NONDETERMINISTIC AND STOCHASTIC CELLULAR AUTOMATA AND VIRUS DYNAMICS

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Abstract. We discuss topological dynamical properties of stochastic cellular automata and nondeterministic cellular automata in the context of virus dynamics models. We consider surjectivity and topological transitivity, and we apply our definitions and results to existing models of dynamics that exhibit different behavior and capture properties of HIV and Ebola virus, labelling the behavior as H-dynamics (surjective and topologically transitive) and E-dynamics (neither).

1. Introduction

A cellular automaton (CA) is a continuous shift commuting map of a symbolic shift space on a finite alphabet $\mathcal{A}$, and is a deterministic dynamical system of great interest in mathematics, physics, and computer science. While CAs are used to model complex dynamics and there remain many deep open questions about them, stochastic CA (SCA) and nondeterministic CA (NCA) are closely related to these and provide better models of physical phenomena when great uncertainty or extreme complexity is involved. SCA and NCA are dynamical systems that are homogeneous in time and space, modeling parallel processes like CA, but at each point in time and space, there are multiple local rules to choose from to update a state (always from the same finite list of rules). More precisely an SCA is a dynamical system that randomly chooses one of n different CAs on $\mathcal{A}^{Z^D}$ to apply at each time and at each coordinate [9]. While frequently SCAs are defined probabilistically (see [1] and the references therein), in [9, 3, 2] SCA are defined and used as pointwise maps with measures that give the probabilistic data; as processes the two definitions are equivalent. An NCA is a multivalued continuous shift-commuting map on $\mathcal{A}^{Z^D}$ [4]. We compare SCA (studied eg., in [2, 3, 5]) with NCA [12, 15]; in each setting the underlying dynamical system is based on iterating finitely many cellular automata, chosen independently at each coordinate of the space under consideration, and at each time.

Indeed SCA and NCA are the same object, viewed from different points of view and therefore share topological dynamical properties. The goal in this paper is to prove some topological dynamical differences between the SCA models given by the authors to model HIV and Ebola (EBV), and more generally, virus dynamics. The models originally arose using clinical data, but we are able to articulate the differences in their viral dynamics using some topological dynamical properties. The paper is organized as follows. In Section 2 we give the basic definitions and properties of SCAs and NCAs and show that these dynamical systems are in one-to-one correspondence with each other. In Section 2.4 we

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discuss surjectivity and topological transitivity of SCAs and NCAs, showing connections between them. Section 3 is devoted to a discussion of the application to viral dynamics and proofs of the dynamical properties discussed in Section 2.4. In many existing virus models [2, 3, 10, 7, 8, 18] for example, the dynamics for HIV models are much less predictable and tend to a Markov shift. However the corresponding models for EBV which appear to be similar, either progresses to a healthy configuration or deteriorates quickly to an unsurvivable steady state. We call these H-type and E-type dynamics respectively and we prove results that support their differences using topological transitivity and surjectivity, and discuss the interpretation of these differences in the models.

2. Stochastic and nondeterministic cellular automata

2.1. **Cellular Automata.** We begin with a finite state space $\mathcal{A} = \{0, 1, \ldots, m-1\}$ and for any integer $D \geq 1$ we consider the lattice $\mathbb{Z}^D$, consisting of vectors $\vec{\imath} = (i_1, i_2, \ldots, i_D)$, $i_j \in \mathbb{Z}$ for each $j = 1, \ldots, D$. On $\mathbb{Z}^D$ we define $\|\vec{\imath}\| = \max\{|i_j|, j = 1, \ldots, D\}$. A CA acts on the product space $X = \mathcal{A}^{\mathbb{Z}^D}$. The shift is given coordinatewise by: $\forall x \in X$ and $\vec{\imath} \in \mathbb{Z}^D$,

$$\forall\, \vec{\imath} \in \mathbb{Z}^D, \forall\, x \in X, \quad \sigma_{\vec{\imath}}(x)_{(j_1,\ldots,j_D)} = x_{(j_1+i_1,\ldots,j_D+i_D)}. \tag{2.1}$$

Definition 2.1. A *D*-**dimensional cellular automaton (CA)** is a continuous map F on X such that for every $\vec{\imath} \in \mathbb{Z}^D$, $F \circ \sigma_{\vec{\imath}} = \sigma_{\vec{\imath}} \circ F$.

We define a neighborhood of radius $k \in \mathbb{N} \cup \{0\}$ about $\overline{0} \in \mathbb{Z}^D$, by $N_k = \{\vec{\imath} = (i_1, i_2, \ldots, i_D) : |i_j| \leq k\} = \{\vec{\imath} : \|\vec{\imath}\| \leq k\}$; $|N_k| = (2k+1)^D$. With respect to the metric on X given by: $d_X(x, v) = \dfrac{1}{2^k}$, where $k = \min\{|i| \mid x_{N_i} \neq v_{N_i}\}$, X is compact in the metric topology. Let u denote any fixed $(2k+1)^D$ pattern of symbols from $\mathcal{A}$, arranged in a $(2k+1) \times \cdots \times (2k+1)$ D-cube centered at $\overline{0} \in \mathbb{Z}^D$. These patterns give the basis elements of the topology on X; define $C_{u,k} = \{x \in X : x_{N_k} = u\}$ to be a *cylinder of length* k (centered at $\overline{0}$).

It is a classical result [11] that a map F on X is a CA if and only if there exists an integer $r \geq 0$ and a local rule $f : \mathcal{A}^{(2r+1)^D} \to \mathcal{A}$ such that for every $x \in X$,

$$F(x)_{\vec{\imath}} = f(x_{N_r+\vec{\imath}}).$$

The value of r for the local rule f for F is called the *radius* of F. A point in X is called a *configuration*, and X is the *configuration space*.

2.2. **Stochastic cellular automata.** In the applications of interest there are multiple CAs $F_1, \cdots, F_n$, acting on the same space X. At each iteration, we choose from among these CAs independently and randomly at each lattice site; the choices might not be equally weighted, but the probabilities are the same at each site. For the SCA analyzed in [2, 3, 9] we use a skew product construction over the configuration space to model the random choice made at each coordinate. This approach turns the random selection into a continuous map (on a higher dimensional space); dynamical results are projected down onto the configuration space from the bigger space.

The random choice space. The finite index set $J = \{0, 1, \cdots, n-1\}$ gives $\Omega = J^{\mathbb{N}\cup\{0\}}$ used at each site in our integer lattice $\mathbb{Z}^D$, if we choose randomly from among n different local rules. The random selection at each coordinate is modeled by the dynamical system (Ω, s),

the one sided shift space with metric $d_\Omega(\omega,\zeta) = \frac{1}{2^k}$ where $k = \min\{i \mid \omega_i \neq \zeta_i\}$; we have the shift map:

$$s(\omega)_j = \omega_{j+1}, \forall \omega \in \Omega. \tag{2.2}$$

We next extend (Ω, s) to all coordinates in the space $\mathbb{Z}^D$.

For each $\vec{\imath}$, we set $\Omega_{\vec{\imath}} = \Omega$, and define $\overline{\Omega} = \prod_{\vec{\imath}\in\mathbb{Z}^D} \Omega_{\vec{\imath}}$. Each coordinate of a point $\overline{\omega}$ is given by $\overline{\omega}_{\vec{\imath}} \equiv \omega^{(\vec{\imath})}$, with $\omega^{(\vec{\imath})} = \{\omega_j^{(\vec{\imath})}\}_{j\in\mathbb{N}\cup\{0\}}$ a one-sided sequence from Ω. A basis for the topology of $\overline{\Omega}$ is the collection of cylinders: fix a finite central block $Z \subset \mathbb{Z}^D$, and for each $Z_{\vec{\imath}}$ we specify a length $m_{\vec{\imath}}$ string of symbols from J; denote the resulting cylinder by $\mathcal{C}_Z$, which is a slight abuse of notation.

On $\overline{\Omega}$ we need the shift map of (2.2) coordinatewise, denoted $\overline{s}$, so that $[\overline{s}\,(\overline{\omega})]_j^{(\vec{\imath})} = \omega_{j+1}^{(\vec{\imath})} = [s(\omega^{(\vec{\imath})})]_j$.

The SCA on X. We have n CA's $F_1, \ldots, F_n$, on $X = \mathcal{A}^{\mathbb{Z}^D}$ with associated local rules $f_1, \ldots, f_n$ respectively. We assume each $F_j : X \to X$ has radius at most r. We define the SCA generated by $F_1, F_2, \ldots, F_n$ by defining a local rule $g : \Omega \times \mathcal{A}^{|N_r|} \to \mathcal{A}$, that depends only on the 0^{th} coordinate of ω and finitely many coordinates in X by:

$$g(\omega, x_{N_r}) = \pi_A\left(s(\omega), f_{\omega_0}(x_{N_r})\right) = f_{\omega_0}(x_{N_r}),$$

where π_A denotes projection onto the second coordinate (which is in $\mathcal{A}$).

This allows us to define a random choice of local rule at every x and for each infinite roll of the die $\overline{\omega} \in \overline{\Omega}$ to give the global SCA:

$$[F_{\overline{\omega}}(x)]_{\vec{\imath}} = g(\omega^{(\vec{\imath})}, x_{N_r+\vec{\imath}}) = f_{\omega_0^{(\vec{\imath})}}(x_{N_r+\vec{\imath}}) \in \mathcal{A}. \tag{2.3}$$

For each $\vec{\imath}$, we consider only the coordinate $\omega^{(\vec{\imath})}$ of $\overline{\omega}$ and the coordinate block $x_{N_r+\vec{\imath}}$ to choose and apply one of the n local rules.

For each fixed pair $(\overline{\omega}, x)$ we iterate $F_{\overline{\omega}}(x)$ by: $F^n_{\overline{\omega}}(x) \equiv F_{\overline{s}^{n-1}\overline{\omega}} \circ \cdots \circ F_{\overline{s}\,\overline{\omega}} \circ F_{\overline{\omega}}(x)$.

The skew product structure of an SCA. There is a skew product structure to the SCA given above. We endow the space $Y = (\Omega \times \mathcal{A})^{\mathbb{Z}^D}$ with the product topology, so a point in Y has coordinates $y_{\vec{\imath}} = (\omega^{(\vec{\imath})}, x_{\vec{\imath}})$, for each $\vec{\imath} \in \mathbb{Z}^D$, with $\omega^{(\vec{\imath})} \in \Omega$, $x_{\vec{\imath}} \in \mathcal{A}$. The topology is given by a natural metric [3], making Y a compact metric space. A ball in Y is of the form $B(y,\varepsilon) = \mathcal{C}_Z \times C_{u,k}$, a product of cylinders from $\overline{\Omega}$ and X, and Z and k depend on ε.

Proposition 2.2. [3] Let $F_1, \ldots, F_n$ be n CAs on $X = \mathcal{A}^{\mathbb{Z}^D}$ with associated local rules $f_1, \ldots, f_n$ respectively, where each $F_j : X \to X$ has radius bounded above by r. With the notation above, the map: $\overline{F} : Y \to Y$ defined using the local rule $\overline{g} : (\Omega \times \mathcal{A})^{|N_r|} \to \Omega \times \mathcal{A}$, given by:

$$\overline{g}(\omega^{(N_r)}, x_{N_r}) = (s(\omega^{(\vec{0})}), f_{\omega_0^{(\vec{0})}}(x_{N_r})); \tag{2.4}$$

so that

$$\overline{F}(y)_{\vec{\imath}} = \overline{g}(\omega^{(N_r+\vec{\imath})}, x_{N_r+\vec{\imath}}) = (s(\omega^{(\vec{\imath})}), f_{\omega_0^{(\vec{\imath})}}(x_{N_r+\vec{\imath}}) \tag{2.5}$$

is a continuous shift commuting map of Y.

Moreover, for each $x \in X$, and for each $\overline{\omega} \in \overline{\Omega}$, $[F_{\overline{\omega}}(x)]_{\vec{\imath}} = \pi_{A_{\vec{\imath}}} \circ \overline{F}(y)$, where $\pi_{A_{\vec{\imath}}}$ denotes projection of $X = \mathcal{A}^{\mathbb{Z}^D}$ onto the copy of $\mathcal{A}$ at the $\vec{\imath}^{th}$ coordinate.

We denote by $\pi_X : Y \to X$ the projection map given by $\pi_X(\overline{\omega}, x) = x$. By casting an SCA as a single shift commuting map, many results about CA can be extended to this setting and iteration becomes a simple operation.

Proposition 2.3. [9] For every $x \in X$, $\overline{\omega} \in \overline{\Omega}$, and $y \in (\Omega \times A)^{\mathbb{Z}^d}$ with $y_{\vec{\imath}} = (\omega^{(\vec{\imath})}, x_{\vec{\imath}})$,
(1) $F^n_{\overline{\omega}}(x) = \left\{\pi_{A_{\vec{\imath}}} \overline{F}^n(y)\right\}_{\vec{\imath} \in \mathbb{Z}^d}$;
(2) For all $n, m \in \mathbb{N}$, $F^{n+m}_{\overline{\omega}}(x) = F^n_{\overline{s}^m \overline{\omega}} \circ F^m_{\overline{\omega}}(x)$.

2.3. **Nondeterministic cellular automata.** NCA are multivalued cellular automata, which were studied by Richardson in 1972 [15] (see also [17]), then not explored much further until the recent papers [1] and [4]. We assume X and Y are metric spaces. A *binary relation* $R(X, Y)$ is a subset of $X \times Y$, which defines a *multi-valued map*, or a *multimap* from X to Y, and we use the function notation: $F : X \multimap Y$.

Definition 2.4. If X, Y are metric spaces and $R(X, Y)$ is a binary relation on $X \times Y$, then $F : X \multimap Y$ is given by:
(1) For every $x \in X$, $F(x) = \{y \in Y : (x, y) \in R(X, Y)\}$ and
(2) For every $y \in Y$, $F^{-1}(y) = \{x \in X : (x, y) \in R(X, Y)\}$.

We only consider maps $F : X \multimap Y$ such that $F(x) \neq \emptyset$ for each $x \in X$. Continuity requires a definition in this setting as it is not exactly the same as for single valued maps.

Definition 2.5. [4] If $F : X \multimap Y$ is a multimap, then F is *continuous* if for each $x \in X$, and for any open set $V \subset Y$, with $F(x) \subset V$, there exists $\delta > 0$ such that for all $w \in B(x, \delta)$, $F(w) \subset V$ as well. (F is *upper semicontinuous*); also for any x, and for each open set V, such that $F(x) \cap V \neq \emptyset$, there exists $\delta > 0$ such that for all $w \in B(x, \delta)$, $F(w) \cap V \neq \emptyset$. ($F$ is *lower semicontinuous.*)

An NCA is defined by its local rule, and it was shown in [4] that each NCA is continuous and shift commuting.

Definition 2.6. Let $X = \mathcal{A}^{\mathbb{Z}^D}$ as above. The multimap $F : X \multimap X$ is a *nondeterministic CA* (also called a *nondeterministic sliding block code*) if there exists an integer $r \geq 0$ (the radius) and multimap local rule $f : \mathcal{A}^{(2r+1)^D} \multimap \mathcal{A}$ (with $f(w) \neq \emptyset$ for all $w \in \mathcal{A}^{(2r+1)^D}$) such that for every $x, y \in X$, $y \notin f(x)$ if and only if there exists some $\vec{\imath} \in \mathbb{Z}^D$ such that $y_{\vec{\imath}} \notin f(x_{N_r + \vec{\imath}})$.

The proofs given for shift commuting and continuity of an NCA in [4] are for $D = 1$, but similar proofs work in higher dimensions. In fact a stronger result was proved, characterizing NCAs. For $x \in X$ and multimap $F : X \multimap X$, we set

$$F(x)_{\vec{\imath}} = \{a \in \mathcal{A} \mid \exists z \in F(x),\, z_{\vec{\imath}} = a\}.$$

Proposition 2.7. A multimap $F : \mathcal{A}^{\mathbb{Z}^D} \multimap \mathcal{A}^{\mathbb{Z}^D}$ is an NCA if and only if: F is continuous (upper and lower semicontinuous), for each $\vec{\imath} \in \mathbb{Z}^D$, the shift $\sigma_{\vec{\imath}}$ commutes with F, and for each $x \in X$, $y \in F(x)$ iff for all $\vec{\imath} \in \mathbb{Z}^D$, $y_{\vec{\imath}} \in F(x)_{\vec{\imath}}$.

Definition 2.8. Given an NCA F on X, an *orbit* of $x_0 \in X$ is a sequence $\Theta_F(x_0) = \{x_0, x_1, x_2, \ldots\}$ with $x_i \in F(x_{i-1})$ for all $i \in \mathbb{N}$. The set $\Theta^+_F(x_0)$ denotes the set of all forward orbits of x_0, which can be viewed as all possible paths through configuration space, starting from x_0, under the multimap F. We call $\Theta^+_F(x_0)$ the *grand forward orbit* of x_0.

2.4. **Dynamical properties of SCAs and NCAs.** Even though they act on spaces of different dimension, any NCA can be realized as an SCA and vice versa. While this may be intuitively clear, we outline the proof. Suppose $\overline{F}$ is an SCA on $\overline{\Omega} \times A^{\mathbb{Z}^D}$ generated by deterministic CAs $F_1, \cdots, F_k$ on $\mathcal{A}^{\mathbb{Z}^D}$, with local rules $f_1, \cdots, f_k$, respectively, each having radius $r_i \leq r = \max\{r_1, \cdots, r_k\}$. We define the multimap $f : A^{(2r+1)^D} \multimap A$ by

$$f(u) = \{f_1(u), \cdots, f_k(u)\}. \tag{2.6}$$

Since each f_i is defined on every block $u \in A^{(2r+1)^D}$ and returns an element of A, $f(u) \neq \emptyset$ for each $u \in A^{(2r+1)^D}$. Now, let $F : \mathcal{A}^{\mathbb{Z}^D} \multimap \mathcal{A}^{\mathbb{Z}^D}$ be the resulting NCA given by

$$y \in F(x) \iff \forall\, \vec{\imath} \in \mathbb{Z}^D,\ y_{\vec{\imath}} \in f(x_{\vec{\imath}+N_r}). \tag{2.7}$$

It is not difficult to show that

$$F(x) = \{\pi_X(\overline{F}(\overline{\omega}, x)) : \overline{\omega} \in \overline{\Omega}\} \text{ for all } x \in \mathcal{A}^{\mathbb{Z}^D}. \tag{2.8}$$

For the other direction, we let F be an NCA on $\mathcal{A}^{\mathbb{Z}^D}$ with radius r and local multimap $f : A^{(2r+1)^D} \multimap A$. Since $f(u)$ has finitely many values for each $u \in A^{(2r+1)^D}$ and there are finitely many such u, let $k = \max\{|f(u)| : u \in A^{(2r+1)^D}\}$. For each $u \in A^{(2r+1)^D}$, enumerate $f(u)$ in any way as $v_{u_1}, v_{u_2}, \cdots,\ v_{u_{\ell_u}}$, where $\ell_u \leq k$. Define k deterministic block maps f_j, $1 \leq j \leq k$, on $A^{(2r+1)^D}$ as follows:

$$f_j(u) = \begin{cases} v_{u_j} & \text{if } 1 \leq j \leq \ell_u \\ v_{u_1} & \text{if } \ell_u < j \leq k \end{cases}. \tag{2.9}$$

Then each f_j corresponds to a CA $F_j : \mathcal{A}^{\mathbb{Z}^D} \to \mathcal{A}^{\mathbb{Z}^D}$, and these k CAs in turn generate a stochastic CA $\overline{F}$ on $\overline{\Omega} \times \mathcal{A}^{\mathbb{Z}^D}$, where $\Omega = \{1, \cdots,\ k\}^{\mathbb{N} \cup \{0\}}$, such that Eqn (2.8) holds.

Surjectivity of SCA and NCA. Surjectivity of CAs is extremely important due to the theorems of Moore and Myhill [13, 14], which state that a CA on $\mathcal{A}^{\mathbb{Z}^D}$ is surjective if and only if it is injective on the set of finite configurations (a finite configuration is a point in $\mathcal{A}^{\mathbb{Z}^D}$ with finitely many nonzero entries). We show this result is not true for every NCA (see Rk 3.3(2) below).

Nevertheless an SCA is a surjective map in a higher dimensional space than the corresponding NCA space; i.e., it involves the infinite die roll and the point in configuration space. An SCA is surjective if and only if its corresponding NCA is surjective. In Section 3 we define and show that an H-type SCA is surjective, and while similar in many respects, an E-type SCA is not, highlighting some of the differences between existing virus models that are revealed by their topological dynamics. To justify the assertion above, assume the SCA $\overline{F}$ is surjective, and let $z \in \mathcal{A}^{\mathbb{Z}^D}$ be any configuration. Then there exists $(\overline{\omega}, x) \in (\overline{\Omega} \times A)^{\mathbb{Z}^D}$ with $\overline{F}(\overline{\omega}, x) = (\overline{1}, z)$, where $\overline{1}$ is the element of $\overline{\Omega}$ consisting of a one-sided infinite sequence of all 1s at each lattice site $\vec{\imath} \in \mathbb{Z}^D$. Thus $\pi_X(\overline{F}(\overline{\omega}, x)) = z$, and so $z \in F(x)$ and the NCA F is surjective. Conversely if F is surjective, we let $y = \{(\omega^{(\vec{\imath})}, z_{\vec{\imath}}) : \vec{\imath} \in \mathbb{Z}^D\}$ be an arbitrary element of $(\Omega \times A)^{\mathbb{Z}^D}$. Then as $z = \{z_{\vec{\imath}} : \vec{\imath} \in \mathbb{Z}^D\} \in \mathcal{A}^{\mathbb{Z}^D}$, there exists $x \in \mathcal{A}^{\mathbb{Z}^D}$ with $z \in F(x)$. So, for each $\vec{\imath} \in \mathbb{Z}^D$, $z_{\vec{\imath}} \in f(x_{\vec{\imath}+N_r})$, and thus there exists $m_{\vec{\imath}} \in \{1, \cdots, k\}$ with $z_{\vec{\imath}} = f_{m_{\vec{\imath}}}(x_{\vec{\imath}+N_r})$. Define $\overline{\xi} \in \overline{\Omega}$ by $\xi_0^{(\vec{\imath})} = m_{\vec{\imath}}$ for each $\vec{\imath} \in \mathbb{Z}^D$ and $\xi_t^{(\vec{\imath})} = \omega_{t-1}^{(\vec{\imath})}$ for each $t > 0$ and $\vec{\imath} \in \mathbb{Z}^D$. Then $\overline{F}(\overline{\xi}, x)_{\vec{\imath}} = (s(\xi^{(\vec{\imath})}), f_{\xi^{(\vec{\imath})}_0}(x_{\vec{\imath}+N_r})) = (\omega^{(\vec{\imath})}, z_i)$ for each $\vec{\imath} \in \mathbb{Z}^D$, so that $\overline{F}(\overline{\xi}, x) = y$.

Topological transitivity of SCA and NCA. We assume that X is a complete nonempty metric space.

Definition 2.9. (1) For any continuous map $T : X \to X$, we say T is *topologically transitive* if for any pair of non-empty open sets, $U, V \subseteq X$, there exists $n \geq 1$ with $T^n(U) \cap V \neq \emptyset$.
(2) If $S : X \multimap X$ is a multimap the definition is the same: S is *topologically transitive* if for any non-empty open sets, $U, V \subseteq X$, there exists $n \geq 1$ with $S^n(U) \cap V \neq \emptyset$.
(3) A multimap $S : X \multimap X$ is *strongly transitive* if for any nonempty open sets U, V, the set of points $\{z \in U : \Theta_S(z) \cap V \neq \emptyset\}$ is of the second category

If X is a compact metric space and $T : X \to X$ is continuous, then T topologically transitive implies that T is surjective (see e.g. [16]). For a single transformation T, Defn 2.9 is equivalent to the statement that T has a dense forward orbit, which is sometimes used as the definition (see eg., [16]). In particular, if an SCA $\overline{F} : Y \to Y$ is topologically transitive, then the set of points $\{y = (\overline{\omega}, x) | \overline{F}^n(y) \text{ is dense in } Y\}$ contains a dense G_δ set.

For an NCA $F : X \multimap X$ topological transitivity implies that there is at least one point $x \in X$ such that $\overline{\Theta_F^+(x)} = X$ ([12], Thm 5). However, a topologically transitive multimap does not necessarily have a point x_0 with a single dense forward path (as shown in [12]).

Proposition 2.10. Let $F : \mathcal{A}^{\mathbb{Z}^D} \multimap \mathcal{A}^{\mathbb{Z}^D}$ be a nondeterministic CA and let $\overline{F} : (\Omega \times A)^{\mathbb{Z}^D} \to (\Omega \times A)^{\mathbb{Z}^D}$ be a stochastic CA such that $F(x) = \{\pi_X(\overline{F}(\overline{\omega}, x)) : \overline{\omega} \in \overline{\Omega}\}$ for all $x \in \mathcal{A}^{\mathbb{Z}^D}$. Then F is topologically transitive if $\overline{F}$ is topologically transitive.

Proof. It is enough to show the result on cylinder sets since they form a basis for the topology. Let $u \in A^{(2\ell+1)^D}$ and $v \in A^{(2g+1)^D}$ be patterns giving cylinder sets $C_{u,\ell}$ and $C_{v,g}$. Then $U = \left(\prod_{|\vec{\imath}|<\ell} 1\right) \times C_{u,\ell}$ and $V = \left(\prod_{|\vec{\imath}|\leq g} 1\right) \times C_{v,g}$ are non-empty, open sets in $\overline{\Omega} \times \mathcal{A}^{\mathbb{Z}^D}$, and so there exists $n > 0$ such that $\overline{F}^{-n}(V) \cap U \neq \emptyset$. That is, there exists $y = (\overline{\xi}, x) \in (\Omega \times A)^{\mathbb{Z}^D}$ with $\overline{F}^n y \in V$ and $y \in U$. As $y \in U$, $\pi_X(y) = x \in C_{u,\ell}$, and as $\overline{F}^n(y) \in V$, we have $\pi_X(\overline{F}^n y) = F_{\overline{\xi}}^n x \in C_{v,g}$. And since $F_{\overline{\xi}}^n(x) \in \{F_{\overline{\omega}}^n(x) : \overline{\omega} \in \overline{\Omega}\} = F^n(x)$, we have $F^n(x) \cap C_{v,g} \neq \emptyset$. Therefore, the NCA F is transitive. □

As a corollary we can show if an SCA $\overline{F} : (\Omega \times A)^{\mathbb{Z}^D} \to (\Omega \times A)^{\mathbb{Z}^D}$ is topologically transitive, then the corresponding NCA F is strongly topologically transitive. Given cylinders $\mathcal{C}_{u,k}$, $\mathcal{C}_{v,\ell}$ in X by transitivity of $\overline{F}$, there is a dense G_δ set in $U = \overline{\Omega} \times \mathcal{C}_{u,k}$ with the property that every point has a dense orbit in Y. Each such point has the property that it enters $V = \overline{\Omega} \times \mathcal{C}_{v,\ell}$ infinitely often under $\overline{F}$, so there is a set of second category in $\mathcal{C}_{u,k} = \pi_X(U)$ of points x_0 such that $\{x_0, F(x_0), \cdots, F^j(x_0) \cdots\} = \Theta_F(x_0)$ intersects $\mathcal{C}_{v,\ell}$. Since every open set in X contains a cylinder set, this shows that F is strongly transitive.

A partial converse to Prop 2.10 is not hard to show. If an NCA $F : \mathcal{A}^{\mathbb{Z}^D} \to \mathcal{A}^{\mathbb{Z}^D}$ is strongly topologically transitive, then given open sets $U, V \subset \mathcal{A}^{\mathbb{Z}^D}$ there exists an $n \in \mathbb{N}$ and an $\overline{\omega} \in \overline{\Omega}$ such that the associated SCA $F_{\overline{\omega}}$ defined in (2.3) satisfies $F_{\overline{\omega}}^n U \cap V \neq \emptyset$.

3. Applications to Viral Dynamics

In [3] and [2] SCA models were given to simulate and analyze the dynamics of the Human Immunodeficiency Virus (HIV) in a lymph node and Ebola virus (EBV) in an organ,

respectively. While the models were based on similar SCAs, the outcomes from running the models were very different, each picking up the characteristic timing and variation in outcome of the respective diseases. The differences and similarities are more apparent when we look at the topological dynamical properties of each SCA. There are some common features among viral infections that we list here, so we can incorporate them into our models.

- An initial dose of the virus is delivered to an individual through external means.
- After infection, there is a brief acute illness phase and the virus increases sharply for 1-5 days. After 5 days antibodies have typically developed.
- Virions typically attack and infect contiguous cells. A cell is infected by a contiguous virus particle independently of space and time within an organ or a system; the system infected is specific to the virus.
- The immune system's response is not always adequate and depends on many factors.
- An effective or even partially effective immune response allows a healthy cell(s) to replace an infected cell (site).
- An incomplete response can leave residual areas of virus behind (sanctuary sites).
- If "cured", there remains a possibility of reinfection coming from virus hiding in sanctuary sites of the body.

We model the viral dynamics using symbolic dynamics and SCA. The symbol 0 always represents a cell (or cell site) in a healthy state, and 1 represents a newly infected cell. Depending on the virus, we model the passage of time by using states 2 through $m-1$ to represent still infected cells, finishing with state m being used for a dead cell. For example, when modeling HIV, in [3] we used states $\{0, 1, 2, \dots 6\}$ to mirror clinical data, while in [2] we needed only 4 states to model the faster progressing EBV. Often in the models, state $m-1$ represents an infected cell that is no longer infectious, but this gets reflected in our definition of the CA rules. The viral CAs $\overline{H}$ and $\overline{E}$ are each given by three local rules of radius 1, $h_1,\ h_2,\ h_3$ and $e_1,\ e_2,\ e_3$, respectively, defined below. Under $\overline{H}$, a healthy cell becomes infected if there are infectious sites nearby, infectious states advance one at a time to the dead state, and a dead state can come back either healthy (h_1) or infected (h_2), potentially after a delay (h_3).

$$h_1, h_2, h_3 \begin{pmatrix} * & * & * \\ * & 0 & * \\ * & * & * \end{pmatrix} = \begin{cases} 1 & \text{if any } * \text{ has } 1 \le * < m-1 \\ & \text{or if at least } R \text{ *s are } m-1 \ ; \\ 0 & \text{otherwise} \end{cases}$$

$$h_1, h_2, h_3 \begin{pmatrix} * & * & * \\ * & a & * \\ * & * & * \end{pmatrix} = a+1, \text{ for } a \in \{1, \cdots, m-1\};$$

$$h_1 \begin{pmatrix} * & * & * \\ * & m & * \\ * & * & * \end{pmatrix} = 0, \quad h_2 \begin{pmatrix} * & * & * \\ * & m & * \\ * & * & * \end{pmatrix} = 1, \quad h_3 \begin{pmatrix} * & * & * \\ * & m & * \\ * & * & * \end{pmatrix} = m.$$

Under $\overline{E}$, a healthy cell becomes infected if there are infectious sites nearby, infectious states either are depleted by the system immediately (e_1 and e_3) or progress one state at a time to the dead state (e_2), and a dead state comes back healthy (e_1 and e_2), potentially after a delay (e_3).

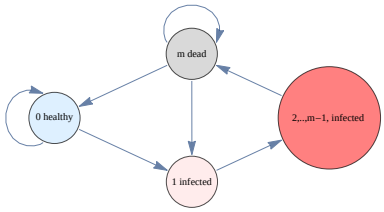

FIGURE 1. A typical adjacency graph for a chronic illness like HIV, which we call H-type dynamics

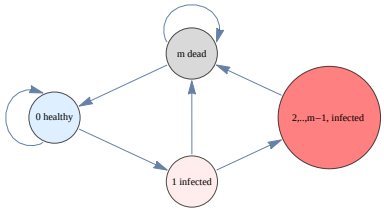

FIGURE 2. A typical adjacency graph for an acute viral illness, which we call E-type dynamics

$$e_1, e_2, e_3\begin{pmatrix} * & * & * \\ * & 0 & * \\ * & * & * \end{pmatrix} = \begin{cases} 1 & \text{if any } * \text{ has } 1 \le * < m-1 \\ & \text{or if at least } R \text{ *s are } m-1 \\ 0 & \text{otherwise} \end{cases};$$

$$e_1, e_3\begin{pmatrix} * & * & * \\ * & a & * \\ * & * & * \end{pmatrix} = m, \quad e_2\begin{pmatrix} * & * & * \\ * & a & * \\ * & * & * \end{pmatrix} = a+1, \text{ for } a \in \{1, \cdots, m-1\};$$

$$e_1, e_2\begin{pmatrix} * & * & * \\ * & m & * \\ * & * & * \end{pmatrix} = 0, \quad e_3\begin{pmatrix} * & * & * \\ * & m & * \\ * & * & * \end{pmatrix} = m.$$

In Figures 1 - 3 we show directed graphs of the connections among states of cells within a given organ that has been infected by a virus. The nodes of the graph represent different states of cells in the organ. A directed arrow is drawn from node i to node j (i could be the same as j) if and only if a cell can progress from state i to j in one time step; a time step is determined by the virus and the organ. Two different types of viral dynamics are illustrated in Figure 1 and 2. The dynamics illustrated in Fig 1 exhibits an infection which leads more often to chronic disease such as HIV and Hepatitis C; we label the resulting SCA $\overline{H}$, and refer to this type of progression as H-type dynamics. The graph shown in Fig 2 provides a better model of the progression of Ebola and the flu for example; we label this SCA by $\overline{E}$ and refer to it as E-type dynamics.

The only difference between the diagrams reflects whether dead or depleted cells can be replaced by cells that are already (or immediately) infected, as shown by the direction of the arrow connecting the state 1 to the state m in each diagram. In H-type dynamics cells can be replaced by infected cells (showing a fast reinfection or virus mutation), while in E-type dynamics depleted cells are replaced with healthy cells even though there could be a delay before replacing them. We implement the change for the graph in Figure 2 by reversing the arrow connecting the state 1 (recently infected) with the last state (m, dead or depleted cell), which causes large differences in both the clinical outcome of the individual

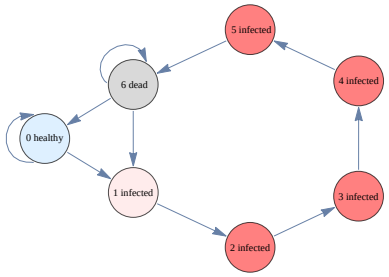

FIGURE 3. The expanded adjacency graph for HIV used in [3]

and the mathematical properties of the resulting SCAs $\overline{H}$ and $\overline{E}$ and the corresponding NCAs, as shown by the next results. While Figure 3 shows the specific adjacency graph used to model HIV in [3], the adjacency diagram does not completely capture the dynamics. A 2-dimensional SCA is needed as the state of a cell at any given moment depends on the state of its nearest neighbors in addition to its current state.

In [3] it was observed that in the long-term, the SCA model for HIV appears to progress to a Markov shift at each coordinate independently. Here as a partial support of that claim, we show that every configuration in $\mathcal{A}^{\mathbb{Z}^2}$ can be reached via the SCA by showing that $\overline{H}$ is topologically transitive and onto.

The idea of the proof of transitivity is to construct a point which agrees with coordinates determined by our choice of V for as long as needed and has the specified central block surrounded by consecutive rings of 0s, 2s, 0s, so that after enough time passes, the coordinates determined by the set U each progress to an m; then, by judicious choice of rule h_1, h_2, or h_3, those coordinates will progress to the starting configuration of U.

Theorem 3.1. *The SCA for the H-type dynamics, $\overline{H} : Y \to Y$, is topologically transitive.*

Proof. To show topological transitivity, we only need to prove it on cylinder sets. We consider $u \in A^{(2k+1)\times(2k+1)}$ for some $k \geq 0$, $v \in A^{(2\ell+1)\times(2\ell+1)}$ for some $\ell \geq 0$, $\mu^{i,j} \in \Omega^q$ for $|i|, |j| \leq k$ and some $q \geq 0$, and $\nu^{i,j} \in \Omega^r$ for $|i|, |j| \leq \ell$ and some $r \geq 0$, so that

$$U = \bigcap_{|i|,|j|\leq k} \mu^{i,j} \times C_{u,k} \text{ and } V = \bigcap_{|i|,|j|\leq \ell} \nu^{i,j} \times C_{v,\ell}$$

are two cylinder sets in Y. Construct $y = (\overline{\omega}, x) \in \overline{\Omega} \times \mathcal{A}^{\mathbb{Z}^2}$ by letting $M = \max\{k, \ell\}$ and setting

$$x_{i,j} = \begin{cases} v_{i,j} & \text{if } (i,j) \in N_\ell \\ 0 & \text{if } (i,j) \in N_{M+1} \setminus N_\ell \text{ or } (i,j) \in \mathbb{Z}^2 \setminus N_{M+2} \\ 2 & \text{if } |i| = M+2 \text{ or } |j| = M+2 \end{cases}.$$

We define $\overline{\omega}$ in stages, beginning with the first r time coordinates:

$$\omega_t^{i,j} = \begin{cases} \nu_t^{i,j} & \text{if } (i,j) \in N_\ell \text{ and } 0 \leq t < r \\ 1 & \text{if } (i,j) \notin N_\ell \text{ and } 0 \leq t < r \end{cases}.$$

We can see that no matter how we define the remaining coordinates $\omega_t^{i,j}$, $t \geq r$, we will have $y = (\overline{\omega}, x) \in V$ by construction. Next, set

$$\omega_t^{i,j} = \begin{cases} 2 & \text{if } (i,j) \in N_M \text{ and } r \leq t < r + M + 4 \\ 1 & \text{if } (i,j) \notin N_M \text{ and } r \leq t < r + M + 4 \end{cases}.$$

At any $(i,j) \in N_M$ where $(H^r_{\overline{\omega}}x)_{i,j} \neq 0$, the state will cycle through 1, 2, $\cdots$, $m-1$, m, 1, 2, $\cdots$, so that no new 0s are introduced in the coordinates of N_M after time r. Any 0s adjacent to a non-zero state will update to state 1 within 3 timesteps, since states 1 through $m-2$ will infect the 0 immediately, state m first updates to a 1 and can then infect the 0, and a state $m-1$ has to first update to m and then to 1 before it can infect the 0. So, non-zero coordinates in N_M will spread throughout the entire region eventually. For an upper bound on timing, suppose that $(H^r_{\overline{\omega}}x)_{N_M} = \overline{0}_{N_M}$; that is, coordinates of N_M are all at state 0. Note that the annular pattern consisting of a ring of 2s placed between rings of 0s, occurring at coordinates $N_{M+3} \setminus N_M$ of x, is an $(m+1)$-periodic pattern under h_1, cycling through rings of 0/2/0, 1/3/1, 2/4/2, $\cdots$, $(m-2)/m/(m-2)$, $(m-1)/0/(m-1)$, and $m/1/m$ (see [3], Lemma 4.6). After time r, it will take at most 3 time steps for an infectious state 1, 2,$\cdots$, $(m-2)$ to be present at the coordinates of $N_{M+1} \setminus N_M$ (since $(m-1) \mapsto m \mapsto 0 \mapsto 1$ based on rule h_1 being applied), which will then propagate a 1 through the neighboring 0s in N_M at a rate of one coordinate per timestep. Thus, each $(i,j) \in N_M$ will have $\left(H^{t*}_{\overline{\omega}}x\right)_{i,j} \neq 0$ for some time $t* \leq r+3+(M+1)$ and further, for all $t* \leq t \leq r+M+4$, $\left(H^t_{\overline{\omega}}x\right)_{i,j} \neq 0$. Next, set

$$\omega^{i,j}_t = 3 \text{ for } r+M+4 \leq t < r+M+3+m.$$

Since each coordinate of N_M has a non-zero value at time $r+M+4$, it will take at most $m-1$ additional time steps to update through the states 1, $\cdots$, $m-1$, m. Once a state becomes m it will then stay m as rule h_3 is being applied, so that $\left(H^{r+M+3+m}_{\overline{\omega}}x\right)_{i,j} = m$ for all $(i,j) \in N_k$. We define the next $m-1$ time coordinates such that the states advance to match block u. Set

$$\omega^{i,j}_{[r+M+3+m,\ r+M+1+2m]} = \begin{cases} 1^{(m-1)} & \text{if } (i,j) \notin N_k \\ 3^{(m-2)}1 & \text{if } (i,j) \in N_k \text{ and } u_{i,j} = 0 \\ 3^{(m-1)-u_{i,j}}\ 2^{u_{i,j}} & \text{if } (i,j) \in N_k \text{ and } u_{i,j} \in \{1, \cdots, m-1\} \\ 3^{(m-1)} & \text{if } (i,j) \in N_k \text{ and } u_{i,j} = m \end{cases}.$$

Thus, we have $\left(H^{r+M+2+2m}_{\overline{\omega}}x\right)_{N_k} = u$. Now, we complete the assignment of $\overline{\omega}$ by setting

$$\omega^{i,j}_{r+M+2+2m+t} = \begin{cases} \mu^{i,j}_t & \text{if } (i,j) \in N_k \text{ and } 0 \leq t < q \\ 1 & \text{if } (i,j) \in N_k \text{ and } q \leq t \\ 1 & \text{if } (i,j) \notin N_k \text{ and } 0 \leq t \end{cases}$$

Therefore, by construction, $\overline{H}^{r+M+2+2m}(\overline{\omega}, x) \in U$, so that $y = (\overline{\omega}, x) \in \overline{H}^{-(r+M+2+2m)}(U) \cap V$, and consequently, $\overline{H}$ is transitive. □

Theorem 3.2. *The SCA $\overline{H}$ is surjective but not injective.*

Proof. Surjectivity follows from Thm 3.1 and the statement following Defn 2.9. To see that $\overline{H}$ is not an injective mapping, we can take $y, y' \in Y$ given by $y_{i,j} = (\overline{1}, 0)$ and $y'_{i,j} = (\overline{1}, m)$; these satisfy $(\overline{H}(y))_{i,j} = (\overline{1}, 0) = (\overline{H}(y'))_{i,j}$.

□

Remark 3.3. (1) Di Lena and Margara show that any injective NCA must be an injective (deterministic) CA [4]. The HIV SCA is clearly not deterministic, giving another argument that it is not injective, nor is its associated NCA.

(2) In the Garden of Eden Theorems by Moore and Myhill [13, 14], it is shown that a deterministic CA is onto if and only if it is one-to-one on the set of finite configurations. In the SCA setting, however, this equivalence does not hold, as we see that $\overline{H}$, the HIV SCA, takes both the three by three blocks of 0's and of m's to 0 when rule h_1 is applied. Therefore $\overline{H}$ is not injective on finite configurations even though it is surjective.

We next show that there is a fundamental distinction between H-type and E-type CA dynamics, as $\overline{E}$ is neither onto nor topologically transitive. In particular, we show that there is an *orphan pattern*; i.e., there is no finite block which maps to it under the SCA $\overline{E}$.

Theorem 3.4. *The SCA for E-type dynamics, $\overline{E} : Y \to Y$, is not surjective.*

Proof. Consider the block $u = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{matrix}$, and suppose v is a finite block which maps to u under $\overline{E}$. The central 1 of u could only come from a 0 in v which is adjacent to a value $a \in \{1, \cdots, m-1\}$, and that a would itself update to a value $b \in \{2, \cdots, m\}$, which does not allow for b to be 0 or 1. Thus, u is an orphan pattern and $\overline{E}$ is not onto. □

Theorem 3.5. *The SCA for E-type dynamics, $\overline{E} : Y \to Y$, is not topologically transitive.*

Proof. Since $\overline{E}$ is a continuous map of a compact metric space, this follows from Thm 3.4 and the remark immediately following Defn 2.9. □

In fact, any point $x \in \{0,1\}^{\mathbb{Z}^2} \setminus \{\overline{0}\}$ is not in the image of any $E_{\overline{\omega}}$. However, all other points in $\mathcal{A}^{\mathbb{Z}^D}$ are, as the next result shows.

Proposition 3.6. The SCA $\overline{E} : \overline{\Omega} \times \mathcal{A}^{\mathbb{Z}^2} \to \overline{\Omega} \times X_{\mathcal{E}}$ is onto, where $X_{\mathcal{E}} \subset \mathcal{A}^{\mathbb{Z}^2}$ is the subshift of finite type with forbidden set $\mathcal{E} = \left\{ \begin{matrix} * & * & * \\ * & 1 & * \\ * & * & * \end{matrix} \middle| * \in \{0,1\} \right\}$.

Proof. Let $y \in \overline{\Omega} \times X_{\mathcal{E}}$ be given by $y_{\vec{\imath}} = (\omega^{(\vec{\imath})}, a_{\vec{\imath}})$. Define $z \in Y$ with $z_{\vec{\imath}} = (\xi^{(\vec{\imath})}, b_{\vec{\imath}})$ by $b_{\vec{\imath}} = a_{\vec{\imath}} - 1 \pmod{m+1}$, $\xi_0^{(\vec{\imath})} = 2$, and $\xi_t^{(\vec{\imath})} = \omega_{t-1}^{(\vec{\imath})}$ for $t \geq 1$. Clearly $s(\xi^{(\vec{\imath})}) = \omega^{(\vec{\imath})}$ and $e_{\xi_0^{(\vec{\imath})}}(b_{\vec{\imath}+N_1}) = a_{\vec{\imath}}$ for $b_{\vec{\imath}} \neq 0$. If $b_{\vec{\imath}} = 0$, then $a_{\vec{\imath}} = 1$ and $\exists\ \vec{\jmath} \in N_1$ with $a_{\vec{\imath}+\vec{\jmath}} \in \{2, \cdots, m\}$. Thus, $b_{\vec{\imath}+\vec{\jmath}} \in \{1, \cdots, m-1\}$ and so $e_{\xi_0^{(\vec{\imath})}}(b_{\vec{\imath}+N_1}) = 1$. □

4. Discussion

In HIV, to date all patients (with one possible exception) have the virus in their system for life; drug treatment has changed HIV to a chronic condition (see [3, 10] and the references therein). The SCA model reveals these dynamics in two different ways; without drug treatment, it was observed that the initial configurations (initial infection) seems to tend over time to an independent Markov chain at each site [3]. The topological transitivity and surjectivity of the H-dynamics supports the idea that the equilibrium might be the chronically infected condition since every configuration is possible, and most configurations have many infected cells present. The subjectivity of the NCA shows that returning to a healthy state is essentially impossible, and the topological transitivity shows that each cell site is likely to be moving through all possible states as time passes, with recurring infection due to the cycling neighboring cells. However, an equilibrium percentage of viral infection is possible and observed; drug therapy brings this equilibrium to a survivable level. No matter

how small the probabilities are that are assigned to the arrows from m to 1 in Figures 1 and 3 (and they are small [3]), these dynamics occur mathematically and observed clinically.

For the Ebola virus, as we know the outcomes seem to split into one of two cases: either the virus seems to disappear and the patient returns to good health, or the virus takes over and damages one or more organs beyond recovery. Proposition 3.6 shows the E-type dynamics can never return to its starting configuration; this rules out cycling of states, which is not clinically present. In Figure 2, we have an arrow going from state 1 (infected) to states 2 –$(m-1)$ modeling a slower immune reaction time. In [2] it was shown that no matter how small the probability is assigned to that arrow, the existence of this transition leads to a fatal outcome. On the other hand, due to the arrow going from 1 to m, if the probability of the other arrow is 0 (i.e., there is no arrow from 1 to 2), leads to a recovery. In either case the failure to be surjective stems from the fact that one can never return to any configuration that resembles the initial state (1s sprinkled throughout surrounded by 0s); in the NCA certain configurations will never occur. In this way the topological dynamical properties make clear the distinction between H-type and E-type virus dynamics.

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