Rohlin Factors, Product Factors, and Joinings for n-to-One Maps

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ABSTRACT. In this paper we show that every nonsingular conservative ergodic n-to-one endomorphism of a Lebesgue probability space has a factor which is isomorphic to the Cartesian product of an automorphism with a one-sided shift. The measure on the product space is always a nonsingular joining of the factor measures on each factor, but is not in general a product measure. It decomposes over the automorphic factor into a family of measures each of which is trivial on the tail σ-algebra of the shift. We give necessary and sufficient conditions under which the measure is a product measure and study the marginal measures in the general case. Necessary and sufficient conditions are given under which the marginal measure on the shift space is exact. We show that the theorem cannot be strengthened by constructing a finite measure-preserving ergodic two-to-one endomorphism whose product structure cannot be given product measure.

1. Introduction and background. In this paper we show that every nonsingular conservative ergodic n-to-one endomorphism of a Lebesgue probability space has a factor which is isomorphic to the Cartesian product of an automorphism with a one-sided shift. The measure on the product space is always a nonsingular joining of the factor measures on each factor, but is not in general a product measure. It decomposes over the automorphic factor into a family of measures each of which is trivial on the tail σ-algebra of the shift. We give necessary and sufficient conditions under which the measure is a product measure and study the marginal measures in the general case. The automorphic factor is always conservative, nonsingular, and ergodic, and we give necessary and sufficient conditions under which the marginal measure on the shift space is exact.

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We give an example to show that the theorem cannot be strengthened; we construct a finite measure-preserving ergodic two-to-one endomorphism whose product structure cannot be given product measure.

The main theorem of this paper is the following.

Theorem. *Assume that T is an n-to-one nonsingular ergodic conservative endomorphism of $(X, \mathcal{B}, \mu)$. Then T has a factor which is isomorphic to the Cartesian product of an automorphism Φ with an n-to-one shift σ on the product space $(Y \times Z, \mathcal{F} \times \mathcal{D}, m)$. The measure m is of the following form: for any $C \in \mathcal{F} \times \mathcal{D}$,*

$$m(C) = \int_Y \int_{\{z:(y,z)\in C\}} d\rho_y(z)\, d\nu(y),$$

where ν is a nonsingular conservative ergodic measure for the automorphism Φ, and each ρ_y is a tail trivial measure for σ on Z. Furthermore, m is a nonsingular joining for $\Phi \times \sigma$ with respect to the factor measures ν on $(Y, \mathcal{F})$ and ρ on $(Z, \mathcal{D})$.

We recall that a measure m on $(Y \times Z, \mathcal{F} \times \mathcal{D})$ has *marginal* measures on Y and Z, defined by $m(A \times Z)$ and $m(Y \times B)$, respectively, for $A \in \mathcal{F}$, $B \in \mathcal{D}$. Generally speaking, any measure on $(Y \times Z, \mathcal{F} \times \mathcal{D})$ whose marginal measures are ν on $(Y, \mathcal{F})$ and ρ on $(Z, \mathcal{D})$ is called a *joining* of ν and ρ.

Throughout this paper we will assume that $(X, \mathcal{B}, \mu)$ is a Lebesgue probability space and $T : X \to X$ is a nonsingular conservative ergodic endomorphism which is surjective and countable-to-one almost everywhere. (The assumption that $\mu(X) < \infty$ results in no loss of generality [4].) By a result of Rohlin [13], [19] we can assume by replacing X by a measurable T-invariant subset of full measure if necessary that T is forward nonsingular as well so that T satisfies: for all $A \in \mathcal{B}$, $\mu(A) = 0 \Leftrightarrow \mu(T^{-1}A) = 0 \Leftrightarrow \mu(TA) = 0$. We apply a well-known result of Rohlin [13] to obtain a measurable partition $\zeta = \{A_1, A_2, A_3, \dots\}$ of X into at most countably many pieces, called atoms, satisfying:

(i) $\mu(A_i) > 0$ for each i;
(ii) the restriction of T to each A_i, which we will write as T_i, is one-to-one;
(iii) each A_i is of maximal measure in $X \setminus \bigcup_{j<i} A_j$ with respect to property (ii);
(iv) T_1 is one-to-one and onto X (by numbering the atoms so that $\mu(TA_i) \geq \mu(TA_{i+1})$ for $i \in \mathbb{N}$).

When we say that an endomorphism T is *n-to-one*, we mean that every partition $\zeta = \{A_1, A_2, A_3, \dots\}$ satisfying (i)–(iv) contains precisely n atoms and that T_i is one-to-one and onto X for each $i = 1, \dots, n$. Equivalently, for μ-a.e. $x \in X$, the set $T^{-1}x$ contains exactly n points. An endomorphism T is *conservative* if for every set A of positive measure, there exists an $m \in \mathbb{N}$ such that $\mu(T^{-m}A \cap A) > 0$.

For each $x \in A_i$, let

$$J_{\mu T_i}(x) = \frac{d\mu T_i}{d\mu}(x),$$

and for $x \in X$, let

$$J_{\mu T}(x) = \sum_i J_{\mu T_i}(x)\chi_{A_i}(x).$$

This is the Jacobian function for T, defined by W. Parry [12], and is independent of the choice of ζ. Our nonsingularity assumptions imply that $J_{\mu T} > 0$ μ-a.e. We then have the following identities holding μ-a.e. (see [8] or [15]):

$$\vartheta_{\mu T}(x) \equiv \frac{d\mu T^{-1}}{d\mu}(x) = \sum_{y \in T^{-1}x} \frac{1}{J_{\mu T}(y)}, \tag{1}$$

$$\omega_{\mu T}(x) \equiv \frac{d\mu}{d\mu T^{-1}}(Tx) = \frac{1}{\vartheta_{\mu T}(Tx)}. \tag{2}$$

The function $\omega_{\mu T}$ satisfies for every $f \in L^1(X, \mathcal{B}, \mu)$:

$$\int_X f(Tx) \cdot \omega_{\mu T}(x)\, d\mu(x) = \int_X f(x)\, d\mu(x). \tag{3}$$

For any measurable function ω satisfying (3) in place of $\omega_{\mu T}$, we say that ω is *Markovian* for T and μ, and it was shown in [16] that $\omega_{\mu T}$ is the unique $T^{-1}\mathcal{B}$ measurable function which is Markovian for T and μ. Thus, any ω which is Markovian for T and μ satisfies $\omega \in L^1(X, \mathcal{B}, \mu)$ and $E_\mu(\omega \mid T^{-1}\mathcal{B}) = \omega_{\mu T}$. Here $E_\mu(h \mid T^{-1}\mathcal{B})$ denotes the conditional expectation of $h \in L^1(X, \mathcal{B}, \mu)$ with respect to the sub-σ-algebra $T^{-1}\mathcal{B}$. Similarly,

$$\omega_{\mu T^k}(x) = \frac{d\mu}{d\mu T^{-k}}(T^k x)$$

is the unique $T^{-k}\mathcal{B}$ measurable function which is Markovian for T^k and μ. In particular, $\omega_\mu(k, x) \equiv \omega_{\mu T}(x) \cdot \omega_{\mu T}(Tx) \cdots \omega_{\mu T}(T^{k-1}x)$ is Markovian for T^k and μ, so $E_\mu\big(\omega_\mu(k, \cdot\,) \mid T^{-k}\mathcal{B}\big) = \omega_{\mu T^k}$.

2. The maximal automorphic factor of T. We recall that a nonsingular endomorphism T of $(X, \mathcal{B}, \mu)$ is said to be *exact* if the tail σ-algebra $\bigcap_{n \geq 0} T^{-n}\mathcal{B}$ is trivial, i.e., if it consists of $\{\emptyset, X\}$ mod sets of μ measure 0 (cf. [3]). If T is not exact, there exist nontrivial sets in $\mathcal{F} \equiv \bigcap_{n \geq 0} T^{-n}\mathcal{B}$. Note that $T^{-1}\mathcal{F} = \mathcal{F}$

(μ mod 0); this property defines an *automorphic factor.* We denote by $(Y, \mathcal{F}, \nu; \Phi)$ the factor map induced by T on $\mathcal{F}$; this is the *maximal automorphic factor* of T (cf. [4]).

When we refer to a measure μ' for T as being *tail trivial*, we mean that $\bigcap_{n\geq 0} T^{-n}\mathcal{B} = \{\emptyset, X\}$ mod sets of μ' measure 0; however μ' is not necessarily nonsingular for T. In Section 5 we show that tail trivial measures arise naturally when a nonsingular nonexact measure μ is decomposed over the maximal automorphic factor of T.

In [4], the maximal automorphic factor of a countable-to-one endomorphism was shown to be isomorphic to the quotient relation of the following two natural orbit relations associated to T. We define the amenable equivalence relations R_T and $S_T \subseteq R_T$ (cf. [2] and [6]) as follows:

$$(x,y) \in R_T \subseteq X \times X \Longleftrightarrow T^n x = T^m y \text{ for some } m,\ n \geq 1.$$

We also associate a subrelation $S_T \subseteq R_T$ by:

$$(x,y) \in S_T \Longleftrightarrow T^n x = T^n y \text{ for some } n \geq 1.$$

When the endomorphism T is clearly understood, we write R and S for R_T and S_T. For $x \in X$, let

$$R(x) \equiv \{y \in X : (x,y) \in R\} \text{ and } S(x) \equiv \{y \in X : (x,y) \in S\}.$$

We refer to $R(x)$ as the big orbit of x under T and to $S(x)$ as the lateral orbit of x. One can verify that $S(x) = \bigcup_{n\geq 0} T^{-n}T^n x$ [6]. Similarly we define for each set $A \in \mathcal{B}$,

$$R(A) \equiv \{y : (x,y) \in R \text{ for some } x \in A\},$$

and we say that R is nonsingular provided $\mu(A) = 0 \Leftrightarrow \mu\big(R(A)\big) = 0$ for all $A \in \mathcal{B}$. We say R is ergodic (with respect to μ) provided $R(A) = A \Rightarrow \mu(A) = 0$ or $\mu(X \backslash A) = 0$. We have identical definitions for the subrelation S.

The endomorphism T is one-to-one if and only if S is trivial m-a.e. (each equivalence class consists of exactly one point). For invertible T, R is the usual equivalence relation associated to orbits which is studied in detail in [7] and [10]. For noninvertible T, the following connections were proved in [6] to exist between T and the relations R and S.

1. T is nonsingular $\Leftrightarrow$ R is nonsingular.
2. T is ergodic $\Leftrightarrow$ R is ergodic.
3. T is nonsingular $\Rightarrow$ S is nonsingular (and the converse is false).
4. T is exact $\Leftrightarrow$ S is ergodic.

Also, it is easily checked that:

5. For all $n \in \mathbb{Z}$, $S(T^n x) = T^n(Sx)$ for a.e. $x \in X$.

It was proved in [4] that:

6. $(X, \bigcap_{n\geq 0} T^{-n}\mathcal{B}, \mu) \simeq R_T/S_T$

The following proposition shows that the measure μ on X can always be assumed to be nonatomic if T is n-to-one and conservative. This proposition is false if $n = 1$.

Proposition 2.1. *If $n \geq 2$ and T is an n-to-one endomorphism on $(X, \mathcal{B}, \mu)$ which is ergodic, conservative, and nonsingular, then μ is a nonatomic measure.*

Proof. Suppose that μ is an atomic measure; then there exists some $p \in X$ such that $\mu(\{p\}) > 0$. By ergodicity the support of μ, denoted $\mathrm{supp}(\mu)$, satisfies $\mathrm{supp}(\mu) \subseteq R_T(p)$; by nonsingularity of T and R_T we have $\mathrm{supp}(\mu) = R_T(p)$. This implies that for every $x \in S_T(p)$ we have $\mu(x) > 0$, and by conservativity of T there exists a smallest $m \in \mathbb{N}$ such that $\mu(T^{-m}x \cap x) > 0$; i.e., $T^m x = x$. Similarly we have a smallest $m' \geq 1$ such that $T^{m'}p = p$. However, by our choice of x, for some $r \geq 1$, $T^r x = T^r p$, so the forward orbit of x agrees with the forward orbit of p after some finite number of iterates. This means that the set

$$\{T^q p\}_{q\geq r} = \{p, Tp, \ldots, T^{m'-1}p\} \equiv \{p_0, \ldots, p_{m'-1}\}$$

must be the same as the set

$$\{T^q x\}_{q\geq r} = \{x, Tx, \ldots, T^{m-1}x\} \equiv \{x_0, \ldots, x_{m-1}\},$$

so $m = m'$. Furthermore, if we reorder each set so that $p_0 = T^r p$, $p_1 = T^{r+1}p, \ldots,$ etc., and $x_0 = T^r x$, $x_1 = T^{r+1}x$, etc., then we see that $p_i = x_i$ for i$= 0, \ldots, m-1$, so $x = p$. Therefore $S_T(p) = \{p\}$, and T is not n-to-one. This contradiction shows that μ is nonatomic. □

3. The Rohlin factors of T. The authors showed in [4] that the factor $(Y, \mathcal{F}, \nu; \Phi)$ of the endomorphism $(X, \mathcal{B}, \mu; T)$ is isomorphic to the quotient relation R_T/S_T for T. In this section we look at a related factor which we call a Rohlin factor of X, and show in what sense it represents the kernel of Y, or S_T. We assume from now on that T is n-to-one. We describe in more detail the partitions defined in Section 1.

Definition 3.1. Let ε denote the point partition of X. Let $\mathcal{P}_1 = \{A_0, A_1, \dots, A_{n-1}\}$ be a partition of X into measurable subsets such that $T|_{A_i} = T_i$ is a bijection a.e. between A_i and X. The partition $\mathcal{P}_1$ is defined as in [13] to separate points of $T^{-1}\varepsilon$; i.e., $\mathcal{P}_1 \vee T^{-1}\varepsilon = \varepsilon$, where $\mathcal{P} \vee \mathcal{Q}$ is the partition consisting of intersections of sets in $\mathcal{P}$ and $\mathcal{Q}$. The choice of $\mathcal{P}_1$ is not unique, though Parry gives a canonical (nonunique) method of choosing it [12]. We call $\mathcal{P}_1$ a *Rohlin partition* for T. We define the increasing sequence of partitions $\mathcal{P}_1 \leq \mathcal{P}_2 \leq \cdots$, by $\mathcal{P}_i \equiv \mathcal{P}_1 \vee T^{-1}\mathcal{P}_1 \vee \cdots \vee T^{-(i-1)}\mathcal{P}_1$; clearly $\mathcal{P}_i \vee T^{-i}\varepsilon = \varepsilon$. The *Rohlin factor* associated to $\mathcal{P}_1$ is the factor of $(X, \mathcal{B}, \mu)$ associated to the measurable partition $\bigvee_{i \geq 1} \mathcal{P}_i \equiv \mathcal{P}$. We remark that $\mathcal{P}$ is defined to be the smallest common refinement of all $\mathcal{P}_i$ [13], and satisfies $T^{-1}\mathcal{P} \leq \mathcal{P}$.

Some partial results about uniqueness of Rohlin factors are given in Propositions 3.7 and 3.9. However all Rohlin partitions share some common properties.

Lemma 3.2. *For any Rohlin partition $\mathcal{P}_1 = \{A_0, A_1, \dots, A_{n-1}\}$, for any $k \geq 1$, let $\mathcal{P}_k = \mathcal{P}_1 \vee T^{-1}\mathcal{P}_1 \vee \cdots \vee T^{-(k-1)}\mathcal{P}_1$. Then every atom of $\mathcal{P}_k$ is of the form*

$$A_{i_0 i_1 \cdots i_{k-1}} = T_{i_0}^{-1} \circ T_{i_1}^{-1} \circ \cdots \circ T_{i_{k-1}}^{-1} X = A_{i_0} \cap T^{-1} A_{i_1} \cap \cdots \cap T^{-(k-1)} A_{i_{k-1}}$$

with $i_j \in \{0, 1, \dots, n-1\}$ for $0 \leq j \leq k$, and satisfies:

(1) *T^k restricted to $A_{i_0 i_1 \dots i_{k-1}}$ is one-to-one and onto X (i.e., $\mathcal{P}_k$ is a Rohlin partition for T^k);*
(2) *$TA_{i_0 i_1 \dots i_k} = A_{i_1 \dots i_k}$ $(TA_{i_0} = X)$;*
(3) *$A_{i_0 \dots i_k} = T_{i_0}^{-1} A_{i_1 \dots i_k}$;*
(4) *$x \in A_{i_0 \dots i_k}$ if and only if $T^j x \in A_{i_j}$ for every $0 \leq j \leq k$;*
(5) *The Jacobian function $J_{\mu T^k}$ (defined in Section 1) is independent of the choice of $\mathcal{P}_1$ for all $k \geq 1$;*
(6) *For each $k \geq 0$ and $0 \leq i_0, \dots, i_k \leq n-1$,*

$$\begin{aligned} 0 < \mu(A_{i_0 i_1 \dots i_k}) &= \int_X \frac{1}{J_{\mu T^{k+1}}(T_{i_0}^{-1} T_{i_1}^{-1} \cdots T_{i_k}^{-1} x)} \, d\mu(x) \\ &= \int_X \left[J_{\mu T^{k+1}}(T_{i_k} \circ T_{i_{k-1}} \circ \cdots \circ T_{i_0})^{-1}(x) \right]^{-1} d\mu(x). \end{aligned}$$

If T is n-to-one, we can identify the factor space of X associated to a Rohlin factor with the space of one-sided sequences on n states in the following canonical way. We define

$$Z = \prod_{i=0}^{\infty} \{0, 1, \dots, n-1\}_i,$$

and denote a cylinder of length k by

$$C_{i_0 i_1 \ldots i_{k-1}} \equiv \{z \in Z : z_0 = i_0, z_1 = i_1, \ldots, z_{k-1} = i_{k-1}\}.$$

The cylinders of length ≥ 1 generate $\mathcal{D}$, the σ-algebra of Borel sets on Z. We can write the factor map associated to a Rohlin partition $\mathcal{P}_1$ by $\beta : (X, \mathcal{B}) \to (Z, \mathcal{D})$ by $[\beta(x)]_i = j$ if $T^i x \in A_j \in \mathcal{P}_1$ (using the convention that $T^0 x = x$). Clearly $\beta^{-1}\mathcal{D} \subseteq \mathcal{B}$ and $\beta^{-1}(C_{i_0 i_1 \ldots i_k}) = A_{i_0 i_1 \ldots i_k}$. If by σ we denote the shift map $[\sigma z]_i = z_{i+1}$, then by the definition of β we have that for all $k \in \mathbb{N}$, $[\beta(T^k x)]_i = [\beta x]_{k+i} = [\sigma^k \beta x]_i$. In other words, the following diagram commutes and β is a factor map from $(X, \mathcal{B}, \mu; T)$ to $(Z, \mathcal{D}, \rho; \sigma)$ with $\rho \equiv \mu\beta^{-1}$.

$$\begin{array}{ccc} X & \xrightarrow{T^k} & X \\ \beta \downarrow & & \downarrow \beta \\ Y & \xrightarrow{\sigma^k} & Y \end{array}$$

Since β is completely determined by $\mathcal{P}$, which in turn is determined by the choice of $\mathcal{P}_1$, Definition 3.1 implies $(Z, \mathcal{D}, \rho)$ is isomorphic to the Rohlin factor for $\mathcal{P}_1$. By Proposition 2.1 the measure ρ is continuous; equivalently, for every atom $A \in \mathcal{P}$, $\mu(A) = 0$.

Lemma 3.3. *For μ almost every $x \in X$, $\beta(S_T x) = S_\sigma(\beta x)$.*

Proof. We suppose that $y \in S_T x$; then there exists a minimum $k \geq 1$ with $T^k y = T^k x$. Then $\beta(T^k y) = \beta(T^k x)$ so $\sigma^k(\beta x) = \sigma^k(\beta y)$. It follows that $\beta y \in S_\sigma(\beta x)$.

We now suppose that we are given a $z \in S_\sigma(\beta x)$, $z \neq \beta x$; then there is a minimum $k \geq 1$ with $\sigma^k z = \sigma^k \beta x = \beta T^k x$. This means that there exist two disjoint cylinders of length k, $C_{i_0 i_1 \ldots i_{k-1}} \neq C_{j_0 j_1 \ldots j_{k-1}}$ such that $z \in C_{i_0 i_1 \ldots i_{k-1}}$ and $\beta x \in C_{j_0 j_1 \ldots j_{k-1}}$. This means that $\beta^{-1} z \in A_{i_0 i_1 \ldots i_{k-1}}$, and since $A_{i_0 i_1 \ldots i_{k-1}} \neq A_{j_0 j_1 \ldots j_{k-1}}$, we have $x' \neq x$ for every $x' \in \beta^{-1} z$. Since T^k is one-to-one and onto from $A_{j_0 j_1 \ldots j_{k-1}}$ to X, and $(T_{i_{k-1}} \circ T_{i_{k-2}} \circ \cdots \circ T_{i_0})^{-1}$ is one-to-one and onto from X onto $A_{i_0 i_1 \ldots i_{k-1}}$, we choose

$$x' = (T_{i_{k-1}} \circ T_{i_{k-2}} \circ \cdots \circ T_{i_0})^{-1} \circ T^k x.$$

By our choice of x' it is clear that $T^k x = T^k x'$ so $\sigma^k(\beta x') = \sigma^k z$, and z, $\beta x' \in C_{i_0 i_1 \ldots i_{k-1}}$. But the set $\sigma^{-k}(\sigma^k z)$ intersects each cylinder of length k in exactly one point, so $z = \beta x'$ and $x' \in S_T x$. The lemma is proved. $\square$

Let $(Y,\mathcal{F},\nu;\Phi)$ be the maximal automorphic factor of $(X,\mathcal{B},\mu;T)$. We recall from [4] that this factor is generated by the partition

$$\eta' = \bigcap_{i\geq 0} T^{-i}\varepsilon.$$

We say that the factor $(Z,\mathcal{D},\rho;\sigma)$ of $(X,\mathcal{B},\mu;T)$ is *strictly contained* in $(Y,\mathcal{F},\nu;\Phi)$ if every atom of the partition $\mathcal{P}$ which generates Z is the union of atoms of the partition η' which generates Y. The following proposition characterizes the invertible maps.

Proposition 3.4. *Assume T is a nonsingular n-to-one endomorphism for $n \geq 1$. Then $n = 1$ if and only if some (equivalently every) Rohlin factor $(Z,\mathcal{D},\rho;\sigma)$ is strictly contained in $(Y,\mathcal{F},\nu;\Phi)$.*

Proof. ($\Rightarrow$) If T is an automorphism then $(X,\mathcal{B},\mu;T) = (Y,\mathcal{F},\nu;\Phi)$.

($\Leftarrow$) If Z is a Rohlin factor strictly contained in Y, then each atom of $\mathcal{P}$ is a union of atoms of $\eta' = \bigcap_{i\geq 0} T^{-i}\varepsilon$; then for any $z \in Z$ there exists a measurable subset C of Y such that $\beta^{-1}z = \bigcup_{y\in C}\alpha^{-1}y$. Since $\alpha^{-1}y$ is an S_T-invariant subset of X, it follows that for any $x \in \beta^{-1}z$, for every $x' \in S_T(x) \subseteq \beta^{-1}z$ we must have $\beta(x) = \beta(x')$. However, if $x \neq x'$ satisfies $x' \in S_T(x)$, then $\beta(x) \neq \beta(x')$ by arguments similar to those used in Lemma 3.3. Therefore $S_T(x) = \{x\}$ for μ a.e. x, which implies that $n = 1$. This proves the proposition. □

In Section 5 we will give a list of equivalent necessary and sufficient conditions under which the factor measure ρ is exact with respect to σ for a Rohlin factor Z. To this end, we will consider the join of the factors Y and Z in X. We recall that $\alpha^{-1}\mathcal{F} \equiv \bigcap_{i\geq 0} T^{-i}\mathcal{B}$ is the tail σ-algebra associated to the measurable partition of X given by $\eta' = \bigcap_{i\geq 0} T^{-i}\varepsilon$ and $\beta^{-1}\mathcal{D}$ is the σ-algebra associated to the measurable partition $\mathcal{P} = \bigvee_{i\geq 1}\mathcal{P}_i$. It is easy to show that for any $j \in \mathbb{N}$,

$$\Big(\bigcap_{i=1}^{j} T^{-i}\varepsilon\Big) \vee \Big(\bigvee_{i=1}^{j}\mathcal{P}_i\Big) = \varepsilon \qquad (\mu \bmod 0).$$

Furthermore, we have

$$\cdots \leq \bigcap_{i=1}^{j} T^{-i}\varepsilon \leq \bigcap_{i=1}^{j-1} T^{-i}\varepsilon \leq \cdots \leq T^{-1}\varepsilon \leq \varepsilon$$

and

$$\cdots \geq \bigvee_{i=1}^{j}\mathcal{P}_i \geq \bigvee_{i=1}^{j-1}\mathcal{P}_i \geq \cdots \geq \mathcal{P}_1.$$

Then $\mathcal{P} = \sup \mathcal{P}_i$ and $\eta' = \inf \bigcap_{i=1}^{j} T^{-i}\varepsilon$ and it seems likely that $\mathcal{P} \vee \eta' = \varepsilon$. However, in the case when T is exact, it is not always true that $\mathcal{P} = \varepsilon$ (mod 0). We describe a four-to-one exact endomorphism whose unique Rohlin partition $\mathcal{P}_1$ does not generate $\mathcal{B}$. (The example was given by W. Parry and P. Walters [18] to illustrate a slightly different phenomenon.)

We precede the example with a proposition which gives sufficient conditions under which the Rohlin factor is unique and is a Bernoulli shift.

Proposition 3.5. *Suppose that $T : (X,\mathcal{B},\mu) \to (X,\mathcal{B},\mu)$ is an n-to-one endomorphism with $J_T(x) = n$ a.e. Then T preserves μ, and for any Rohlin partition $\mathcal{P}_1$, the associated Rohlin factor $(Z,\mathcal{D},\rho;\sigma)$ is isomorphic to the $(1/n,\dots,1/n)$ one-sided Bernoulli shift.*

Proof. Clearly $\omega_{\mu T}(x) = 1$, since

$$\frac{1}{\omega_{\mu T}(x)} = \sum_{y\in T^{-1}(Tx)} \frac{1}{J_{\mu T}(y)} = \frac{n}{n} = 1.$$

By the Kolomogorov extension theorem the measure ρ on Z is completely determined by its values on the cylinder sets in $\mathcal{D}$; in particular it is determined by its values on the atoms of $\mathcal{P}_k$ for each $k \geq 1$. By Lemma 3.2 (6) we have that $\mu(A_{i_0 i_1 \dots i_{k-1}}) = 1/n^k$ for any atom $A_{i_0 \dots i_{k-1}} \in \mathcal{P}_k$. Hence for every cylinder set of length k in Z,

$$\rho(C_{i_0 i_1 \dots i_{k-1}}) = \mu(\beta^{-1} C_{i_0 i_1 \dots i_{k-1}}) = \mu(A_{i_0 i_1 \dots i_{k-1}}) = \frac{1}{n^k}.$$

This means that $(Z,\mathcal{D},\rho;\sigma)$ is just the $(1/n,\dots,1/n)$ Bernoulli shift, as claimed. □

Example 3.6. Let $X = \mathbb{R}^2/\mathbb{Z}^2$, the 2-torus, with $\mathcal{B}$ the σ-algebra of Borel sets and μ = 2-dimensional Lebesgue measure. Let T be the endomorphism on X determined by the integer matrix:

$$T = \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}.$$

This gives a four-to-one measure-preserving endomorphsim of X with $J_{\mu T}(x,y) = 4$ for all $(x,y) \in X$. To show that T is exact, we observe that if

$$T_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

then $T = T_1 \circ T_2 = T_2 \circ T_1$. Since T_1 is an automorphism and T_2 is clearly exact, it is easy to see that the tail sets of T are in one-to-one correspondence with the tail sets of T_2, so T is exact. If the Rohlin factor generates $\mathcal{B}$, then, by Proposition 3.5, T is isomorphic to the $(\frac{1}{4},\dots,\frac{1}{4})$ Bernoulli shift (as is T_2). However, one can easily compute that $h_\mu(T) = \log(3+\sqrt{5})$ [17], so T is not isomorphic to T_2.

The same proof gives the following more general version of Proposition 3.5.

Proposition 3.7. *If T is n-to-one and there exists a Rohlin partition*

$$\mathcal{P}_1 = \{A_0, A_1, \dots, A_{n-1}\}$$

such that $J_{\mu T}(x) = a_i$ on each A_i, then T preserves μ and the measure ρ on the Rohlin factor is the i.i.d. Bernoulli measure given by $\{a_1, \dots, a_n\}$.

We calculate the Radon-Nikodym derivative of the factor endomorphism of T on a Rohlin factor. We define, for each $k \geq 1$, $\mathcal{P}_k(x) = A_{i_0 i_1 \dots i_{k-1}}$ if $x \in A_{i_0 i_1 \dots i_{k-1}} \in \mathcal{P}_k$.

Proposition 3.8. *For any Rohlin factor $(Z, \mathcal{D}, \rho; \sigma)$,*

$$\frac{d\rho\sigma^{-1}}{d\rho}(\beta x) = \lim_{k\to\infty} \frac{\mu T^{-1}\big(\mathcal{P}_k(x)\big)}{\mu\big(\mathcal{P}_k(x)\big)}.$$

Proof. Since $(Z, \mathcal{D}, \rho; \sigma)$ is a factor of the probability space $(X, \mathcal{B}, \mu; T)$, we have that

$$\frac{d\rho\sigma^{-1}}{d\rho}(\beta x) = E_\mu\left(\frac{d\mu T^{-1}}{d\mu} \;\middle|\; \mathcal{P}\right)(x)$$

for μ a.e. $x \in X$. We fix x; then for each $k \geq 1$, we have that

$$\begin{aligned} E_\mu\left(\frac{d\mu T^{-1}}{d\mu} \;\middle|\; \mathcal{P}_k\right)(x) &= \frac{1}{\mu\big(\mathcal{P}_k(x)\big)} \cdot \int_X \chi_{\mathcal{P}_k(x)} \cdot \frac{d\mu T^{-1}}{d\mu}(y)\, d\mu(y) \\ &= \frac{1}{\mu\big(\mathcal{P}_k(x)\big)} \cdot \int_X \chi_{\mathcal{P}_k(x)} \circ T d\mu(y) \\ &= \frac{\mu\big(T^{-1}\mathcal{P}_k(x)\big)}{\mu\big(\mathcal{P}_k(x)\big)}. \end{aligned}$$

By the Increasing Martingale Theorem we have,

$$\frac{d\rho\sigma^{-1}}{d\rho}(\beta x) = \lim_{k\to\infty} E_\mu\left(\frac{d\mu T^{-1}}{d\mu} \;\middle|\; \mathcal{P}_k\right)(x) = \lim_{k\to\infty} \frac{\mu\big(T^{-1}\mathcal{P}_k(x)\big)}{\mu\big(\mathcal{P}_k(x)\big)}. \qquad \square$$

Proposition 3.9. *Suppose $\gamma \sim \mu$ on $(X, \mathcal{B})$ satisfies $d\gamma/d\mu = h \in L^1(X, \mathcal{B}, \mu)$. Then for any choice of Rohlin partition $\mathcal{P}_1$, the associated Rohlin factors $(Z, \mathcal{D}, \rho_\mu; \sigma)$ and $(Z, \mathcal{D}, \rho_\gamma; \sigma)$ are isomorphic, and*

$$\frac{d\rho_\gamma\sigma^{-1}}{d\rho_\gamma}(\beta x) = \frac{E_\rho(\hat{h} \mid \sigma^{-1}\mathcal{D})(\sigma^{-1}\beta x)}{\hat{h}(\beta x)} \cdot \frac{d\rho_\mu\sigma^{-1}}{d\rho_\mu}(\beta x)$$

for μ almost everywhere x, where $\hat{h} = d\rho_\gamma/d\rho_\mu = E_\mu(h \mid \mathcal{P})$ a.e.

Proof. We first note that changing to an equivalent measure on X preserves the property that $\mathcal{P}_1$ is a Rohlin partition for T. It suffices to show that $\rho_\mu \sim \rho_\gamma$; then the first statement follows immediately and the second statement follows from [8]. For any set $A \in \mathcal{D}$, $\rho_\mu(A) = \mu(\beta^{-1}A) = 0$ if and only if $0 = \gamma(\beta^{-1}A) = \rho_\gamma(A)$. □

4. The product factors of T. We consider first the join of any factor with the maximal automorphic factor for T, then focus on the specific properties of the join when Z is a Rohlin factor. Let $(X, \mathcal{B}, \mu; T)$ be an endomorphism of $(X, \mathcal{B}, \mu)$ as above. Recall from Section 3 that the measurable partition $\eta' = \inf \bigcap_{i \geq 0} T^{-i}\varepsilon$ satisfies $T^{-1}\eta' = \eta'$; the maximal automorphic factor $(Y, \mathcal{F}, \nu; \Phi)$ is the factor space associated with η'. Let $\alpha : X \to Y$ denote the factor map. Let ζ be any measurable partition of X such that $T^{-1}\zeta \leq \zeta$ mod 0. We will denote by $(Z, \mathcal{D}, \rho; \sigma)$ the factor space associated with ζ and let $\beta : X \to Z$ denote the factor map. (Our choice of notation is justified since later on Z will be restricted to a Rohlin partition.)

Lemma 4.1. *The partition $\eta' \vee \zeta$ is measurable and satisfies $T^{-1}(\eta' \vee \zeta) \leq \eta' \vee \zeta$ mod 0.*

Proof. The measurability of $\eta' \vee \zeta$ follows from [13]. Let $A \in T^{-1}(\eta' \vee \zeta)$; then there exist $C \in \eta'$ and $D \in \zeta$ such that $A = T^{-1}(C \cap D) = T^{-1}C \cap T^{-1}D$. Since $T^{-1}\eta' = \eta'$, then $T^{-1}C \in \eta'$, and $T^{-1}\zeta \leq \zeta$ implies that there exists a family of atoms $\{D_\iota\}_\iota$ in ζ such that $T^{-1}D = \bigcup_\iota D_\iota$. Then $A = (T^{-1}C) \cap \bigcup_\iota D_\iota = \bigcup_\iota (T^{-1}C \cap D_\iota)$, which is a union of atoms of $\eta' \vee \zeta$. □

Let $(\bar{X}, \bar{\mathcal{B}}, \bar{\mu}; \bar{T})$ denote the factor space associated with $\eta' \vee \zeta$. An atom or point in $\bar{X}$ is a set of the form $\bar{x} = C \cap D$ for some $C \in \eta'$ and D$\in \zeta$. Since η' generates the factor Y, C can also be expressed as C$= \alpha^{-1}y$ for some $y \in Y$, and similarly $D = \beta^{-1}z$ for some $z \in Z$. In other words, each $\bar{x} \in \bar{X}$ can be written in a unique way as $\bar{x} = \alpha^{-1}y \cap \beta^{-1}z$. The σ-algebra $\bar{\mathcal{B}} \subseteq \mathcal{B}$ is the subalgebra generated by $\eta' \vee \zeta$, and $\bar{\mu}$ is just the restriction of μ to $\bar{\mathcal{B}}$. The factor endomorphism $\bar{T}$ (on $\bar{\mathcal{B}} \subseteq \mathcal{B}$ in X) is defined as follows [cf. 17] : for each $\bar{x} = C \cap D \in \bar{X}$,

$$\bar{T}(C \cap D) = C' \cap D' \text{ if } T(C \cap D) \subseteq C' \cap D'$$

where C, $C' \in \eta'$ and D, $D' \in \zeta$. $\bar{T}$ (as a map on points in $\bar{X}$) can also be expressed as follows: for $\bar{x} \in \bar{X}$, $\bar{T}(\bar{x}) = \bar{x}'$ where $T(\alpha^{-1}y \cap \beta^{-1}z) \subseteq \bar{x}'$. Since $T(\alpha^{-1}y \cap \beta^{-1}z) \subseteq \alpha^{-1}\Phi y \cap \beta^{-1}\sigma z$, $\bar{x}' = \alpha^{-1}\Phi y \cap \beta^{-1}\sigma z$. We define the canonical projections $\bar{\alpha} : \bar{X} \to Y$ and $\bar{\beta} : \bar{X} \to Z$ by $\bar{\alpha}(\bar{x}) = y$ and $\bar{\beta}(\bar{x}) = z$ where $y \in Y$ and $z \in Z$ are such that $\bar{x} = \alpha^{-1}y \cap \beta^{-1}z$. The following properties follow immediately from the definitions and properties of factor maps.

Proposition 4.2. *The following six assertions hold.*

(1) $\bar{\alpha}(\bar{x}) = \alpha(x)$ *and* $\bar{\beta}(\bar{x}) = \beta(x)$ *for* $x \in X$ *such that* $x \in \bar{x}$ *and* $\bar{\mu}$ *a.e.* $\bar{x}$.
(2) *If* $\bar{x} = \alpha^{-1}y \cap \beta^{-1}z$ *for some* $y \in Y$ *and* $z \in Z$, *then* $\bar{x} = \bar{\alpha}^{-1}y \cap \bar{\beta}^{-1}z$; *i.e., points of* $\bar{X}$ *can be written as* $\bar{\alpha}^{-1}y \cap \bar{\beta}^{-1}z$ *for* $y \in Y$ *and* $z \in Z$.
(3) $\bar{T}(\bar{\alpha}^{-1}y \cap \bar{\beta}^{-1}z) = \bar{\alpha}^{-1}\Phi y \cap \bar{\beta}^{-1}\sigma z$.
(4) $\bar{\alpha} \circ \bar{T} = \Phi \circ \bar{\alpha}$ *and* $\bar{\beta} \circ \bar{T} = \sigma \circ \bar{\beta}$.
(5) $\bar{T}^{-1}\eta' = \eta'$ *and* $\bar{T}^{-1}\zeta \leq \zeta$.
(6) $\bar{\alpha}$, $\bar{\beta}$ *are factor maps for* $(\bar{X}, \bar{\mathcal{B}}, \bar{\mu}; \bar{T})$ *with factor endomorphisms* Φ *and* σ *respectively.*

Let $\{\mu_y\}_{y \in Y}$ and $\{\gamma_z\}_{z \in Z}$ be the disintegrations of μ over the factor spaces Y and Z respectively. Then for any $E \in \mathcal{B}$,

$$\mu(E) = \int_Y \mu_y(E)\, d\nu(y) = \int_Z \gamma_z(E)\, d\rho(z).$$

We now define a measure m on the space $(Y \times Z, \mathcal{F} \times \mathcal{D})$ by: for each $A \in \mathcal{F}$, $B \in \mathcal{D}$,

$$m(A \times B) = \int_A \mu_y(\beta^{-1}B)\, d\nu(y).$$

We extend the measure to all of $\mathcal{F} \times \mathcal{D}$ by the Kolmogorov Extension Theorem (cf. [11]).

Theorem 4.3. *We have*

$$(\bar{X}, \bar{\mathcal{B}}, \bar{\mu}; \bar{T}) \cong (Y \times Z, \mathcal{F} \times \mathcal{D}, m; \Phi \times \sigma).$$

Proof. We define $\psi : (\bar{X}, \bar{\mathcal{B}}, \bar{\mu}, \bar{T}) \to (Y \times Z, \mathcal{F} \times \mathcal{D}, m, \Phi \times \sigma)$ by $\psi(\bar{x}) = (\bar{\alpha}\bar{x}, \bar{\beta}\bar{x})$ i.e.,

$$\psi(\bar{\alpha}^{-1}y \cap \bar{\beta}^{-1}z) = (y, z).$$

From the discussion above, it follows that the map ψ is a measure-preserving isomorphism satisfying $\psi \circ \bar{T} = (\Phi \times \sigma) \circ \psi$. □

Remarks 4.4.

(1) If instead we write $\mu(E) = \int_Z \gamma_z(E)\, d\rho(z)$, then m is also defined by

$$m(A \times B) = \int_B \gamma_z(\alpha^{-1}A)\, d\rho(z).$$

(2) $(Y, \mathcal{F}, \nu; \Phi)$ is the maximal automorphic factor of $(\bar{X}, \bar{\mathcal{B}}, \bar{\mu}; \bar{T})$. □

Let $\{m_y\}_{y \in Y}$ be the disintegration of m on $Y \times Z$ over the factor space Y.

Proposition 4.5. *For every $y \in Y$,*

$$(Y \times Z, \mathcal{F} \times \mathcal{D}, m_y) \cong (\{y\} \times Z, (\mathcal{F} \times \mathcal{D})_y, m_y) \cong (Z, \mathcal{D}, \rho_y).$$

Proof. Using Theorem 4.3, by a slight abuse of notation, we will consider the homomorphisms $\bar{\alpha}$ ($\equiv \bar{\alpha} \circ \psi^{-1}$) and $\bar{\beta}$ ($\equiv \bar{\beta} \circ \psi^{-1}$) to be the canonical projection maps from $Y \times Z$ to Y and Z respectively. For each $y \in Y$, the set $\bar{\alpha}^{-1}y \equiv \{y\} \times Z$ can be regarded as a Lebesgue space by setting $(\mathcal{F} \times \mathcal{D})_y = (\mathcal{F} \times \mathcal{D}) \cap \bar{\alpha}^{-1}y$. The map

$$\bar{\beta}\,|_{\{y\}\times Z}\colon \big(\{y\} \times Z, (\mathcal{F} \times \mathcal{D})_y\big) \to (Z, \mathcal{D})$$

is just the identity map on the Z coordinate.

The measure m_y, which is a measure on $Y \times Z$ with support completely contained in $\{y\} \times Z$, trivially induces a measure on $(Z, \mathcal{D})$. We will denote that measure by ρ_y ($\equiv m_y\bar{\beta}^{-1}$) to distinguish it from the same measure considered on the product space. We also have that $\rho_y = \mu_y\beta^{-1}$; for each $D \in \mathcal{D}$ the map $y \to \rho_y(D)$ is measurable in y. □

We now turn to the specific factor $(Z, \mathcal{D}, \rho; \sigma)$ which is generated by a Rohlin partition $\mathcal{P}_1 = \{A_0, A_1, \dots, A_{n-1}\}$ of X. The collection of sets $C_i = \{z \in Z : z_0 = i\} = \beta(A_i)$ for $0 \le i \le n-1$ determine a Rohlin partition of Z (for σ and ρ) which generates the σ-algebra $\mathcal{D}$. Since $\rho(C_i) = \mu\beta^{-1}(C_i) = \mu(A_i) > 0$, we have that σ is n-to-one with respect to ρ.

Proposition 4.6. *The partition*

$$\bar{\mathcal{P}}_1 = \{Y \times C_i : 0 \le i \le n-1\}$$

is a Rohlin partition for $(Y \times Z, \mathcal{F} \times \mathcal{D}, m; \Phi \times \sigma)$; furthermore $\Phi \times \sigma$ is n-to-one with respect to m.

Proof. Clearly $\bar{\mathcal{P}}_1$ is a measurable partition of $Y \times Z$. Let σ_i denote the map $\sigma\,|_{C_i}\colon C_i \to Z$ defined on $(Z, \mathcal{D})$; then for every $w = (i, w_1, w_2, \dots) \in C_i$, $\sigma_i w = (w_1, w_2, \dots)$ and for $z = (z_0, z_1, \dots) \in Z$, $\sigma_i^{-1}(z) = (i, z_0, z_1, \dots)$. Therefore $\sigma_i^{-1} \circ \sigma_i = \mathrm{Id}_{C_i}$ and $\sigma_i \circ \sigma_i^{-1} = \mathrm{Id}_Z$. It suffices to show that a measurable inverse exists for $\Phi \times \sigma_i$; it follows from above that $(\Phi \times \sigma_i) \circ (\Phi^{-1} \times \sigma_i^{-1}) = \mathrm{Id}_{Y \times Z}$ and $(\Phi^{-1} \times \sigma_i^{-1}) \circ (\Phi \times \sigma_i) = \mathrm{Id}_{Y \times C_i}$, m a.e. Clearly the map $\Phi^{-1} \times \sigma_i^{-1}$ is measurable from $(Y \times Z, \mathcal{F} \times \mathcal{D})$ to $\big(Y \times C_i, \mathcal{F} \times (\mathcal{D} \cap C_i)\big)$, so the result follows immediately. Since $m(Y \times C_i) = \mu(A_i) > 0$, $\Phi \times \sigma$ is n-to-one. The proof is done. □

Corollary 4.7. *$(Z, \mathcal{D}, \rho; \sigma)$ is the Rohlin factor of $(Y \times Z, \mathcal{F} \times \mathcal{D}, m; \Phi \times \sigma)$ associated with the Rohlin partition $\bar{\mathcal{P}}_1$.*

We recall that $\{m_y\}_{y\in Y}$ denotes the disintegration of m over the factor space Y. By the nonsingularity of $S_{\Phi\times\sigma}$ with respect to m, we have that each m_y is a tail trivial measure on $(Y\times Z, \mathcal{F}\times\mathcal{D})$ for which the relation $S_{\Phi\times\sigma}$ is nonsingular (and ergodic) (cf. [4] or [14]).

Proposition 4.8. *For every $y \in Y$, $z \in Z$, we have $S_{\Phi\times\sigma}(y,z) = \{y\}\times S_\sigma z$.*

Proof. For each $n \geq 1$, $(\Phi\times\sigma)^{-n}(\Phi\times\sigma)^n(y,z) = (y,\sigma^{-n}\sigma^n z)$. □

Recalling that tail triviality of ρ_y for σ is equivalent to ergodicity with respect to ρ_y of the relation S_σ, we have the following.

Corollary 4.9. *For every $y \in Y$, ρ_y is an ergodic and nonsingular measure for S_σ on $(Z,\mathcal{D})$.*

Proof. We use the fact that m_y is ergodic and nonsingular for $S_{\Phi\times\sigma}$ on $Y\times Z$, and Propositions 4.5 and 4.8, and the result is proved. □

Remarks 4.10.

(1) Corollary 4.9 does not imply that ρ_y is nonsingular for the map σ on Z. In fact, such nonsingularity does not always occur. This will be discussed further in the next two sections.
(2) The nonsingularity of ρ_y for the relation S_σ plus the fact that $S_\sigma(C_i) = Z$ for all i implies that $\rho_y(C_i) > 0$ for all i. However, since σ is not nonsingular with respect to ρ_y in general, it cannot be said to be n-to-one (cf. Section 1).

The discussion above proves the following main theorem of this section.

Theorem 4.11. *Assume that T is an n-to-one nonsingular ergodic conservative endomorphism of $(X,\mathcal{B},\mu)$. Then T has a factor isomorphic to the Cartesian product of an automorphism Φ with an n-to-one shift map σ on $(Y\times Z, \mathcal{F}\times\mathcal{D}, m)$. The measure m is of the form for any $C \in \mathcal{F}\times\mathcal{D}$:*

$$m(C) = \int_Y \int_{\{z:(y,z)\in C\}} d\rho_y(z)\, d\nu(y)$$

where ν is a nonsingular conservative ergodic measure for the automorphsim Φ, and each ρ_y is a tail trivial measure for σ on Z. Furthermore m is a nonsingular joining for $\Phi\times\sigma$ with respect to the factor measures ν on $(Y,\mathcal{F})$ and ρ on $(Z,\mathcal{D})$ given by:

$$\rho(A) = \int_Y \rho_y(A)\, d\nu(y), \qquad \forall\, A \in \mathcal{D}.$$

The marginal measure ρ is ergodic, nonsingular and conservative for the shift σ on $(Z,\mathcal{D})$.

We conclude this section with a formula for the Radon-Nikodym derviative of the factor induced by T on $Y \times Z$ (cf. [4] for proof).

Proposition 4.12. *We have the identity*

$$\frac{dm(\Phi \times \sigma)^{-1}}{dm}(y,z) = \frac{d\nu\Phi^{-1}}{d\nu}(y) \cdot \frac{d(\rho_{\Phi^{-1}y}\sigma^{-1})}{d\rho_y}(z), \ m \text{ almost everywhere.}$$

5. Exactness properties of the marginal and fiber measues. For every choice of Rohlin partition, the factor measure ρ on the associated Rohlin factor $(Z,\mathcal{D})$ is defined by $\rho(A) = \mu\beta^{-1}(A) = m\bar{\beta}^{-1}(A)$ and disintegrates over Y by

$$\rho(A) = \int_Y \rho_y(A)\ d\nu(y) \equiv \int_Y \mu_y\beta^{-1}(A)\ d\nu(y)$$

for all $A \in \mathcal{D}$. The measure ρ is always nonsingular for σ by the nonsingularity of T with repsect to μ while the ρ_y's are always tail trivial for σ by the tail triviality of μ_y for T. We would like to determine under which conditions ρ reflects various properties of a generic ρ_y; in the strongest case we will see that for ν-a.e. $y \in Y$, ρ_y is equivalent to ρ. This can fail to be true as the example in Section 6 shows, but under weaker hypotheses we still have exactness of ρ. We give necessary and sufficient conditions under which ρ is an exact measure for σ. The second author and Stanley Eigen have constructed examples of n-to-one endomorphisms T which are ergodic but not totally ergodic with nonexact Rohlin factors; in all known examples, ν is an atomic measure.

We say that two factors of X, $(Y,\mathcal{F},\nu)$ and $(Z,\mathcal{D},\rho)$, *overlap* if there is some nontrivial factor $(W,\mathcal{S},\gamma)$ strictly contained in both Y and Z. Otherwise, Y and Z are *nonoverlapping*; in this case, for any $A \in \mathcal{F} \times \mathcal{D}$ such that $A \in \bar{\alpha}^{-1}\mathcal{F}$ and $A \in \bar{\beta}^{-1}\mathcal{D}$ we have $m(A) = 0$ or 1.

Theorem 5.1. *For any Rohlin factor $(Z,\mathcal{D},\rho)$, the following nine properties are equivalent:*

(1) *ρ is an exact measure for σ.*
(2) *The space $(Z,\mathcal{D},\rho)$ contains no nontrivial automorphic factor for σ.*
(3) *The spaces $(Y,\mathcal{F},\nu)$ and $(Z,\mathcal{D},\rho)$ have no nontrivial common automorphic factor.*
(4) *The factors $(Y,\mathcal{F},\nu)$ and $(Z,\mathcal{D},\rho)$ are nonoverlapping.*
(5) *$\forall\ A \in \mathcal{D}$, either $\rho_y(A) = 0$ for ν a.e. $y \in Y$ or $\rho_y(A) > 0$ for ν a.e. $y \in Y$.*
(6) *$\forall\ A \in \bigcap_{n\geq 0}\sigma^{-n}\mathcal{D}$, either $\rho_y(A) = 0$ for ν a.e. $y \in Y$ or $\rho_y(A) = 1$ for ν a.e. $y \in Y$.*
(7) *$\forall\ A \in \mathcal{D}$, either $\rho_y(S_\sigma A) = 0$ for ν a.e. $y \in Y$ or $\rho_y(S_\sigma A) = 1$ for ν a.e. $y \in Y$.*

(8) $\forall\ A \in \mathcal{D}$, *and for* ν *a.e.* $y \in Y$, *either* $\rho_y(A) = 0$ *and* $\rho_y(\sigma^{-1}A) = 0$, *or* $\rho_y(A) > 0$ *and* $\rho_y(\sigma^{-1}A) > 0$.

(9) $\forall\ A \in \mathcal{D}$, *either* $\rho_y(A) = 0$ *and* $\rho_y(\sigma^{-1}A) = 0$ *for* ν *a.e.* $y \in Y$, *or* $\rho_y(A) > 0$ *and* $\rho_y(\sigma^{-1}A) > 0$ *for* ν *a.e.* $y \in Y$.

Proof. (1) $\Leftrightarrow$ (2): The tail sub-σ-algebra of $(Z, \mathcal{D})$ is the maximal automorphic factor for σ. Therefore the measure ρ is exact for σ if and only if Z has a trivial tail sub-σ-algebra if and only if Z has no nontrivial automorphic factor.

(2) $\Rightarrow$ (3) trivially.

(4) $\Rightarrow$ (1): Let $A \in \bigcap_{n\geq 0} \sigma^{-n}\mathcal{D}$. Then

$$\bar{\beta}^{-1}A \in \bigcap_{n\geq 0} (\Phi \times \sigma)^{-n}(\mathcal{F} \times \mathcal{D}) = \bar{\alpha}^{-1}\mathcal{F},$$

i.e., $\bar{\beta}^{-1}A \in \mathcal{F} \times \mathcal{D}$ belongs to $\bar{\alpha}^{-1}\mathcal{F}$ and $\bar{\beta}^{-1}\mathcal{D}$. Since Y and Z are nonoverlapping in $Y \times Z$ it follows that $\rho(A) = m(\bar{\beta}^{-1}A) = 0$ or 1. Therefore, ρ is exact.

(3) $\Rightarrow$ (2): The automorphic factors for T on $(X, \mathcal{B}, \mu)$ are ordered by inclusion [13], with Y the maximal automorphic factor for T. Any automorphic factor of σ in Z is an automorphic factor of T since σ is just the factor endomorphism of T on Z. Consequently, if Z has a nontrivial automorphic factor for σ, it is contained in Y.

(9) $\Rightarrow$ (5) is trivial.

(5) $\Rightarrow$ (4): Suppose that (5) holds and Y and Z both strictly contain the factor $(W, \mathcal{S}, \gamma)$. Let π_Z and π_Y denote the factor maps from Z and Y to W. Then there exists some set $D \in \mathcal{S}$ with $0 < \gamma(D) = \rho({\pi_Z}^{-1}D) = \nu({\pi_Y}^{-1}D) \leq 1$. Since $\rho({\pi_Z}^{-1}D) > 0$, (5) implies that for ν a.e. $y \in Y$, $\rho_y({\pi_Z}^{-1}D) > 0$; that is, $\nu\{y : \mu_y(\beta^{-1}{\pi_Z}^{-1}D) > 0\} = 1$. But

$$\begin{aligned}\nu\{y : \mu_y(\beta^{-1}{\pi_Z}^{-1}D) > 0\} &= \nu\{y : \ \mu_y(\alpha^{-1}{\pi_Y}^{-1}D) > 0\} \\ &= \nu\{y : \mu_y(\alpha^{-1}{\pi_Y}^{-1}D) = 1\} = 1,\end{aligned}$$

since $\mu_y(\alpha^{-1}{\pi_Y}^{-1}D) = \chi_{\pi_Y^{-1}D}(y)$. This implies that $\gamma(D) = 1$, so the factor is trivial.

(5) $\Rightarrow$ (9): By [14], for each $A \in \mathcal{D}$ the set $Y_A \equiv \{y : \rho_y(A) > 0\}$ is measurable; so is the set $\Phi^{-1}Y_A = \{y : \rho_{\Phi y}(A) > 0\} = \{y : \rho_y(\sigma^{-1}A) > 0\} = Y_{\sigma^{-1}A}$. By (5), $\nu(Y_A) = 1$ for all A such that $\rho(A) > 0$; by nonsingularity of Φ, $\nu(Y_{\sigma^{-1}A}) = 1$ as well. Consequently $1 = \nu(Y_A \cap Y_{\sigma^{-1}A}) = \nu\big(\{y : \rho_y(A) > 0 \text{ and } \rho_y(\sigma^{-1}A) > 0\}\big)$. A similar statement holds if $\rho(A) = 0$, so (9) is proved.

(9) $\Rightarrow$ (8) is trivial.

(8) $\Rightarrow$ (5): As defined above, for each $A \in \mathcal{D}$ the sets

$$Y_A \equiv \{y : \rho_y(A) > 0\},$$
$$Y_A^0 \equiv \{y : \rho_y(A) = 0\},$$
$$\Phi^{-1}Y_A = Y_{\sigma^{-1}A} = \{y : \rho_y(\sigma^{-1}A) > 0\},$$
$$\Phi^{-1}Y_A^0 = Y_{\sigma^{-1}A}^0 = \{y : \rho_y(\sigma^{-1}A) = 0\}$$

are all measurable sets in Y. By (8) we have that

$$\nu\big(\{(Y_A \cap Y_{\sigma^{-1}A}) \cup (Y_A^0 \cap Y_{\sigma^{-1}A}^0)\}\big) = 1.$$

To show (5), for each $A \in \mathcal{D}$ we define on Y the measurable function $F_A(y) = 0$ if $\rho_y(A) = 0$ and $F_A(y) = 1$ if $\rho_y(A) > 0$. The assumption implies that $F_A(\Phi y) = F_A(y)\nu$ a.e., by ergodicity of Φ, F_A is constant ν a.e., so (5) holds.

(6) $\Leftrightarrow$ (7): This follows immediately since $\{S_\sigma A : A \in \mathcal{D}\} = \bigcap_{n\geq 0} \sigma^{-n}\mathcal{D}$ (mod 0) with respect to every ρ_y.

(1) $\Rightarrow$ (6): Suppose that $A \in \bigcap_{n\geq 0}\sigma^{-n}\mathcal{D}$. By exactness of ρ, either $\rho(A) = 0$ or $\rho(A) = 1$. If $\rho(A) = 0$, then clearly $\rho_y(A) = 0$ for ν a.e. y. Let $\rho(A) = 1$; by tail triviality of each ρ_y, for each $y \in Y$, either $\rho_y(A) = 0$ or $\rho_y(A) = 1$. Let $\mathcal{N} = \{y \in Y : \rho_y(A) = 0\}$; then $1 = \rho(A) = \int_{Y\setminus\mathcal{N}} \rho_y(A)\, d\nu(y) = \nu(Y \setminus \mathcal{N})$, so $\rho_y(A) = 1$, ν a.e.

(7) $\Rightarrow$ (5): Let $A \in \mathcal{D}$; by the nonsingularity of S_σ with respect to each ρ_y, if $\rho_y(S_\sigma A) = 0$ for ν a.e. y then $\rho_y(A) = 0$ for ν a.e. y. If $\rho_y(S_\sigma A) = 1$ for ν a.e. y then $\rho_y(A) > 0$ for ν a.e. y. Since (7) says that one or the other holds, (5) is proved.

(6) $\Rightarrow$ (1): Let $A \in \bigcap_{n\geq 0}\sigma^{-n}\mathcal{D}$ with $\rho(A) > 0$. It suffices to show that $\rho(A) = 1$. By (2) we must have that $\rho_y(A) > 0$ for ν a.e. y, and tail triviality of ρ_y implies that $\rho_y(A) = 1$ for these y. Thus $\rho(A) = 1$. The proof of Theorem 5.1 is complete. □

We now define

$$\Sigma = \{y \in Y : \rho_y \sim \rho_y\sigma^{-1}\}.$$

One can show, using the nonsingularity of S_σ, that

$$\Sigma = \{y : \sigma \text{ is (both forward and backward) nonsingular with respect to } \rho_y\}.$$

It is clear from Theorem 5.1 (8) and (9) that if Σ has full measure in Y, then ρ is exact. However we give an example in Section 6 to show that nonsingularity of σ for ν a.e. ρ_y is not a necessary condition for exactness of ρ, i.e., conditions (8) and (9) in Theorem 5.1 are strictly weaker than nonsingularity of σ with respect to ν a.e. ρ_y.

Lemma 5.2. *The set Σ satisfies $\Phi^{-1}(\Sigma) = \Sigma$; if $\Sigma \in \mathcal{F}$, then $\nu(\Sigma) = 0$ or $\nu(\Sigma) = 1$.*

Proof. We first show that $\Sigma \subseteq \Phi^{-1}\Sigma$. Suppose $y \in \Sigma$; then $\rho_y \sim \rho_y\sigma^{-1} \sim \rho_{\Phi y}$. To show $\Phi y \in \Sigma$ as well, it is enough to show that $\rho_y\sigma^{-1} \sim \rho_y\sigma^{-2}$. This follows from nonsingularity of σ for ρ_y.

We now assume that $y \in \Phi^{-1}\Sigma$; then $\Phi y \in \Sigma$, so σ is nonsingular with respect to $\rho_{\Phi y}$. Since $\rho_{\Phi y} \sim \rho_y\sigma^{-1}$, it follows that σ is nonsingular with respect to $\rho_y\sigma^{-1}$. By nonsingularity of S_σ, $\rho_y\sigma^{-1}(A) = 0 \Leftrightarrow \rho_y\sigma^{-1}\sigma(A) = 0 \Leftrightarrow \rho_y(A) = 0$, so $\rho_y \sim \rho_y\sigma^{-1}$, hence $y \in \Sigma$. If $\Sigma \in \mathcal{F}$, the ergodicity of Φ implies the last statement. □

Corollary 5.3. *If Σ contains any $B \in \mathcal{F}$ of positive measure, then $\Sigma \in \mathcal{F}$ and $\nu(\Sigma) = 1$ and ρ is exact.*

We state a theorem giving necessary and sufficient conditions under which the factor we have constructed is a product (i.e., m is equivalent to product measure). The equivalence of (1) and (2) appears in a paper by G. Brown and A. Dooley [1].

Theorem 5.4. *The following are equivalent:*

(1) *For ν a.e. $y \in Y$, $\rho_y \sim \rho$;*
(2) *m is equivalent to the product measure $\nu \times \rho$;*
(3) *The factor endomorphism of $(X, \mathcal{B}, \mu; T)$ induced on $(Y \times Z, \mathcal{F} \times \mathcal{D}, m)$ is isomorphic to the Cartesian product, with product measure, of an exact n-to-one endomorphism and an ergodic automorphism.*

Proof. (1) $\Rightarrow$ (2): For each $C \subseteq Y \times Z$, the function $\delta_C(y) = m_y(C)$ is measurable on Y. Since $m_y(C) = \rho_y(C_y)$ with $C_y = \{z : (y,z) \in C\} \in \mathcal{D}$, we have $\delta_C(y) = \rho_y(C_y)$ is a measurable real-valued function on Y. We also have that $\delta'_C(y) = \rho(C_y)$ is a measurable real-valued function on Y, by Fubini's Theorem applied to $Y \times Z$ with $\nu \times \rho$. Assumption (1) says that for each fixed $C \in \mathcal{F} \times \mathcal{D}$, for ν a.e. $y \in Y$, $\delta_C(y) = 0$ if and only if $\delta'_C(y) = 0$. Our notation allows us to write, for every measurable $C \subseteq Y \times Z$,

$$m(C) = \int_Y m_y(C)\, d\nu(y) = \int_Y \delta_C(y)\, d\nu(y),$$

and

$$\nu \times \rho(C) = \int_Y \rho(C_y)\, d\nu(y) = \int_Y \delta'_C(y)\, d\nu(y).$$

We now have that

$$0 = \int_Y \delta_C(y)\, d\nu(y) \Longleftrightarrow \int_Y \delta'_C(y)\, d\nu(y) = 0,$$

and this implies that $m \sim \nu \times \rho$.

(2) $\Rightarrow$ (3): We have shown that $\bar{T}$ on $(\bar{X}, \bar{\mathcal{B}}, \bar{\mu})$ is isomorphic to $\Phi \times \sigma$ on $(Y \times Z, \mathcal{F} \times \mathcal{D}, m)$. Now (2) implies that $\bar{T}$ is isomorphic to $\Phi \times \sigma$ on $(Y \times Z, \mathcal{F} \times \mathcal{D}, \nu \times \rho)$. To show that ρ is exact, we will prove that Theorem 5.1 (5) holds. For any $A \in \mathcal{D}$, $\rho(A) = 0$ implies trivially that $\rho_y(A) = 0$ for almost every y. We assume now that $\rho(A) > 0$, and set $\tilde{A} = \{y : \rho_y(A) = 0\}$. Since $m(\tilde{A} \times A) = \int_{\tilde{A}} \rho_y(A)\, d\nu(y) = 0$, then by hypothesis, $\nu \times \rho(\tilde{A} \times A) = 0$, so $\nu(\tilde{A}) = 0$. Therefore $\rho_y(A) > 0$ ν a.e.

(3) $\Rightarrow$ (1): We can assume without loss of generality that $\bar{T} = \Phi \times \sigma$ with Φ an automorphism of $(Y, \mathcal{F}, \nu)$ and σ an exact endomorphism of $(Z, \mathcal{D}, \rho)$. Then clearly R_T / S_T gives Φ on Y so each $\rho_y = \rho$. □

Corollary 5.5. *Under any of the equivalent assumptions of Theorem 5.4,*

$$\Sigma = Y \qquad (\nu \bmod 0).$$

Proof. Assume that statement 1 of Theorem 5.4 holds and define $B = \{y \in Y : \rho_y \sim \rho\}$. If $y \in B$, then $y \in \Sigma$, since ρ is a nonsingular measure for σ. Since $\nu(B) = 1$ by hypothesis, the completeness of ν implies $\Sigma \in \mathcal{F}$ and $\nu(\Sigma) = 1$. □

Remark 5.6. In an earlier version of the paper the authors posed the following question. Is every ergodic nonsingular shift measure ρ on the n-state space $(Z, \mathcal{D})$ which gives positive measure to every finite length cylinder also ergodic for the odometer on the same space? If ρ is a product measure, then the answer is yes. Of course, in general ρ cannot be assumed to be a product measure; by results of G. Brown and A. Dooley on ergodic odometer measures [1], an equivalent question is whether every measure with the above mentioned properties for the shift is of weak product type. Examples have been constructed by Stanley Eigen and the second author of Rohlin factors $(Z, \mathcal{D}, \rho)$ for which ρ is not exact. The examples give explicit measures which are ergodic for the shift but not for the odometer. A paper on this is in preparation.

6. An example of an endomorphism with non-product measure on the product space. In this section we construct an endomorphism whose product decomposition into an automorphism and a family of exact endomorphisms does not give a measure equivalent to any product measure. The existence of a non-product measure example, due to W. Parry and P. Walters, is asserted in [17] but does not appear there.

We will construct a two-to-one endomorphism T on the product space

$$Y \times Y^+ \equiv \prod_{n=-\infty}^{\infty} \{0,1\}_n \times \prod_{k=0}^{\infty} \{0,1\}_k$$

with the following properties:

(1) T is the product of the two-sided shift with the one-sided shift;
(2) T is constructed to be finite measure-preserving and ergodic with respect to the following measure;
(3) On Y we put $\nu =$ the $(\frac{1}{2}, \frac{1}{2})$ i.i.d. Bernoulli measure (we denote by $\mathcal{B} \times \mathcal{B}^+$ the product Borel σ-algebra on $Y \times Y^+$), and the measure on $Y \times Y^+$ is of the form

$$\mu(C) = \int_Y \rho_y(C \cap \alpha^{-1} y)\, d\nu(y).$$

We specify the measures ρ_y on $(Y^+, \mathcal{B}^+)$ according to the following algorithm. We fix any $\lambda \in (0,1)$. We define two measures ρ^0 and ρ^1 on the space $\{0,1\}$ by $\rho^0(0) = \rho^0(1) = \frac{1}{2}$, and $\rho^1(0) = 1/(1+\lambda)$, $\rho^1(1) = \lambda/(1+\lambda)$. For each $y \in Y$, we define ρ_y to be the infinite product measure given by

$$\rho_y = \prod_{i=0}^{\infty} \rho^{y_i}.$$

That is, we consider $y = (\dots, y_{-1}, y_0, y_1, \dots, y_n, \dots)$ and the ith factor in the measure ρ_y is ρ^j if and only if $y_i = j$. Each ρ_y is an infinite product of factors of two different measures on Y^+ and, by results of Kakutani [9], for ν a.e. $y \in Y$, ρ_y is singular with respect to the shift σ.

As mentioned above, the invertible action we put on Y is the two-sided shift. To avoid confusion, we denote this invertible map Φ as in Section 2; then $(\Phi y)_i = y_{i+1}$. It follows that

$$\rho_{\Phi y} = \prod_{i=0}^{\infty} \rho^{y_{i+1}} = \prod_{i=1}^{\infty} \rho^{y_i}.$$

We now restrict our attention to a single fiber $\{y\} \times Y^+$ with ρ_y a measure on $\mathcal{D}$, and we compute $\rho_y \sigma^{-1}$. Suppose $C \in \mathcal{D}$ is any cylinder set; then we can write

$$C = \{z \in Y^+ : z_0 = i_0, z_1 = i_1, \dots, z_n = i_n\},$$

and

$$\sigma^{-1}(C) = \{z : z_0 = 0,\ z_1 = i_0, \dots, z_{n+1} = i_n\}$$
$$\cup \{z : z_0 = 1,\ z_1 = i_0, \dots, z_{n+1} = i_n\}.$$

Therefore $\rho_y(\sigma^{-1}C)$ is equal to

$$\left[\rho^{y_0}(0)\cdot\rho^{y_1}(i_0)\cdot\rho^{y_2}(i_1)\cdots\rho^{y_{n+1}}(i_n)\right]+\left[\rho^{y_0}(1)\cdot\rho^{y_1}(i_0)\cdot\rho^{y_2}(i_1)\cdots\rho^{y_{n+1}}(i_n)\right]$$
$$=\prod_{j=0}^{n}\rho^{y_{j+1}}(i_j)=\rho_{\Phi y}(C).$$

Since an infinite product measure is completely determined by its values on cylinder sets, we have that the two measures are equal. Consequently,

$$\frac{d\rho_y\sigma^{-1}}{d\rho_{\Phi y}}=1 \text{ and } \frac{d\nu\Phi}{d\nu}=1, \text{ almost everywhere,}$$

so $T=\Phi\times\sigma$ is measure preserving by Proposition 2.11 in [4] and hence conservative. To prove that T is ergodic, we first remark that computing R_T, S_T, and R_T/S_T will give the same product structure we started with. That T is tail trivial for each ρ_y then follows immediately. We next note that any T-invariant set is contained in the tail of $\mathcal{B}$, where $\mathcal{B}$ denotes the product σ-algebra on $Y\times Y^+$. This implies that if $T^{-1}A=A$ mod 0, then A is of the form $A=B\times Y^+$ where B is some measurable set in Y. The ergodicity of Φ on Y with respect to ν implies that $\nu(B)=0$ or 1; the ergodicity of T with respect to μ follows immediately.

If we compute the Rohlin factor measure $\rho=\int_Y\rho_y$, we see that ρ is the infinite product measure determined by the probability vector $\left(\frac{3+\lambda}{4(1+\lambda)},\frac{1+3\lambda}{4(1+\lambda)}\right)$. That is, σ on $(Y^+,\mathcal{B}^+,\rho)$ is a Bernoulli shift and hence measure preserving and exact. On the other hand, using the notation of Section 5, we see that

$$\nu(\Sigma)=\nu\big(\{y:\rho_y\sim\rho_y\sigma^{-1}\}\big)=0,$$

so, by Corollary 5.7, μ is not equivalent to any measure on $Y\times Y^+$ of the form $\nu\times\gamma$ with γ a measure on Y^+. However, by uniqueness of Y, any other representation of the system $(Y\times Y^+,\mathcal{B}\times\mathcal{B}^+,\mu)$ into the product of an automorphism with an exact endomorphism (with product measure) must be of the form $(Y\times W,\mathcal{B}\times\mathcal{C},\nu\times\gamma)$. This proves the assertion.

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