm{II}_\infty$) extensions is residual.

If we look at cocycles taking values in any locally compact connected abelian Lie group ($\mathbf{R}^n \times T^p$ with $n \geqslant 1$, $p \geqslant 1$), the result simply is not true. It is well known that for certain irrational numbers $\alpha \in T^1$, the diffeomorphism R_α given by rotation through α has no ergodic $\mathbf{R}$-extensions. This follows from the fact that using Fourier series one can show that any $\psi \in C^\infty(T^1, \mathbf{R})$ has a solution η for $\psi(z) = \eta \circ R_\alpha(z) - \eta(z)$ (which is at least measurable). However, it was proved by the author in **[H]** that any ergodic type III diffeomorphism (one which does not preserve any smooth measure) admits ergodic real line extensions, in fact orbit equivalent ones.

The compactness of T^n allows us to use a much simpler argument than in **[H]**. In fact, the idea for the proof of Theorem 2.1 was motivated by a paper of R. Jones and W. Parry **[J]**. Klaus Schmidt is gratefully acknowledged for helpful discussions.

1. Notation and definitions. We introduce some concepts from the study of nonsingular group actions on measure spaces. For the purposes of this paper, we will give the definitions in the differentiable context; for the most general definitions, we refer the reader to **[S]**.

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Let $(X, \mathfrak{B}, \mu)$ denote a C^∞ manifold with $\mathfrak{B}$ the σ-algebra of Borel sets and μ a smooth probability measure. By $\mathrm{Diff}^\infty(X)$ we denote the set of C^∞ diffeomorphisms of X. Assume $f \in \mathrm{Diff}^\infty(X)$ is μ-ergodic, and let H be a locally compact second countable abelian group. The action $(n, x) \mapsto f^n x$ of $\mathbf{Z}$ on X is nonsingular since for each $n \in \mathbf{Z}$, the map $x \to f^n x$ is a Borel automorphism (in fact a diffeomorphism) of X which leaves μ quasi-invariant.

DEFINITION 1.1. A Borel map a: $\mathbf{Z} \times X \to H$ is called a *cocycle for* f if for every $n, m \in \mathbf{Z}$ and for μ-a.e. $x \in X$, we have

$$a(n, f^m x) - a(n+m, x) + a(m, x) = 0.$$

If H is a Lie group, and if for each $n \in \mathbf{Z}$, $a(n, \cdot)$: $X \to H$ is a smooth map, then we say a is a *smooth cocycle* for f. The cocycle a: $\mathbf{Z} \times X \to H$ is called a *coboundary* if there exists a Borel map b: $X \to H$ with $a(n, x) = b(f^n x) - b(x)$, for each $n \in \mathbf{Z}$ and for μ-a.e. $x \in X$. Two cocycles a_1 and a_2 are said to be *comhomologous* if their difference is a coboundary. The following defines a cohomology invariant.

DEFINITION 1.2. With $(X, \mathfrak{B}, \mu)$ and f as above, consider a cocycle a: $\mathbf{Z} \times X \to H$. An element $\lambda \in \overline{H} = H \cup \{\infty\}$ is called an *essential value* of a, if for every Borel set $B \in \mathfrak{B}$ with $\mu(B) > 0$ and for every neighbourhood $N(\lambda)$ of λ in $\overline{H}$,

$$\mu(B \cap f^{-n}B \cap \{x\colon a(n, x) \in N(\lambda)\}) > 0$$

for some $n \in \mathbf{Z}$. The set of essential values, called the *essential range*, is denoted by $\overline{E}(a)$, and we put $E(a) = \overline{E}(a) \cap H$. We state a few well-known properties of $\overline{E}(a)$:

(1) $\overline{E}(a)$ is a nonempty closed subset of $\overline{H}$.

(2) $E(a)$ is a closed subgroup of H.

(3) $\overline{E}(a) = \{0\}$ if and only if a is a coboundary.

(4) $\overline{E}(a_1) = \overline{E}(a_2)$ whenever a_1 and a_2 are cohomologous.

We now consider the smooth cocycle $\log(d\mu f/d\mu)$: $\mathbf{Z} \times X \to \mathbf{R}$ defined by $\log(d\mu f/d\mu)(n, x) = \log(d\mu f^{-n}/d\mu)(x)$, the log of the Radon-Nikodym derivative of the measure μf^{-n} with respect to μ. We define the *ratio set of* f to be the essential range for this cocycle and we write $\bar{r}(f) = \overline{E}(\log(d\mu f/d\mu))$. We say f *is of type*:

(i) II if $\bar{r}(f) = \{0\}$ (in this case f admits an invariant measure, either finite or σ-finite, equivalent to μ);

(ii) III_1 if $\bar{r}(f) = \mathbf{R} \cup \{\infty\}$;

(iii) III_λ if $\bar{r}(f) = \{n\lambda\}_{n \in \mathbf{Z}}$ for some $\lambda \in \mathbf{R} \setminus \{0\}$;

(iv) III_0 if $\bar{r}(f) = \{0, \infty\}$.

2. Smooth cocycles with values in T^1. We begin with results about smooth cocycles for an ergodic diffeomorphism taking values in the circle $T^1 = \{z \in \mathbf{C} \mid |z| = 1\}$. In this case we will use multiplicative notation for the cocycle a: $\mathbf{Z} \times X \to T^1$, and we see that for each fixed ergodic $f \in \mathrm{Diff}^\infty(X)$ every C^∞ map from X to T^1 determines a smooth multiplicative cocycle for f as follows. If $f \in \mathrm{Diff}^\infty(X)$ is μ-ergodic, and ψ: $X \to T^1$ is C^∞, we define

$$a_\psi(n, x) = \begin{cases} \prod_{k=0}^{n-1} \psi(f^k x) & \text{if } n \geqslant 1, \\ 1 & \text{if } n = 0, \\ -a_\psi(-n, f^n x) & \text{if } n \leqslant -1. \end{cases}$$

Similarly, any smooth T^1-valued cocycle for f determines a unique C^∞ map ψ: $X \to T^1$ by $\psi_a(x) = a(1, x)$ for every $x \in X$. We remark that a smooth cocycle a is a coboundary if and only if $\psi_a(x) = \eta \circ f/\eta(x)$ μ-a.e., with η a Borel map (not necessarily even continous) from X to T^1. In what follows we will write any smooth cocycle for f as an element of $C^\infty(X, T^1) = \{C^\infty$ maps from X to $T^1\}$ and we will write $\overline{E}(\psi)$ instead of $\overline{E}(a_\psi)$ for its essential range.

The main theorem of this section is the following

THEOREM 2.1. *Let $(X, \mathfrak{B}, \mu)$ be a connected paracompact manifold with $\mathfrak{B}$ the σ-algebra of Borel sets and μ any smooth probability measure. If $f \in \mathrm{Diff}^\infty(X)$ is μ-ergodic, then it is generic in $C^\infty(X, T^1)$ (i.e., true on a dense G_δ with respect to the C^∞ topology) that the skew product extension (f, ψ) defined on $X \times T^1$ and given by $(f, \psi)(x, z) = (fx, z \cdot \psi x)$ with $\psi \in C^\infty(X, T^1)$ is $\mu \otimes m$ ergodic. (Here, m denotes Haar measure on T^1.)*

PROOF. We will use the result from [S] which states that (f, ψ) is $\mu \otimes m$ ergodic if and only if $\overline{E}(\psi) = T^1$. By properties of the essential range given in §1, it follows that (f, ψ) is ergodic if and only if $\psi^k \neq \eta \circ f/\eta$ a.e. for any $k \geqslant 1$, and for any Borel map η: $X \to T^1$. (Because if $E(\psi) \neq T^1$, then $E(\psi) = \{0\}$ or $E(\psi) = \{\omega_i\}_{i=1}^k$ $= k$th roots of unity.)

We define the set $G = \{\eta$: $X \to T^1 | \eta$ is Borel and $\eta \circ f/\eta = \psi$ μ-a.e. for some $\psi \in C^\infty(X, T^1)\}$. We identify two maps in G if they are equal μ-a.e., and we see that G is a group under pointwise multiplication. We define the map ρ: $G \to C^\infty(X, T^1)$ by $\rho(\eta) = \eta \circ f/\eta$. We see that ρ is a homomorphism and that $\ker \rho =$ constant maps $\cong T^1$. We define a metric on G by

$$\delta(\eta_1, \eta_2) = \int_X |\eta_1 - \eta_2| d\mu + d_\infty(\rho\eta_1, \rho\eta_2),$$

where

$$d_\infty(f, g) = \sum_{r=1}^{\infty} 2^{-r} \frac{d_r(f, g)}{1 + d_r(f, g)}$$

for every f, $g \in C^\infty(X, T^1)$, and d_r denotes the usual C^r metric on $C^\infty(X, T^1)$. We see that G is complete and separable with respect to δ, and that ρ is a continuous group homomorphism. Therefore it suffices to show that ρG is of the first category in $C^\infty(X, T^1)$; that is, it can be written as the countable union of nowhere dense sets. We suppose that ρG is of the second category in $C^\infty(X, T^1)$; then its closure has nonempty interior in $C^\infty(X, T^1)$. By the Open Mapping Theorem [**K**], ρ is continuous implies that ρ: $G \to C^\infty(X, T^1)$ is open. Then ρG contains the connected component of the identity $e \equiv 1$ (the constant map), since it is a subgroup which is both open and closed. Therefore ρG contains all the constant maps from X to T^1, as they are homotopic to e. However, there exists $\beta \in (0, 1)$ such that $e^{2\pi i\beta}$ is not a coboundary for f; we just choose β not in the L^∞ spectrum of f. Therefore $e^{2\pi i\beta} \notin \rho G$. This contradiction proves that the coboundaries are nowhere dense in $C^\infty(X, T^1)$. Then the complement is residual, and the theorem is proved.

COROLLARY 2.2 (TYPE II CASE). *Suppose* $f \in \mathrm{Diff}^{\infty}(X)$ *is a type* II_1 (II_{∞}) *ergodic diffeomorphism. If the map* $\psi \in C^{\infty}(X, T^1)$ *is such that* $(f, \psi)(x, z) = (fx, z \cdot \psi x)$ *is* $\mu \otimes m$ *ergodic, then* $(f, \psi) \in \mathrm{Diff}^{\infty}(X \times T^1)$ *is also of type* II_1 (II_{∞}).

PROOF. This follows from the fact that $\bar{r}(f, \psi) \subset \bar{r}(f)$, and we note that (f, ψ) will preserve a finite (infinite σ-finite) measure if f does.

COROLLARY 2.3 (TYPE III_1 CASE). *If* $f \in \mathrm{Diff}^{\infty}(X)$ *is of type* III_1, *then it is generic in* $C^{\infty}(X, T^1)$ *that* (f, ψ) *is of type* III_1 *with* $\psi \in C^{\infty}(X, T^1)$.

PROOF. In this case, the map F: $X \times \mathbf{R} \to X \times \mathbf{R}$ defined by $(x, s) \mapsto (fx, s + \log(d\mu f^{-1}/d\mu)(X))$ is $d\mu \otimes e^{-z}\,dz$ ergodic [**S**]. A smooth T^1-valued cocycle for F is given by a C^{∞} map ζ: $X \times \mathbf{R} \to T^1$ and defined as before. In particular, the subset of $C^{\infty}(X \times \mathbf{R}, T^1)$ defined by $J = \{\zeta \in C^{\infty}(X \times \mathbf{R}, T^1) | \zeta(x, s_1) = \zeta(x, s_2)$ for all $x \in X$, $s_1, s_2 \in \mathbf{R}\}$ describes the set of C^{∞} maps which do not depend on the $\mathbf{R}$-coordinate. It is clear that J forms a topological group under pointwise multiplication (with respect to the C^{∞} topology), and that $J \cong C^{\infty}(X, T^1)$. We define $G' = \{\xi$: $X \times \mathbf{R} \to T^1$ such that ξ is Borel and $\xi \circ F/\xi = \varphi$ a.e. for some $\varphi \in J\}$, ρ' such that $\rho'(\xi) = \xi \circ F/\xi$, for each $\xi \in G'$, and we define the metric on G'

$$\delta'(\xi_1, \xi_2) = \int_{X \times \mathbf{R}} |\xi_1 - \xi_2|\, d\nu' + d_{\infty}\big(\rho'\xi_1, \rho'\xi_2\big).$$

Here ν' denotes a smooth probability measure on $X \times \mathbf{R}$. We then use the proof of Theorem 2.1 to see that $\rho' G'$ is meager in J, and the result follows.

REMARK 2.4. In the type III_{λ} case, with $0 \leqslant \lambda < 1$, it is known that there exist orbit equivalent circle extensions for each type III_{λ}, $f \in \mathrm{Diff}^{\infty}(X)$ [**H**]. However due to the nontrivial ergodic decomposition of the smooth measure ν' under F as in Corollary 2.3, we do not immediately get an analogous corollary to Theorem 2.1; the theorem itself is still true in this case though.

3. Higher-dimensional extensions. We prove a generalization of Theorem 2.1 for T^n-valued cocycles of an ergodic diffeomorphism. As before, we assume $f \in \mathrm{Diff}^{\infty}(X)$ is a conservative ergodic diffeomorphism of a connected paracompact manifold X.

THEOREM 3.1. *For each* $n \geqslant 1$, *there is a dense* G_{δ} *in the space* $C^{\infty}(X, T^n)$, $\mathfrak{E}_n$, *having the property that if* $\psi \in \mathfrak{E}_n$, *then the diffeomorphism* (f, ψ): $X \times T^n \to X \times T^n$ *given by* $(x, z) \mapsto (fx, z \cdot \psi x)$ *is ergodic with respect to the smooth probability measure on* $X \times T^n$.

PROOF. For $n = 1$, we have proved the set $\mathfrak{E}_1$ is a dense G_{δ} (Theorem 2.1). By m_k we will denote Haar (Lebesgue) measure on T^k, and therefore $\mu \otimes m_k$ will be a smooth probability measure on $X \times T^k$. Assume we have shown the existence of a dense G_{δ} in $C^{\infty}(X, T^k)$ such that, for each $\psi \in \mathfrak{E}_k$, (f, ψ) is $\mu \otimes m_k$ ergodic. Since $C^{\infty}(X, T^k)$ is separable, we can fix a countable dense sequence in $C^{\infty}(X, T^k)$, call it $\{\varphi_i\}_{i \in \mathbf{N}}$, such that (f, φ_i) is $\mu \otimes m_k$ ergodic. Using an argument like that of Corollary 2.3, we can prove the existence of a dense G_{δ} in $C^{\infty}(X, T^1)$ for each

$i \in \mathbf{N}$, call it $\mathfrak{F}_k^i$, such that for each $\psi \in \mathfrak{F}_k^i$, the map (f, φ_i, ψ): $X \times T^{k+1} \circlearrowleft$ defined by $(x, z, z_{k+1}) \to (fx, z \cdot \varphi_i x, z_{k+1} \cdot \psi x)$ with $x \in X$, $z = (z_1, \dots, z_k) \in T^k$, and $z_{k+1} \in T^1$ is $\mu \otimes m_{k+1}$ ergodic. If we now consider $\mathfrak{F}_k = \bigcap_{i \in \mathbf{N}} \mathfrak{F}_k^i$, we obtain a dense G_δ. $\mathfrak{F}_k$ is a set of $\psi \in C^\infty(X, T^1)$ such that (f, φ_i, ψ) is ergodic for all i. By identifying $C^\infty(X, T^{k+1})$ with $C^\infty(X, T^k) \times C^\infty(X, T^1)$, we have just proved that there is a dense set in $C^\infty(X, T^{k+1})$ of ergodic T^{k+1}-valued cocycle extensions for f.

We now show that the set $\{\varphi \in C^\infty(X, T^{k+1}) | E(\varphi) = T^{k+1}\}$ can be written as the countable intersection of open sets.

We fix a countable generating sequence of sets $\{B_i\}_{i \in \mathbf{N}}$ in $\mathfrak{B}$, the σ-algebra of Borel sets of X. We fix a countable dense set in T^{k+1}, call it $\{\zeta_j\}_{j \in \mathbf{N}}$. Now for each fixed quadruple (i, j, l, m) we define the set

$$\Lambda(i, j, l, m) = \left\{ \varphi \in C^\infty(X, T^{k+l}) \mid \mu\left(\bigcup_{p \in \mathbf{Z}} \left(B_i \cap f^{-p} B_i \cap \left\{ x\colon \prod_{k=0}^{p-1} \varphi \circ f^{-k}(x) \in N_{1/l}(\zeta_j) \right\} \right) \right) > (1 - 1/m)\mu(B_i)/2 \right\}.$$

A routine argument and the continuity of the map

$$\varphi \mapsto \mu\left(\bigcup_{p=-J}^{J} \left(B_i \cap f^{-p} B_i \cap \left\{ x\colon \prod_{0}^{p-1} \varphi \circ f^{-k}(x) \in N_{1/l}(\zeta_j) \right\} \right) \right)$$

for each fixed (i, J, j, l, m) shows that $\Lambda(i, j, l, m)$ is open. Now

$$\bigcap_m \bigcap_l \bigcap_j \bigcap_i \Lambda(i, j, l, m) = \mathfrak{E}_{k+1},$$

and the theorem is proved.

COROLLARY 3.2. *The set of smooth T^n-valued coboundaries for the $\mathbf{Z}$-action of f on X is meager in $C^\infty(X, T^n)$.*

Results analogous to those of Corollaries 2.2 and 2.3 hold in the higher-dimensional case as well.

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