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Rachel Bayless-Rossetti & Jane Hawkins

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A SPECIAL CLASS OF INFINITE MEASURE-PRESERVING QUADRATIC RATIONAL MAPS

RACHEL BAYLESS-ROSSETTI
AND JANE HAWKINS

Abstract. We show the existence of a set of positive measure in parameter space, each element corresponding to a rational map of degree $d \geq 2$, for which there exists an invariant infinite σ-finite measure equivalent to a conformal measure. These maps are conservative and exact with respect to the invariant measure, and are perturbations of generalized Boole maps and inner functions. For quadratic Boole maps of with real coefficients we show that the Krengel entropy provides a complete invariant for c-isomorphism classes of quadratic maps of this type.

1. Introduction

We prove the existence of a set of positive measure in a slice of parameter space of degree d rational maps, $d \geq 2$, with the property that each parameter in the set corresponds to a map for which there exists an invariant infinite σ-finite measure equivalent to a uniquely determined conformal probability measure. These maps are ergodic and exact with respect to this conformal measure, and are related to generalized Boole maps and classical inner functions. If $d = 2$ and the coefficients are real we calculate the Krengel entropy, shown to exist by the first author in [5], and prove that Krengel entropy provides a complete invariant for c-isomorphism classes of these maps.

Rational maps that exhibit extremely chaotic behavior by virtue of being ergodic and exact with respect to an infinite invariant measure, are somewhat rare; moreover any rational map of degree $d \geq 2$ has an invariant probability measure of entropy $\log d$. However, certain real analytic maps of the real line that preserve Lebegue measure provide natural infinite measure preserving examples, and the earliest study is due to Boole [8], with ergodicity being established over one hundred years later [4]. These examples extend to inner functions on $\mathbb{C}$, and their properties have been studied by many, including [1], [3], [11], and [12]. Boole functions can be viewed as maps of the Riemann sphere, which we will denote by $\widehat{\mathbb{C}}$.

Our goal is to extend some measure theoretic properties of Boole functions to natural complex analogs. Negative generalized Boole functions, which we will call

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Boole functions for ease of notation, are defined by

$$S(z) = -z - \beta - \sum_{k=1}^{N} \frac{p_k}{t_k - z}, \quad \beta, t_k, p_k \in \mathbb{R}, p_k > 0, \tag{1.1}$$

with all t_k distinct. Any Boole function of the form (1.1), is a "square root" of an inner function, that is, $S^2 = S \circ S$ maps the upper half plane to itself and preserves the real line, providing natural examples of transformations that preserve Lebesgue measure, λ, on $\mathbb{R}$. The first author studied Boole functions of the form (1.1) in [5] and [6], and proved that they are ergodic and exact with respect to λ. Furthermore, a formula for their Krengel entropy, an entropy that is defined for infinite measure preserving maps, was established in [6].

We begin by showing that functions of the form (1.1), as maps of $\widehat{\mathbb{C}}$, have Julia set $\mathbb{R} \cup \{\infty\}$ which we denote by $\widehat{\mathbb{R}}$. We then restrict to the real line and investigate Krengel entropy with respect to λ, as an isomorphism invariant of Boole functions. Since infinite measures can be scaled, we must consider a less restrictive type of isomorphism called a *c*-isomorphism (defined in Section 2, where c is a constant and refers to the measure $c\lambda$.) We observe that since quadratic Boole functions admit no periodic orbits of period 2, they are conformally conjugate to maps in the unique parameter space of rational maps with this property ([7], [9]). We further use this conformally conjugate form to provide a characterization of the *c*-isomorphism classes of quadratic Boole functions and pinpoint in Theorems 2.11 – 2.13 in what way Krengel entropy provides a complete *c*-isomorphism invariant in this setting.

In Section 3 we consider Boole functions where the parameters, β, t_k and p_k are complex. We call these maps *complex Boole functions*. When the parameters are close to ones of the form in (1.1), the Julia set changes from a circle ($\widehat{\mathbb{R}}$), to a homeomorphic image of the circle, and the infinite invariant measure changes from λ to an infinite invariant measure, μ, that is equivalent to an appropriate conformal measure. We apply results from [2] to prove that many complex Boole functions are conservative, exact and ergodic with respect to μ, and we show there is an open set of parameters corresponding to maps with these properties. In the quadratic case, these maps occur in a well-studied slice of quadratic parameter space. We also establish some results of independent interest about the structure of the Julia set of the complex Boole functions of arbitrary degree.

1.1. **Background and notation.** By $(X, \mathcal{B}, \mu, T)$ we denote a nonsingular measurable dynamical system; we always assume $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$, where X is a topological space, $\mathcal{B}$ is the σ-algebra of Borel sets, and μ is a σ-finite measure on X. In the definitions that follow, all sets A are measurable and each statement holds up to sets of μ measure 0. We assume T maps X onto X and $\mu(A) = 0$ if and only if $\mu(T^{-1}A) = 0$ (i.e., T is *nonsingular*). We say T is *conservative* if there exists some $n \in \mathbb{N}$ such that $\mu(T^{-n}A \cap A) > 0$ for every set A with $\mu(A) > 0$. The map T

is *ergodic* if $T^{-1}(A) = A$ implies $\mu(A) = 0$ or $\mu(X \setminus A) = 0$. We say T is *exact* if $X = \bigcup_{n\geq 1} T^{-n} \circ T^n(A)$ for every A with $\mu(A) > 0$.

In [6] it was shown that $(\mathbb{R}, \mathcal{B}, \lambda, S)$, with S any real Boole function of the form (1.1), is λ-preserving, conservative, ergodic, and exact; the proof uses similar results about inner functions [1], and the fact that S^2 is inner.

A nonsingular map T is *n-to-1* if for μ-almost every $x \in X$, the set $T^{-1}(x)$ contains precisely n distinct points. Given a nonsingular n-to-1 transformation, we define $\mathcal{P} = \{P_i\}_{i=1}^n$ to be a *Rohlin partition* of X if $T : P_i \to X$ is one-to-one and onto for each $i = 1, ..., n$ and $\bigcup_{i=1}^n P_i = X \mod \mu$. Given $(X, \mathcal{B}, \mu, T)$ with Rohlin partition $\mathcal{P} = \{P_i\}_{i=1}^n$, we denote each branch $T|_{P_i}$ by T_i. The *Jacobian* of T is defined by $J_T(x) = \sum_{i=1}^n \mathbb{1}_{P_i}(x) \frac{d\mu T_i}{d\mu}(x)$, where $\mathbb{1}_A$ denotes the characteristic function of the set A. If $X = \mathbb{R}$, $\mu = \lambda$, and T is piecewise C^1, then $J_T(x) = |T'(x)|$.

A set $A \in \mathcal{B}$ is called a *sweep-out set* for T if $\bigcup_{n=0}^\infty T^{-n}A = X \mod \mu$. For $x \in A$ we let $\phi_A(x)$ be the *first-return-time* of x to A, so $\phi_A(x) = \min\{n : T^n(x) \in A\}$. The *induced transformation*, $T_A : A \to A$, is defined by

$$T_A(x) = T^{\phi_A(x)}(x) \text{ for } x \in A.$$

If $(X, \mathcal{B}, \mu, T)$ is a measure-preserving system and A is a sweep-out set for T, then T_A is a measure-preserving transformation of $(A, \mathcal{B}|_A, \mu|_A)$, where $\mathcal{B}|_A = \{B \cap A : B \in \mathcal{B}\}$ and $\mu|_A(B) = \mu(A \cap B)$.

In [10] Krengel gave the following definition of entropy for infinite measure-preserving transformations. Let $(X, \mathcal{B}, \mu, T)$ be a conservative σ-finite measure-preserving system. If $A \in \mathcal{B}$ is such that $0 < \mu(A) < \infty$, and A is a sweep-out set for T, then $h_{\mathrm{Kr}}(T) = h(T_A, \mu|_A)$ (i.e. the traditional Kolmogorov-Sinai entropy of the induced system).

Letac proved that if $G : (\mathbb{R}, \mathcal{B}, \lambda) \to (\mathbb{R}, \mathcal{B}, \lambda)$ is a rational function that preserves λ on $\mathbb{R}$, then G is of the form $G = \pm S$ for some S of the form (1.1) [11]. The following formula for the Krengel entropy of $(\mathbb{R}, \mathcal{B}, \lambda, S)$, where S is of the form (1.1), was proved in [6].

Theorem 1.1 ([6]). Any rational function $G : (\mathbb{R}, \mathcal{B}, \lambda) \to (\mathbb{R}, \mathcal{B}, \lambda)$ which is λ-preserving and conservative has Krengel entropy given by:

$$h_{\mathrm{Kr}}(G) = \int_{\mathbb{R}} \log |G'(x)| d\lambda(x). \tag{1.2}$$

We make the following useful observation, proved in [5].

Lemma 1.2. For each S of the form (1.1), every point in $\mathbb{R}$ has precisely $\deg(S)$ distinct preimages in $\mathbb{R}$.

Proof. We relabel the set of poles $\{t_1, t_2, \ldots, t_N\}$ so that $t_i < t_{i+1}$ for $i = 1, \ldots, N-1$. Then it is straightforward to check that $\mathcal{P} = \{(\infty, t_1), (t_1, t_2), \ldots, (t_{N-1}, t_N), (t_N, \infty)\}$ is a Rohlin partition for S restricted to $\mathbb{R}$ and S maps each interval in $\mathcal{P}$ diffeomorphically onto $\mathbb{R}$. There are $N + 1$ intervals in $\mathcal{P}$, and $\deg(S) = N + 1$. ∎

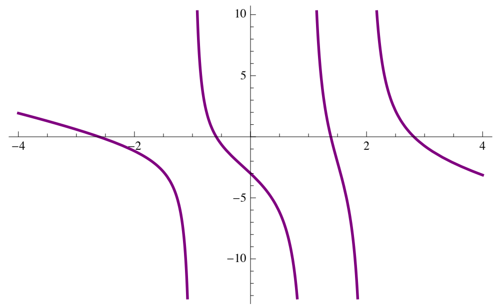

FIGURE 1. A typical graph of a negative Boole function restricted to $\mathbb{R}$

Lemma 1.2 is illustrated in Figure 1. We now shift our focus from restricting the domain of Boole functions to $\mathbb{R}$, and allow the domain to be the entire Riemann sphere $\widehat{\mathbb{C}}$. We present some basic results about the Julia sets of Boole functions.

Definition 1.3. Let R be rational function of degree $d \geq 2$, defined on $\widehat{\mathbb{C}}$. The *Fatou set* of R, $F(R)$, is the open subset of $\widehat{\mathbb{C}}$ defined by: $\{z \in \widehat{\mathbb{C}} : \{R^n\}$ is equicontinuous at $z\}$. The *Julia set* of R, $J(R)$, is the complement of $F(R)$ in $\widehat{\mathbb{C}}$.

We extend a Boole function, S, to a map of the Riemann sphere in what follows; by $\widehat{\mathbb{R}}$ we denote the great circle on $\widehat{\mathbb{C}}$ whose image is the real line under stereographic projection.

Remark 1.4. The point at ∞ is fixed under any map of the form (1.1), and we easily compute that the fixed point at ∞ has multiplier -1, where the multiplier at ∞ is given by $\frac{d}{dz}(1/S(1/z))|_{z=0}$. There are N other preimages of ∞, namely the set of poles, $\{t_1, \ldots, t_N\}$. We refer the reader to a standard reference like [7] or [16] for background on parabolic fixed points for rational maps. However, it is a classical result that if a rational map R has a fixed point $z_0 \in \widehat{\mathbb{C}}$ with multiplier $e^{2\pi i/k}$, $k \in \mathbb{N}$, then $z_0 \in J(R)$. Therefore $\infty \in J(S)$ for any rational map of the form (1.1); additionally ∞ is a nonisolated point of the Julia set because there are no isolated points in $J(S)$.

Proposition 1.5. Every Boole function of the form (1.1) satisfies $J(R) = \widehat{\mathbb{R}}$.

Proof. By Remark 1.4 above, we have that $\infty \in J(S)$. Moreover by Lemma 1.2, each $x \in \mathbb{R}$ has $\deg(S)$ preimages in $\mathbb{R}$. Therefore $\widehat{\mathbb{R}}$ is closed and completely invariant under S, so $J(S) \subset \widehat{\mathbb{R}}$.

If we assume $J(S) \neq \widehat{\mathbb{R}}$, then $F(S) \cap \widehat{\mathbb{R}}$ is open and nonempty, and Remark 1.4 implies that

$$F(S) \cap \widehat{\mathbb{R}} = F(S) \cap \mathbb{R}. \tag{1.3}$$

Therefore $F(S)$ must contain an open interval $I = (a, b)$; we fix some $x_0 \in I$. Then given any $0 < \varepsilon < 1$, there exists a $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |S^n(x) - S^n(x_0)| < \varepsilon$ for all $n \in \mathbb{N}$. Consider the interval $U = [x_0, x_0 + \delta/2]$; then $\overline{U} \subset I$, and $\lambda(U) = \delta/2$. Also U cannot contain a pole or a prepole of S since $\infty \in J(S)$. By invariance of λ, $\lambda(S^{-n}A) = \lambda(A)$ for all Borel sets $A \subset \mathbb{R}$, and by exactness,

$$\bigcup_{n \in \mathbb{N}} (S^{-n}(S^n(U))) = \mathbb{R} \tag{1.4}$$

up to a set of λ measure 0. We have that $\lambda(S^n(U)) = \lambda(S^{-n}(S^n(U)))$, and $S^n(U)$ is an interval in $\mathbb{R}$.

Therefore by (1.4) there exists some N such that $\lambda(S^N(U)) = \lambda(S^{-N}(S^N(U))) > 1$; by piecewise monotonicity of S, $S^N(U) = (S^N(x_0), S^N(x_0 + \delta/2))$ or $(S^N(x_0 + \delta/2), S^N(x_0))$. This contradicts equicontinuity, so there is no interval $I \subset F(S) \cap \mathbb{R}$, and Eqn (1.3) implies there cannot be any interval in $\widehat{\mathbb{R}} \cap F(S)$ either.

□

We have the following consequence of Proposition 1.5; (Theorem 3.3 below is a more general version of this).

Corollary 1.6. If S is a rational map of the form (1.1), then the parabolic fixed point at ∞ has two petals.

Proof. There are at least two petals since by exactness of S there are two repelling directions along $\mathbb{R}$, one on each side of the point at ∞. If there were more petals, then there would be more repelling directions, and $J(S)$ would have to contain more points than just points in $\mathbb{R}$. □

2. Isomorphisms of negative Boole functions

In ergodic theory, entropy is a useful isomorphism invariant for probability-preserving transformations, but in the infinite setting we must account for a less restrictive type of isomorphism called a c-isomorphism.

Definition 2.1. Let $(X_1, \mathcal{B}_1, m_1, T_1)$ and $(X_2, \mathcal{B}_2, m_2, T_2)$ be two infinite-measure-preserving systems. Suppose we have two sets $M_1 \in \mathcal{B}_1$ and $M_2 \in \mathcal{B}_2$ with $m_1(X_1 \setminus M_1) = 0$ and $m_2(X_2 \setminus M_2) = 0$ such that $T_1(M_1) \subseteq M_1$ and $T_2(M_2) \subseteq M_2$. For $c \in (0, \infty]$ we say $(X_1, \mathcal{B}_1, m_1, T_1)$ is *c-isomorphic* to $(X_2, \mathcal{B}_2, m_2, T_2)$ if there exists an invertible map $\phi : M_1 \to M_2$ such that for all $A \in \mathcal{B}_2|_{M_2}$,

(1) $\phi^{-1}(A) \in \mathcal{B}_1|_{M_1}$,
(2) $m_1(\phi^{-1}(A)) = c \cdot m_2(A)$, and
(3) $(\phi \circ T_1)(x) = (T_2 \circ \phi)(x)$ for all $x \in M_1$.

If (1)-(3) hold, we write $\phi : T_1 \to^c T_2$, and call ϕ a *c-isomorphism.*

In this section, we consider maps of the form (1.1), and we restrict the domain to $\mathbb{R}$. We have the following relationship between c-isomorphisms and Krengel entropy for Boole maps on $(\mathbb{R}, \lambda)$.

Proposition 2.2. If S_1 and S_2 are two Boole functions of the form (1.1), and $\phi : S_1 \to^c S_2$ is a c-isomorphism, then

$$h_{\mathrm{Kr}}(S_1) = c \cdot h_{\mathrm{Kr}}(S_2). \tag{2.1}$$

Proof. Suppose $\phi : S_1 \to^c S_2$ is a c-isomorphism, and let $\phi_*\lambda$ denote the pushforward measure; i.e., $\phi_*\lambda(A) = \lambda(\phi^{-1}A)$ for all measurable A. By Definition 2.1, we have $S_2 = \phi \circ S_1 \circ \phi^{-1}$ and $\phi_*\lambda = c\lambda$; equivalently, $\frac{d\phi_*\lambda}{d\lambda}(x) = c$ for λ-a.e. $x \in \mathbb{R}$. Furthermore, ϕ^{-1} exists and is measurable, so on a set of full measure, $\frac{d\phi_*^{-1}\lambda}{d\lambda}(x) = \frac{1}{d\phi_*\lambda/d\lambda}(y) = \frac{1}{c}$, with $y = \phi(x)$. Since $|S_j'(x)| = J_{S_j}(x)$, $j = 1, 2$, and $J_\phi(x) = \frac{d\phi_*^{-1}\lambda}{d\lambda}(x)$, by the chain rule and (1.2) we have,

$$\begin{aligned}
c \cdot h_{\mathrm{Kr}}(S_2) &= c \cdot \int_{\mathbb{R}} \log |S_2'(x)| d\lambda(x) \\
&= c \cdot \int_{\mathbb{R}} \log |J_{\phi \circ S_1 \circ \phi^{-1}}(x)| d\lambda(x) \\
&= c \cdot \int_{\mathbb{R}} \left(\log \left| \frac{d\phi_*^{-1}\lambda}{d\lambda}(S_1(\phi^{-1}x)) \right| + \log |S_1'(\phi^{-1}x)| + \log \left| \frac{d\phi_*\lambda}{d\lambda}(x) \right| \right) d\lambda(x).
\end{aligned}$$

This in turn gives

$$\begin{aligned}
c \cdot h_{\mathrm{Kr}}(S_2) &= c \cdot \int_{\mathbb{R}} \left(\log \left| \frac{1}{c} \right| + \log |S_1'(\phi^{-1}x)| + \log |c| \right) d\lambda(x) \\
&= c \cdot \int_{\mathbb{R}} \log |S_1'(\phi^{-1}x)| d\lambda(x) \\
&= c \cdot \int_{\mathbb{R}} \log |S_1'(u)| d\lambda(\phi u) \\
&= c \cdot \frac{1}{c} \int_{\mathbb{R}} \log |S_1'(u)| d\lambda(u),
\end{aligned}$$

since $c > 0$ and setting $u = \phi^{-1}(x)$, so $\frac{d\lambda \circ \phi}{d\lambda}(u) = \frac{d\phi_*^{-1}\lambda}{d\lambda}(u) = \frac{1}{c}$. The last line is $h_{\mathrm{Kr}}(S_1)$ as claimed. □

Corollary 2.3. Krengel entropy is an isomorphism invariant for Boole functions. That is, if S_1 and S_2 are 1-isomorphic, then $h_{\mathrm{Kr}}(S_1) = h_{\mathrm{Kr}}(S_2)$.

Corollary 2.4. If S_1 and S_2 are two Boole functions, then there is at most one $c \in (0, \infty]$ such that $\phi : S_1 \to^c S_2$ is a c-isomorphism.

Proof. If S_1 is c-isomorphic to S_2, then $h_{\text{Kr}}(S_1) = c \cdot h_{\text{Kr}}(S_2)$. The Krengel entropy of S_1 with respect to λ is a fixed value (similarly for S_2), so c must be unique. □

To summarize, if $h_{\text{Kr}}(S_1) \neq h_{\text{Kr}}(S_2)$, then S_1 and S_2 are not 1-isomorphic, but they may still be c-isomorphic for some $c \neq 1$. In the event that they are c-isomorphic, then the c is unique.

2.1. **Quadratic case.** In this section, we focus on quadratic Boole functions, which we write as:

$$S_{(\beta,p,t)}(x) = -x - \beta - \frac{p}{t-x}, \tag{2.2}$$

where $\beta, p, t \in \mathbb{R}$ and $p > 0$.

Lemma 2.5. The Krengel entropy of $S_{(\beta,p,t)}$ is $2\pi\sqrt{p}$.

Proof. By Theorem 1.1 we have

$$h_{\text{Kr}}(S_{(\beta,p,t)}) = \int_{\mathbb{R}} \log\left(1 + \frac{p}{(t-x)^2}\right) d\lambda(x).$$

Integration by parts followed by a substitution yields

$$\left[-2p \arctan\left(\frac{t-x}{\sqrt{p}}\right) + \log\left(1 + \frac{p}{(t-x)^2}\right)(x-t)\right]_{-\infty}^{\infty},$$

and evaluating the above limits yields the result.

□

The next result shows that there is some symmetry with regard to the placement of the poles and constant.

Lemma 2.6. The Boole function, $S_{(\beta,p,t)}$, is 1-isomorphic to $S_{(-\beta,p,-t)}$.

Proof. If we set $\eta(x) = -x$, then $\eta = \eta^{-1}$ and $\eta \circ S_{(\beta,p,t)} \circ \eta = S_{(-\beta,p,-t)}$. □

We say a quadratic Boole function is *normalized* if it has the form $S_{(\beta,1,0)}$, with $\beta \in \mathbb{R}$.

Theorem 2.7. Every quadratic Boole function, $S_{(\beta,p,t)}$, is $\sqrt{p}$-isomorphic to a unique normalized map $S_{(\hat{\beta},1,0)}$, where $\hat{\beta} = \left|\frac{2t+\beta}{\sqrt{p}}\right| \geq 0$.

Proof. We first move the pole from t to 0 via a 1-isomorphism with conjugating map

$$\psi_t(x) = x - t \quad \text{and} \quad \psi_t^{-1}(x) = x + t. \tag{2.3}$$

That is, $(\psi_t \circ S_{(\beta,p,t)} \circ \psi_t^{-1}) = S_{(2t+\beta,p,0)}$. We now change the multiplier from p to 1 via a $\sqrt{p}$-isomorphism with conjugating map

$$\zeta_p(x) = \frac{x}{\sqrt{p}} \quad \text{and} \quad \zeta_p^{-1}(x) = \sqrt{p} \cdot x. \tag{2.4}$$

That is, $\zeta_p \circ S_{(2t+\beta,p,0)} \circ \zeta_p^{-1} = S_{(\frac{2t+\beta}{\sqrt{p}},1,0)}$.

Lemma 2.6 implies that $S_{(\beta,1,0)}$, is 1-isomorphic to $S_{(-\beta,1,0)}$, so we choose $\hat{\beta} = \pm\left(\frac{2t+\beta}{\sqrt{p}}\right)$ to be nonnegative.

□

Corollary 2.8. Every quadratic Boole function, $S_{(\beta,p,t)}$, is $\sqrt{p}$-isomorphic to a Boole function T with $h_{\text{Kr}}(T) = 2\pi$.

Proof. Consider the normalized form: $T = S_{(\hat{\beta},1,0)}$ with $\hat{\beta} = \frac{2t+\beta}{\sqrt{p}}$ from Theorem 2.7, which satisfies $h_{\text{Kr}}(T) = 2\pi$. □

Corollary 2.8 shows that Krengel entropy appears to be unable to distinguish among maps of the form (2.2) if all 3 parameters are allowed to move. However, in order to link this study to parametrized families of rational maps, we show that the values of $\hat{\beta}$ distinguish c-isomorphism classes, which is equivalent to showing Krengel entropy can distinguish c-isomorphism classes of quadratic negative Boole functions.

Proposition 2.9. Let $S_{(\beta,1,0)}$ and $S_{(\gamma,1,0)}$ be two quadratic Boole functions. Then $S_{(\beta,1,0)}$ is c-isomorphic to $S_{(\gamma,1,0)}$ if and only if $\beta = \pm\gamma$. In this case, $c = 1$, and the conjugating map $\phi(x) = x$ λ-a.e., or $\phi(x) = -x$ λ-a.e.

Proof. ($\Longrightarrow$) If $\phi : S_{(\beta,1,0)} \to^c S_{(\gamma,1,0)}$ is a c-isomorphism, then

$$(\phi \circ S_{(\beta,1,0)})(x) = (S_{(\gamma,1,0)} \circ \phi)(x), \tag{2.5}$$

for almost every $x \in \mathbb{R}$. If we consider the Jacobian of both sides of (2.5) at x, and note that $\frac{d\phi_*^{-1}\lambda}{d\lambda}(x) = 1/c$, for λ-a.e. x, then we have $|S'_{(\beta,1,0)}(x)| = |S'_{(\gamma,1,0)}(\phi(x))|$, λ-a.e. Therefore,

$$1 + \frac{1}{x^2} = 1 + \frac{1}{(\phi(x))^2}, \quad \text{for } \lambda\text{-a.e. } x \in \mathbb{R},$$

so $(\phi(x))^2 = x^2$, and therefore $\phi(x) = \pm x$ λ-a.e.

Let $M \subseteq \mathbb{R}$ be a measurable set such that $\phi \circ S_{(\beta,1,0)} = S_{(\gamma,1,0)} \circ \phi$ on M and $\lambda(\mathbb{R} \setminus M) = 0$. Define

$$\begin{aligned} A &= \{x \in M : \phi(x) = x\} \text{ and} \\ B &= \{x \in M : \phi(x) = -x\}. \end{aligned}$$

Note that $\lambda(\mathbb{R} \setminus (A \cup B)) = 0$.

Without loss of generality assume $\lambda(A) > 0$. If $y \in S^{-1}_{(\beta,1,0)}(B) \cap A$, then writing out each side of the equation $(\phi \circ S_{(\beta,1,0)})(y) = (S_{(\gamma,1,0)} \circ \phi)(y)$, we have

$$y + \beta - \frac{1}{y} = -y - \gamma + \frac{1}{y}. \tag{2.6}$$

Simplifying yields $2y^2 + (\beta+\gamma)y - 2 = 0$, and there are at most two points y for which (2.6) holds. Thus, there are at most two points in $S^{-1}_{(\beta,1,0)}(B) \cap A$, so $S^{-1}_{(\beta,1,0)}(B) \subset B$ λ-a.e. A similar calculation shows that $B \subset S^{-1}_{(\beta,1,0)}(B)$, so B is invariant modulo sets of λ measure 0. By ergodicity of $S_{(\beta,1,0)}$, either $\lambda(B) = 0$ or $\lambda(\mathbb{R} \setminus B) = 0$. By assumption the first case holds; therefore, $\beta = \gamma$. If we assume instead that $\lambda(B) > 0$, then a similar argument proves that $\beta = -\gamma$.

($\Longleftarrow$) This direction is obvious using Lemma 2.6. □

Corollary 2.10. The representation of $S_{(\beta,p,t)}$ in the form $S_{(\beta',1,0)}$, where $\beta' > 0$ is unique.

Proof. Assume $\phi' : S_{(\beta,p,t)} \to^{c'} S_{(\beta',1,0)}$ is a c'-isomorphism and $\bar{\phi} : S_{(\beta,p,t)} \to^{\bar{c}} S_{(\bar{\beta},1,0)}$ is a $\bar{c}$-isomorphism, where both $\beta' > 0$ and $\bar{\beta} > 0$. Then $\phi' \circ \bar{\phi}^{-1} : S_{(\bar{\beta},1,0)} \to S_{(\beta',1,0)}$ is a $\frac{c'}{\bar{c}}$-isomorphism, and by Proposition 2.9 $c' = \bar{c}$ and $\beta' = \bar{\beta} = \left|\frac{2t+\beta}{\sqrt{p}}\right|$. □

2.2. **Quadratic maps lacking period two orbits.** Assume S is quadratic and of the form (1.1).

(1) Since S has a fixed point at ∞ with multiplier -1, by Milnor [15], S lacks period two orbits (apart from the fixed points).
(2) Viewing S as a map on the sphere, we know by [9] that S is conformally conjugate to a map of the form $R_a = \frac{z^2-z}{1+az}$, for $a \in \mathbb{C} \setminus \{-1\}$.
(3) If $a < -1$ then $J(R_a) = \widehat{\mathbb{R}}$, and if $a < -3$, then R_a is conformally conjugate to an $R_{a'}$ with $a' \in [-3, -1)$, (see [7],[9]).
(4) Using the spherical metric on $\widehat{\mathbb{C}}$ and the isometric involution $\psi(z) = \frac{1}{z}$, for each $a \in [-3, -1)$,
$$\psi \circ R_a \circ \psi^{-1} = H_b(z) = S_{(-b,b,1)}(z) = -z + b - \frac{b}{1-z},$$
where $b = -(a + 1) \in (0, 2]$.
(5) If $b \neq b'$, then $h_{\mathrm{Kr}}(H_b) \neq h_{\mathrm{Kr}}(H_{b'})$.
(6) The normalization of H_b is $S_{(\frac{2-b}{\sqrt{b}},1,0)}$. As b decreases from 2 to 0, $\frac{2-b}{\sqrt{b}}$ increases monotonically from 0 to $+\infty$. Conversely, if $\beta \geq 0$, $S_{(\beta,1,0)}$ is $\frac{1}{\sqrt{b}}$-isomorphic to H_b, where $b = \frac{1}{2}\left(\beta^2 + 4 - \beta\sqrt{\beta^2+8}\right) \in (0, 2]$.

When $b \neq b' \in (0, 2]$, we already know that H_b and $H_{b'}$ are not conformally conjugate rational maps of $\widehat{\mathbb{C}}$. We now show that each H_b corresponds to a unique c-isomorphism class of quadratic negative Boole functions.

Theorem 2.11. Each map $H_b = S_{(-b,b,1)}(z) = -z + b - \frac{b}{1-z}$, with $b \in (0, 2]$, satisfies the following:

(1) $h_{\mathrm{Kr}}(H_b) = 2\pi\sqrt{b}$;
(2) every quadratic negative Boole function is c-isomorphic to a function H_b;
(3) $S_{(\beta_1,p_1,t_1)}$ is c-isomorphic to $S_{(\beta_2,p_2,t_2)}$ if and only if they are both c-isomorphic to the same map H_b.

Proof. (1) follows immediately from Lemma 2.5. (2) follows using Theorem 2.7 and Section 2.2(6). (3) ($\Rightarrow$) By Theorem 2.7, for $j = 1, 2$, $S_{(\beta_j,p_j,t_j)}$ is $\sqrt{p_j}$-isomorphic to $S_{(\hat{\beta}_j,1,0)}$ with $\hat{\beta}_j = \pm\left(\frac{2t_j+\beta_j}{\sqrt{p_j}}\right)$, chosen so that $\hat{\beta}_j \geq 0$. Then by Proposition 2.9, $\hat{\beta}_1 = \hat{\beta}_2$, and (3) follows using Section 2.2(6). ($\Leftarrow$) If $S_{(\beta_1,p_1,t_1)}$ and $S_{(\beta_2,p_2,t_2)}$ are each c_j-isomorphic to H_b, then the composition of one isomorphism with the inverse of the other gives the c-isomorphism between them. □

The following result, one of the main results of the paper, is obtained as a corollary of Theorem 2.11.

Theorem 2.12. Krengel entropy of H_b is a complete invariant for c-isomorphism classes of quadratic negative Boole functions.

We conclude this section by realizing the c-isomorphism invariant from Theorem 2.12, as a ratio of values in the non-normalized setting.

Theorem 2.13. Two quadratic Boole maps $S_{(\beta,p,t)}$ and $S_{(\gamma,q,s)}$ are c-isomorphic if and only if

$$\left|\frac{2t+\beta}{\sqrt{p}}\right| = \left|\frac{2s+\gamma}{\sqrt{q}}\right|. \tag{2.7}$$

When (2.7) holds, $c = \dfrac{h_{\mathrm{Kr}}(S_{(\beta,p,t)})}{h_{\mathrm{Kr}}(S_{(\gamma,q,s)})} = \dfrac{\sqrt{p}}{\sqrt{q}}$.

Proof. ($\Longrightarrow$) Assume there exists a c-isomorphism, $\xi : S_{(\beta,p,t)} \to^c S_{(\gamma,q,s)}$. To show (2.7) we conjugate $S_{(\beta,p,t)}$ and $S_{(\gamma,q,s)}$ to their completely normalized forms. Using the notation from Theorem 2.7, we have

$$\zeta_p \circ \psi_t : S_{(\beta,p,t)} \to^{\sqrt{p}} S_{(\hat{\beta},1,0)} \quad \text{and} \quad \zeta_q \circ \psi_s : S_{(\gamma,q,s)} \to^{\sqrt{q}} S_{(\hat{\gamma},1,0)}, \tag{2.8}$$

where $\hat{\beta} = \left|\frac{2t+\beta}{\sqrt{p}}\right|$ and $\hat{\gamma} = \left|\frac{2s+\gamma}{\sqrt{q}}\right|$. Thus, the map $\phi = \zeta_q \circ \psi_s \circ \xi \circ \psi_t^{-1} \circ \zeta_p^{-1} : S_{(\hat{\beta},1,0)} \to^{\hat{c}} S_{(\hat{\gamma},1,0)}$, and by Proposition 2.9 $\hat{c} = 1$ and $\hat{\beta} = \hat{\gamma}$, which gives (2.7).

($\Longleftarrow$) If $\left|\frac{2t+\beta}{\sqrt{p}}\right| = \left|\frac{2s+\gamma}{\sqrt{q}}\right|$, we conjugate $S_{(\beta,p,t)}$ and $S_{(\gamma,q,s)}$ to their completely normalized forms. Since $\hat{\beta} = \hat{\gamma}$, by Proposition 2.9 there exists a 1-isomorphism $\phi : S_{(\hat{\beta},1,0)} \to^1 S_{(\hat{\gamma},1,0)}$. Therefore, using the notation from Theorem 2.7, $\xi = \psi_s^{-1} \circ \zeta_q^{-1} \circ$

$\phi \circ \zeta_p \circ \psi_t : S_{(\beta,p,t)} \to^c S_{(\gamma,q,s)}$. By Proposition 2.2 and Lemma 2.5 the c-isomorphism of the maps implies that $c = \frac{h_{Kr}(S_{(\beta,p,t)})}{h_{Kr}(S_{(\gamma,q,s)})} = \frac{\sqrt{p}}{\sqrt{q}}$. □

3. Negative Boole functions with complex coefficients

In this section we consider degree $N+1$ maps of the form:

$$S(z) = -z - \gamma - \sum_{k=1}^{N} \frac{\rho_k}{\tau_k - z}, \quad \gamma, \tau_k, \rho_k \in \mathbb{C}, \ \mathrm{Re}(\rho_k) > 0. \tag{3.1}$$

We call these *complex (negative) Boole functions.* If $\eta = \{\gamma, \rho_1, \tau_1, \ldots, \rho_N, \tau_N\} \in \mathbb{C}^{2N+1}$ is as in (3.1), then S_η denotes the map $S_\eta(z) = -z - \gamma - \sum_{k=1}^{N} \frac{\rho_k}{\tau_k - z}$. When the imaginary parts of the coefficients above are not too large, S_η retains many of the dynamical and measure theoretic properties of real Boole functions, using an appropriately modified measure. The main results of this section are Theorems 3.8 and 3.10; we show that there is an open set of parameters in parameter space that correspond to complex Boole functions with an infinite invariant, exact, and conservative measure equivalent to conformal measure.

We say a rational map R of degree ≥ 2 is *parabolic* if $J(R)$ contains rationally indifferent periodic points but contains no critical points. All Boole functions of the form (1.1) are parabolic; we are interested in holomorphic perturbations of these maps that remain parabolic. We note that it is possible to move the coefficients enough so that a critical point lands on a prepole; for example, the map H_{2i}, (from Section 2.2 (4)) is not parabolic since the critical point $c = i$ lands on the fixed point at ∞.

Proposition 3.1. Suppose $S(z) = -z - \beta - \sum_{k=1}^{N} \frac{p_k}{t_k - z}$, $\beta, t_k, p_k \in \mathbb{R}, p_k > 0$, with all t_k distinct (S satisfies (1.1)). Then denoting the set of critical points of S by $\mathcal{C} = \{s_1, s_2, \ldots, s_{2N}\}$, we have:

1. $\mathcal{C} \cap \widehat{\mathbb{R}} = \emptyset$, there are N points in $\mathcal{C}$ such that $\mathrm{Im}(s_j) > 0$, and N points in $\mathcal{C}$ such that $\mathrm{Im}(s_j) < 0$;
2. for each $2N+1$-tuple $\Phi = \{\beta, p_1, t_1, \ldots, p_N, t_N\}$ satisfying the hypotheses there is a small ball $B_\varepsilon(\Phi) \subset \mathbb{C}^{2N+1}$, such that for any $\eta \in B_\varepsilon(\Phi)$ there are no attracting periodic cycles for S_η;
3. there exists $\varepsilon > 0$ such that for each $\eta \in B_\varepsilon(\Phi)$, the critical point $s_j(\eta) \notin J(S_\eta)$ for any $j = 1, \cdots, 2N$.
4. In a neighbourhood of Φ, the Julia set $J(S_\eta)$ depends continuously on η in the Hausdorff topology.

Proof. If S satisfies the hypotheses, then $S'(z) = -1 - \sum_{k=1}^{N} \frac{p_k}{(t_k - z)^2} < -1$ for all $z \in \mathbb{R}$, so the derivative never vanishes on $\mathbb{R}$. Since $\deg(S) = N+1$, there are $2N$

critical points in $\widehat{\mathbb{C}}$ counting multiplicity, and $\infty \in J(S)$ is fixed; so all critical points in $\mathcal{C}$ are in $\mathbb{C}$. Because all coordinates of Φ are real, and no s_j is real, critical points occur in conjugate pairs, so (1) holds.

By Proposition 1.5 and Corollary 1.6 for $S_\Phi := S$, and each $s_j \in \mathcal{C}$, $\lim_{n\to\infty} S^n(s_j) = \infty$ with each $s_j \in F(S)$. It also follows that all periodic cycles are repelling apart from the parabolic fixed point at ∞; this parabolic fixed point persists for every map S_η of the form (3.1). Because the critical points $s_j(\eta)$ vary continuously (holomorphically) as Φ varies, for some small enough $\varepsilon > 0$ the critical points for S_η will still converge to ∞ under iteration for all $\eta \in B_\varepsilon(\Phi) \subset \mathbb{C}^{2N+1}$. This is because $\mathcal{C}_\eta$ remains in $F(S_\eta)$, and all cycles other than the point at ∞ remain repelling. From this, (2) and (3) will follow immediately. To prove (4), we apply Theorem 4.2 in [13], and we see that (3) and (4) are equivalent statements in our setting since we have a holomorphic family of mappings.

□

More generally, whenever S satisfies (3.1) the fixed point at ∞ is a simple parabolic fixed point of S and has multiplicity $2p+1$ ($p \in \mathbb{N}$) as a fixed point of S^2; the map S^2 has multiplier 1 at ∞. Then ∞ as a fixed point of S has p distinct immediate basins of attraction, each of which consists of 2 disjoint Fatou components forming a period 2 cycle. We call each component of the immediate basin of attraction a *Leau domain* $L_1, L_2, \ldots, L_{2p}$ (see, eg. [16]). As in the real coefficient case, the attracting basin of ∞ is the open set of points in $\widehat{\mathbb{C}}$ converging to the fixed point. The rays of symmetry of the attracting petals of a rational map of the form (3.1) near ∞ are given by the next two theorems, (see, e.g., [16], Lemma 10.1 and its proof).

Theorem 3.2. [16] Suppose $R(z) = z(1 + \alpha z^p + \text{(higher order terms)})$, for some $\alpha \neq 0$, $p \in \mathbb{N}$, near the origin. Then there exist p attracting petals, or equivalently p evenly spaced attracting directions at the origin and any point that approaches the rationally neutral fixed point at 0 must approach it in one of these p directions. The vectors v_j, $j = 1, \ldots, p$, that give the attracting directions are determined by the solutions v to the equation: $p\alpha v^p = -1$; i.e., each v_j satisfies $v_j = \left(\dfrac{-1}{p\alpha}\right)^{1/p}$.

There are also p repelling petals which are defined to be the attracting petals for R^{-1} near a neutral fixed point, and the repelling directions are obtained by finding w such that $p\alpha w^p = 1$. The following result shows that Theorem 3.2 yields two petals for the Boole functions considered in this paper.

Theorem 3.3. If S is of the form (3.1), then the fixed point at ∞ has two petals, with attracting directions determined by the two vectors satisfying $v = (-1/(4\sum_{k=1}^N \rho_k))^{1/2}$.

Proof. For the first part of the result we want to show that ∞ is a fixed point of S^2 of multiplicity 3. If we set $g(z) = S \circ S(z)$, and then consider $G(z) = 1/g(\frac{1}{z})$, then the theorem is equivalent to showing that near 0,

$$G(z) = z(1 - c_1 z^2 - c_2 z^3 \cdots), \tag{3.2}$$

with $c_1 = -2\sum_1^N \rho_k \neq 0$. Then using $\alpha = -c_1$ and $p = 2$, we apply Theorem 3.2 to obtain the second statement of the theorem.

We claim Eqn (3.2) is equivalent, for $|z| < 1$, to

$$G(z) = \frac{z}{1 + (c_1 z^2 + c_2 z^3 + c_3 z^4 \cdots)}. \tag{3.3}$$

This is since for $|w| < 1$, $\dfrac{1}{1+w} = 1 - w + w^2 - w^3 + \cdots$. Inverting the expression in Eqn (3.3), it is equivalent to show for all $0 < |z| < 1$,

$$g\left(\frac{1}{z}\right) = \frac{1}{z}(1 + c_1 z^2 + c_2 z^3 + \cdots) = \frac{1}{z} + c_1 z + c_2 z^2 + \cdots. \tag{3.4}$$

Therefore to show Eqn (3.2), it is enough to show that for $|z| \geq R$, i.e., near ∞,

$$g(z) = z + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots, \tag{3.5}$$

and to compute c_1. Eqn (3.5) is just the Laurent series expansion of S^2 about the fixed point at ∞.

We use this simple fact about series: for $p > 0$, $\tau \in \mathbb{C}$, and $|z| > R$, R large,

$$\frac{p}{\tau - z} = -\frac{p}{z}\sum_{n=0}^{\infty}\left(\frac{\tau}{z}\right)^n = -\frac{p}{z}(1 + \frac{\tau}{z} + \cdots) \tag{3.6}$$

Using Eqn (3.6) we start with the Laurent series centred at ∞ of S; it is not hard to show there is an $R > 0$ such that for $|z| > R$:

$$S(z) = -z - \gamma + \frac{\sum_k \rho_k}{z} + \frac{\sum_k \rho_k \tau_k}{z^2} + \frac{\sum_k \rho_k \tau_k^2}{z^3} + \cdots \tag{3.7}$$

By definition we have: $g(z) = z + \displaystyle\sum_{k=1}^{N} \frac{\rho_k}{\tau_k - z} - \sum_{k=1}^{N} \frac{\rho_k}{\tau_k - S(z)}$; and for z large, $S(z) = -z - \gamma + O\left(\dfrac{1}{z}\right)$.

Defining $\delta_k = (\tau_k + \gamma + \epsilon)$, for small $\epsilon = \epsilon(R)$, and applying Eqn(3.6)

$$g(z) = z - (\frac{\sum_k \rho_k}{z} + \frac{\sum_k \rho_k \tau_k}{z^2} + \frac{\sum_k \rho_k \tau_k^2}{z^3} + \cdots) - \sum_{k=1}^{N} \frac{\rho_k}{\tau_k - S(z)}, \tag{3.8}$$

which, after applying (3.6) once more to the right most term, gives

$$g(z) = z - (\frac{\sum_k \rho_k}{z} + \frac{\sum_k \rho_k}{z}) - (\frac{\sum_k \rho_k \tau_k}{z^2} - \frac{\sum_k \rho_k \delta_k}{z^2}) + \cdots; \tag{3.9}$$

this establishes Eqn (3.5) with $c_1 = -2\sum_1^N \rho_k \neq 0$. □

Parabolic rational maps admit natural dynamical measures on $J(S)$, called conformal measures. Let R be a rational map of degree ≥ 2; for $h > 0$ a probability measure m_h on $J(R)$ is called *h-conformal* for R if

$$m_h(R(A)) = \int_A |R'|^h dm_h \tag{3.10}$$

for every Borel set $A \subset J(R)$ such that $R|_A$ is injective. The following result combines several results from [2] (including Theorems 3.1, 8.7, and 8.8).

Theorem 3.4. Assume that R is a parabolic rational map. Then there exists a unique h-conformal measure m_h for R with the following properties:

(1) m_h is nonatomic;
(2) there exists an R-invariant measure $\mu \sim m_h$ that is σ-finite and ergodic;
(3) h = HD(J(R)) < 2, where HD(J(R)) denotes the Hausdorff dimension of $J(R)$.

For $\omega \in J(R)$ a rationally indifferent periodic point of period k with $p(\omega)$ petals, let

$$\alpha = \min_\omega \frac{p(\omega)+1}{p(\omega)} h. \tag{3.11}$$

Theorem 3.5 ([2], Theorems 9.8 and 9.11)**.** For a parabolic map R, there exists a conservative and exact σ-finite infinite invariant measure, $\mu \sim m_h$, if and only if $\alpha \leq 2$.

Theorem 3.6. Assume S is a complex Boole function (of the form (3.1)), where the fixed point at ∞ is the only parabolic cycle. If HD(J(S)) $< \frac{4}{3}$, then there exists a conservative and exact σ-finite infinite invariant measure, $\mu \sim m_h$, with $h =$ HD(J(R)).

Proof. By Theorem 3.3 there are two petals at ∞, so we apply Eqn (3.11) and Theorem 3.5 to obtain the result. □

It remains to show that many complex Boole functions satisfy the hypotheses of Theorem 3.6. Since for Boole functions of the form (1.1), HD(J(S)) = 1, it suffices to establish some continuity properties of the Hausdorff dimension of $J(S_\eta)$.

3.1. **Sequences converging to real Boole functions.** We fix a real Boole function of the form (1.1) with parameters $(\beta, t_1, \ldots, t_N, p_1, \ldots, p_N) \subset \mathbb{R}^{2N+1}$ and $p_k > 0$. We choose any sequence $\{S_n\}$ of complex Boole functions, each of degree $N+1$, satisfying for each $n \in \mathbb{N}$:

- $S_n = -z - \gamma^{(n)} - \sum_{k=1}^N \frac{\rho_k^{(n)}}{\tau_k^{(n)} - z}$, $\quad \gamma^{(n)}, \tau_k^{(n)}, \rho_k^{(n)} \in \mathbb{C}, \mathrm{Re}(\rho_k^{(n)}) > 0$, $\tau_j^{(n)} \neq \tau_k^{(n)}$ if $j \neq k$;

- $S_n \to S$ in the sense that
$$(\gamma^{(n)}, \tau_1^{(n)}, \ldots, t_N^{(n)}, \rho_1^{(n)}, \ldots, \rho_N^{(n)}) \to (\beta, t_1, \ldots, t_N, p_1, \ldots, p_N)$$
(S_n converges to S *algebraically* [14]);
- S_n has no critical points in $J(S_n)$ and no other parabolic points apart from the one at $z_0 = \infty$.

Proposition 3.7. Under the assumptions above, $J(S_n) \to J(S)$ in the Hausdorff topology, and $\mathrm{HD}(J(S_n)) \to 1 = \mathrm{HD}(J(S))$. In particular for n large enough, $\mathrm{HD}(J(S_n)) < 4/3$.

Proof. This result follows from Proposition 3.1 and from [14], (Theorems 1.3 and 1.4]). □

As a corollary we obtain the following result.

Theorem 3.8. Let S be any Boole function satisfying (1.1) with coefficients $\Phi = \{\beta, p_1, t_1, \ldots, p_N, t_N\} \in \mathbb{R}^{2N+1}$. Then there exists an $\varepsilon > 0$ such that for every $\eta = \{\gamma, \rho_1, \tau_1, \ldots, \rho_N, \tau_N\} \in B_\varepsilon(\Phi) \subset \mathbb{C}^{2N+1}$, the map S_η admits a nonatomic conservative and exact σ-finite infinite measure equivalent to the conformal measure m_h, with $h = HD(J(S_\eta))$.

Proof. We know that Lebesgue measure λ satisfies the conclusion of the theorem for Φ. The hypotheses on S and an application of Proposition 3.1 imply that the hypotheses of Proposition 3.7 hold. Then by Theorem 3.6, the result holds. □

3.2. **The quadratic case.** We now turn to quadratic complex Boole functions of the form

$$S(z) = -z - \gamma - \frac{\rho}{\tau - z}, \quad \gamma, \tau, \rho \in \mathbb{C}, \mathrm{Re}(\rho) > 0. \tag{3.12}$$

Theorem 3.3 shows there are two petals at ∞ and Proposition 3.1 shows that the Julia set varies continuously near the set of real coefficients. We prove that on a large set of parameters the measure μ from Theorem 3.4 is infinite and has highly mixing measure theoretic properties (eg., conservative, exact, and ergodic), sharing the properties of Boole functions. This set is shown in Figure 2.

By Theorem 2.11(2) and Section 2.2(2), every quadratic negative Boole function with real coefficients is c-isomorphic to precisely one map of the form $S_{(-b,b,1)}$, where $b \in (0, 2]$, and each $S_{(-b,b,1)}$ is conformally conjugate to exactly one map $R_a(z) = \frac{z^2 - z}{1 + az}$, where $a \in [-3, -1)$. We set $\Omega = \mathbb{C} \setminus \{-1\}$.

Proposition 3.9. Given any $a \in [-3, -1)$, there exists a neighbourhood $V_a \subset \Omega$ of a such that every $\eta \in V_a$ corresponds to a Boole function of the form (3.12).

Proof. For any $\eta \in V_a$, set $b = -(1+\eta)$; we conjugate R_η to $H_b(z) = -z + b - \frac{b}{1-z}$ using $\psi(z) = \frac{1}{z}$. It is easy to see that we can choose V_a such that $\mathrm{Re(b)} = \mathrm{Re}(-(1+\eta)) > 0$. □

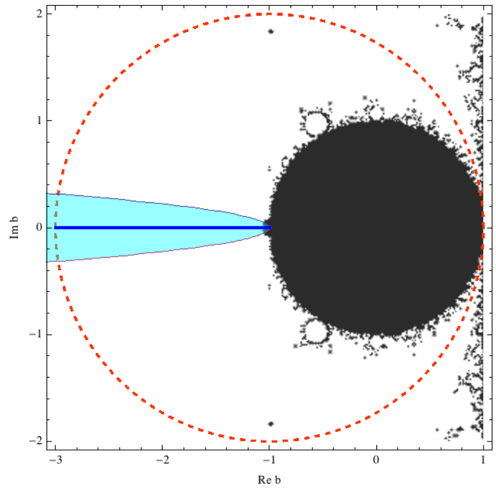

FIGURE 2. In a-space for the family of maps R_a or equivalently $H_{-(1+a)}$, the blue line represents the real Boole functions and the light shaded region shows where parameters correspond to maps with infinite invariant measures. The parameters coloured black show maps with an attracting fixed point. The dotted red circle encloses the region containing precisely one conformal equivalence class of each map.

Theorem 3.10. There exists an open set $\mathcal{U} \subset \Omega$ such that for each $a \in \mathcal{U}$, the complex Boole map $H_{-(1+a)}(z) = -z-(1+a)-\frac{-(1+a)}{1-z}$ admits a nonatomic conservative and exact σ-finite infinite measure equivalent to the conformal measure m_h, with $h = \mathrm{HD}(H_{-(1+a)})$.

Proof. We consider $a \in \Omega$, and set $b = -(1+a)$; we use Proposition 3.9 to parametrize Boole functions H_b with this parameter. By Theorem 3.7 there exist open sets $U_a \subset \Omega$ around each $a < -1$ where the Hausdorff dimension of H_b varies continuously with the parameter, and $\mathrm{HD(H_b)} < 4/3$ for all $a \in U_a$. Therefore, by Theorem 3.5 we have the result for all $a \in U_a \cap V_a$, and we set $\mathcal{U} = \bigcup_{a\in[-3,-1)} (U_a \cap V_a)$. □

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Department of Mathematics, Agnes Scott College, 141 E. College Ave. Box 1093 Decatur, Georgia 30030

E-mail address: rbayless@agnesscott.edu

Department of Mathematics, University of North Carolina at Chapel Hill, CB #3250, Chapel Hill, North Carolina 27599-3250,

E-mail address: jmh@math.unc.edu