

Rational families converging to a family of exponential maps

Joanna Furno, Jane Hawkins, Lorelei Koss

Abstract. We analyze the dynamics of a sequence of families of non-polynomial rational maps, $\{f_{a,d}\}$, for $a \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $d \geq 2$. For each d , $\{f_{a,d}\}$ is a family of rational maps of degree d of the Riemann sphere parametrized by $a \in \mathbb{C}^*$. For each $a \in \mathbb{C}^*$, as $d \rightarrow \infty$, $f_{a,d}$ converges uniformly on compact sets to a map f_a that is conformally conjugate to a transcendental entire map on \mathbb{C} . We study how properties of the families $f_{a,d}$ contribute to our understanding of the dynamical properties of the limiting family of maps. We show all families have a common connectivity locus; moreover the rational maps contain some well-studied examples.

Mathematics Subject Classification (2010). 37F10, 37F45, 30D05.

Keywords. complex dynamics, Julia sets, rational maps, entire functions.

1. Introduction

We analyze the dynamics of a sequence of families of non-polynomial rational maps, $\{f_{a,d}\}$, for $a \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $d \geq 2$. For each fixed d , $\{f_{a,d}\}$ is a family of rational maps of degree d ; $f_{a,d} : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$, where \mathbb{C}_∞ denotes the Riemann sphere. For each $a \in \mathbb{C}^*$, as $d \rightarrow \infty$, $f_{a,d}$ converges uniformly on compact sets on \mathbb{C}^* to a map f_a that is conformally conjugate via a Möbius map to a transcendental entire map on \mathbb{C} . We study how properties of the families $f_{a,d}$ contribute to our understanding of the dynamical properties of the limiting family of maps. The usefulness of this approach is that lower degree families have been studied earlier ([6, 12]), and all families have a common connectivity locus (Theorem 2.1 below). Other authors have published results about polynomial approximations to entire maps (see e.g., [2, 4, 5]); we show that with one exception in degree 2 our maps are not polynomial maps.

For each $d \geq 2$, and $a \in \mathbb{C}^*$, we define:

$$f_{a,d}(z) = az \left(1 + \frac{1}{zd}\right)^d. \quad (1.1)$$

For each fixed d the family $f_{a,d}$ is conformally conjugate via the conformal map $\psi(z) = dz$

to the family $h_{a,d}(z) = \frac{a(z+1)^d}{z^{d-1}}$, $a \in \mathbb{C}^*$; for example $h_{a,2} = a(z + \frac{1}{z} + 2)$, which has been studied in [6] and [12]. We import some properties of these families to our setting; to our knowledge these families of maps have previously not been connected.

By the pointwise convergence to exponential maps, we have that

$$\lim_{d \rightarrow \infty} f_{a,d}(z) = f_a(z) = aze^{1/z},$$

and the convergence is uniform on compact subsets on \mathbb{C}^* . Moreover, the convergence extends continuously to the point at ∞ , by defining $f_a(\infty) = \infty$. Using the series expansion of $f_a(z)$ centered at $z = \infty$, we see the extension is holomorphic. In fact, the map $f_a(z)$ is conjugate to the more familiar map $g_\lambda(z) = \lambda ze^z$, via the involution on \mathbb{C}_∞ given by $z \mapsto -1/z$ and with $\lambda = \frac{1}{a}$. Therefore, since many of the properties of g_λ are known [4, 8], we show how they emerge as limiting properties of the family of maps $f_{a,d}$.

The paper is organized as follows. In Section 2 we discuss properties that the rational families $f_{a,d}$ have in common, as well as differences among them. In particular, we prove that the Julia set of $f_{a,d}$ is connected if $|a| \leq 1$ and Cantor otherwise, and we prove that for each $d \geq 2$, there exists a positive measure set of parameters $a \in \mathbb{C}^*$ such that $f_{a,d}$ is ergodic with respect to Lebesgue measure. In Section 3, we discuss some relationships between periodic cycles of $f_{a,d}$ and the limiting maps f_a . Finally, in Section 4, we connect the results of this paper with previously known results about the limiting family f_a .

2. Properties of the rational families

We begin with a brief discussion of the critical points, fixed points, and fixed point multipliers of $f_{a,d}$ for $d \geq 2$. Taking the derivative, we find three distinct critical points: $c_1 = 0$ has order $d - 2$, $c_2 = -\frac{1}{d}$ has order $d - 1$, and $c_3 = \frac{d-1}{d}$ is a simple critical point. For every $f_{a,d}$ with $a \in \mathbb{C}^*$ and $d \geq 2$, we have the orbit

$$c_2 \mapsto c_1 \mapsto \infty \curvearrowright \tag{2.1}$$

and ∞ is a fixed point with multiplier $1/a$. The connectivity of the Julia set is determined by the modulus of this multiplier. Assume throughout the rest of this paper that $d \geq 2$.

Theorem 2.1. *If $|a| > 1$, then $J(f_{a,d})$ is a Cantor set. If $|a| \leq 1$ and $a \neq 0$, then $J(f_{a,d})$ is connected.*

Proof. If $|a| > 1$, then the point ∞ is an attracting fixed point. The immediate attracting basin B at ∞ must contain a critical point with an infinite forward orbit, so the critical point c_3 and the critical value $f_{a,d}(c_3)$ are also in B . We claim that both critical values from Eqn (2.1), c_1 and ∞ , are in B . Assuming the claim holds (and we only need to prove it for $c_1 = 0$), B contains all of the critical values. Therefore, in this case $J(f_{a,d})$ is a Cantor set (see e.g., [12], Thm B 1).

To prove the claim, we consider paths in the basin B . Since c_3 and $f_{a,d}(c_3)$ are in B and B is path-connected, there exists a path γ_1 from c_3 to $f_{a,d}(c_3)$ and a path γ_2 from

$f_{a,d}(c_3)$ to ∞ , and $\gamma_1, \gamma_2 \subset B$. Consider the preimages of γ_2 . Since 0 is a preimage of ∞ and c_3 is a preimage of $f_{a,d}(c_3)$, one of the paths in $f^{-1}(\gamma_2)$ is a path from c_3 to 0. Let γ_3 be this path, and let $-\gamma_3$ be the path in the opposite direction, from 0 to c_3 . Since γ_2 is contained in $F(f_{a,d})$, so is $-\gamma_3$ as a subset of $f^{-1}(\gamma_2)$. Thus the path composed of $-\gamma_3$ followed by γ_1 followed by γ_2 is a path in B from 0 to ∞ . Therefore, B contains 0.

The fixed point at ∞ of $f_{a,d}$ is a repelling fixed point if and only if $0 < |a| < 1$, and it is a neutral fixed point if and only if $|a| = 1$. In both of these cases, we want to show that there is at most one critical point in each Fatou component. By [16] there cannot be any Herman rings since there is at most one infinite forward critical orbit.

If $0 < |a| < 1$, then ∞ is repelling, hence $\{c_1, c_2\} \in J(f_{a,d})$. Thus, c_3 is the only critical point that could be in $F(f_{a,d})$.

If ∞ is a neutral fixed point and a is a root of unity, then $\{c_1, c_2\} \in J(f_{a,d})$. Similarly, if ∞ is a Cremer fixed point, then $\infty \in J(f_{a,d})$ and $\{c_1, c_2\} \in J(f_{a,d})$ as well. In both these cases, c_3 is the only critical point that could be in $F(f_{a,d})$.

Finally, if ∞ forms the center of a Siegel disk U , then the critical point c_3 must have a forward orbit on ∂U and therefore be in $J(f_{a,d})$. Regarding the remaining critical points: 0 must lie in a preimage of the Siegel disk about ∞ (it cannot be in U , since the restriction of $f_{a,d}$ to U is a homeomorphism). It remains to show that $-1/d$ and 0 cannot occupy the same Fatou component. If there is a component of $F(f_{a,d})$ containing both $-1/d$ and 0, then its image, which is U , contains both image points 0 and ∞ , which is impossible.

In each of these cases, there remains at most one critical point in each Fatou component. Thus, it follows from [14] that $J(f_{a,d})$ is connected. \square

Proposition 2.2. *The map $f_{a,d}$ is not conformally conjugate to a polynomial unless $d = 2$ and $a = 1/4$.*

Proof. The map $f_{1/4,2} = \frac{z}{4} \left(1 + \frac{1}{2z}\right)^2$, has the property that the point $c_3 = 1/2$ is forward and backward invariant; i.e., c_3 is an exceptional point. Therefore $f_{1/4,2}$ is conjugate to the Tchebychev polynomial of degree 2 [6]. A map $f_{a,d}$ is conjugate to a polynomial if and only if there is a critical point which is forward and backward invariant. This is impossible for c_1 and c_2 by Eqn (2.1). Using $c_3 = \frac{d-1}{d}$, we calculate that

$$f_{a,d}(c_3) = \frac{a}{c_3^{d-1}}, \tag{2.2}$$

so if c_3 is fixed, then $a = c_3^d$. However, we also have that there are $d - 2$ other distinct points that map to c_3 , since it is a simple critical point, so there are no exceptional points unless $d = 2$ and $a = 1/4$. \square

The dynamics that can occur for the maps $f_{a,d}$ are limited due to the multiplicities of the critical points, which is summarized in the next result.

Proposition 2.3. *For each map $f_{a,d}$ exactly one of the following can occur.*

- (1) *there exists one attracting or superattracting cycle;*

- (2) there exists one cycle of Siegel disks;
- (3) there exists one parabolic cycle;
- (4) $J(f_{a,d}) = \mathbb{C}_\infty$.

Proof. Since the critical points c_1 and c_2 have finite forward orbits, only c_3 can have an infinite orbit. By [16] there cannot be any Herman rings since there is at most one infinite forward critical orbit. From the proof of Thm 2.1, if $|a| > 1$, then (1) occurs. If $|a| \leq 1$, then at most one of (1)-(3) can occur. If none of them occur, then $c_3 \in J(f_{a,d})$ and (4) holds. \square

2.1. Fixed and prefixed points for $f_{a,d}$. Counting multiplicity, each rational map of degree d has exactly $d + 1$ fixed points on \mathbb{C}_∞ . In addition to the simple fixed point at ∞ , $f_{a,d}$ has d simple fixed points that are related to the d th roots of a . For (nonzero) $a = re^{i\theta}$, let $b_{j,d} = r^{1/d}e^{i(\theta+2\pi j)/d}$ for $j = 0, \dots, d-1$.

Proposition 2.4. *For every $j = 0, \dots, d-1$, each point*

$$z_{j,d} = \frac{b_{j,d}}{(1-b_{j,d})d}$$

is a fixed point of $f_{a,d}$ with multiplier $f'_{a,d}(z_{j,d}) = db_{j,d} - (d-1)$.

Proof. Set $b = b_{j,d}$ for any $j = 0, \dots, d-1$. Since $1 + (1-b)/b = 1/b$ and $a = b^d$,

$$\begin{aligned} f_{a,d} \left(\frac{b}{(1-b)d} \right) &= a \frac{b}{(1-b)d} \left(1 + \frac{(1-b)d}{db} \right)^d \\ &= a \cdot \frac{b}{(1-b)d} \cdot \frac{1}{b^d} = \frac{b}{(1-b)d} \end{aligned}$$

and

$$\begin{aligned} f'_{a,d} \left(\frac{b}{(1-b)d} \right) &= a \left(1 + \frac{(1-b)d}{db} \right)^{d-1} \left(1 + \frac{(1-d)}{d} \cdot \frac{(1-b)d}{b} \right) \\ &= b^d \cdot \frac{1}{b^{d-1}} \cdot \frac{b + (1-d)(1-b)}{b} = db - (d-1). \end{aligned}$$

\square

Corollary 2.5. *For any $a \in \mathbb{C}$ with $|a| \geq 1$, the sequence of fixed points $z_{[d/2],d}$ of $f_{a,d}$ converges to 0 as $d \rightarrow \infty$.*

Proof. Let $a = re^{i\theta}$ with $r \geq 1$. Pick d_0 large enough so that for all $d > d_0$, the point $b_{[d/2],d}$ lies in the sector of the annulus $1 \leq |z| < r$ and $\pi - 0.1 < \text{Arg}(z) < \pi + 0.1$. Then $(1 - b_{[d/2],d})$ lies outside of the circle of radius 1 for every $d > d_0$, and $|(1 - b_{[d/2],d})d| \rightarrow \infty$ as $d \rightarrow \infty$. Since $|b_{[d/2],d}| = r^{1/[d/2]} \rightarrow 1$, we have $z_{[d/2],d} \rightarrow 0$. \square

Corollary 2.5 is used to prove the following result (compare with [3], Thm 4.2 for a similar result for the limiting maps).

Theorem 2.6. *Fix any $a \in \mathbb{C}^*$. Then for any $\varepsilon > 0$, there exists a $d_0 = d_0(a, \varepsilon)$ such that for all $d \geq d_0$, $J(f_{a,d}) \cap B_\varepsilon(0) \neq \emptyset$.*

Proof. If $|a| < 1$, then $0 \in J(f_{a,d})$ by Eqn (2.1) and the fact that ∞ is a repelling fixed point; so the theorem is trivially true for $a \in D^*$. For $|a| > 1$, all fixed points in \mathbb{C} are repelling by the proof of Thm 2.1, and therefore they are in $J(f_{a,d})$. By Corollary 2.5, there exists a d_0 such that for all $d \geq d_0$, there is a fixed point of $f_{a,d}$ contained in $B_\varepsilon(0)$, hence the result holds. The remaining case occurs when $|a| = 1$; if ∞ is a parabolic fixed point, then $\infty \in J(f_{a,d})$, and so is 0. The same is true if ∞ is a Cremer point. Therefore we assume that $|a| = 1$ and ∞ is the center of a Siegel disk for all $d \geq 2$. Then Corollary 2.5 guarantees the existence of a d_0 such that for $d \geq d_0$, there is a fixed point in $B_\varepsilon(0)$, call it p_d . The fixed point p_d cannot be in $F(f_{a,d})$ or it would be nonrepelling, which is impossible since $F(f_{a,d})$ consists of the Siegel disk centered at ∞ and its preimages by Proposition 2.3. \square

Clearly not all d fixed points in Prop 2.4 have the same derivative. Let $D = \{z : |z| < 1\}$, and $D^* = D \setminus \{0\}$. We know that at most one of the $d + 1$ fixed points can be attracting because at most one can attract the free critical point. If $|a| > 1$ by Prop 2.3, ∞ is the only attracting fixed point. If $a \in D^*$, the fixed point z_j is attracting if and only if $|b - (d-1)/d| < 1/d$, i.e., b is in the disk of radius $1/d$, centered at $c_3 = (d-1)/d$. Since b is a d th root of a , the boundary of the attracting fixed point region in the a parameter plane when $a \in D^*$ is a cardioid for degree $d = 2$ and is cardioid-like for higher degrees. Using this idea, the boundary is easy to parametrize.

For $a \in D^*$, we define the *center* of the fixed point region of $f_{a,d}$ to be the parameter for which the critical point c_3 is fixed. This occurs at the parameter

$$a = (1 - 1/d)^d = c_3^d \tag{2.3}$$

using Eqn (2.2). These centers converge to $a = 1/e$ as $d \rightarrow \infty$. For each d , we denote the (open) region in D^* for which $f_{a,d}$ admits an attracting fixed point by Ω_d . We parametrize the boundary of Ω_d based on this discussion.

Lemma 2.7. *For $a \in D^*$, $\partial\Omega_d$ is parametrized by $\alpha_d(t) = \left(\frac{e^{it} + d - 1}{d}\right)^d$, $t \in [0, 2\pi]$.*

Proof. The fixed point $b/((1-b)d)$ for $f_{a,d}$ is attracting if and only if $|b - (d-1)/d| < 1/d$. We parametrize the b values for the circle on the boundary by $\beta_d(t) = (e^{it} + d - 1)/d$. Since $a = b^d$, the boundary of the attracting fixed point region in the a parameter plane is

$$\alpha_d(t) = \left(\frac{e^{it} + d - 1}{d}\right)^d.$$

\square

Lemma 2.8. *For any $d \geq 2$, $\Omega_{d+1} \subset \Omega_d$.*

Proof. The b values for which $b/((1-b)d)$ is a neutral fixed point for $f_{a,d}$ lie on a circle of radius $1/d$ and pass through the point 1; clearly these circles bound nested disks in D^* . We then see that $\partial\Omega_d$ is the image of these disks under the mapping $z \mapsto z^d$, so the nesting persists and the result follows. \square

On the other hand, outside the fixed point region in D^* , there are many parameters for which $J(f_{a,d}) = \mathbb{C}_\infty$; many of them come from a such that $f_{a,d}^n(c_3)$ is fixed for some $n \geq 1$ but c_3 is not fixed; i.e., c_3 is *prefixed* in this case.

Definition 2.9. *We say that a parameter $a \in D^*$ is $-1/d$ -absorbing (or c_2 -absorbing) if the forward orbit of the critical point c_3 under the corresponding map $f_{a,d}$ contains the point $-1/d = c_2$.*

We say that $f_{a,d}$ is postcritically finite if all critical points have finite forward orbit but none are periodic.

If a is a c_2 -absorbing parameter, then $f_{a,d}$ is postcritically finite and c_3 is prefixed; it follows that $J(f_{a,d}) = \mathbb{C}_\infty$ and $f_{a,d}$ is Lebesgue ergodic (cf. [6], [11]).

Theorem 2.10. *For each degree $d \geq 2$, there exist c_2 -absorbing parameters in D^* .*

Proof. Fix $d \geq 2$. If $a = -\frac{(d-1)^{d-1}}{d^d}$, then $f_{a,d}(\frac{d-1}{d}) = -\frac{1}{d}$. In this case, we have the critical orbit diagram

$$c_3 \mapsto c_2 \mapsto c_1 \mapsto \infty \curvearrowright$$

so every critical point is strictly preperiodic. \square

By a result of Aspenberg ([1], Thms A and B) the existence of a c_2 -absorbing parameter implies that there is a positive Lebesgue measure set of parameters a such that $f_{a,d}$ satisfies the Collet-Eckmann property. This property says that there exist constants $C > 0$ and $\gamma > 0$ such that for any critical point $c \in J(f_{a,d})$, whose forward orbit does not contain any other critical point, we have:

$$|(f_{a,d}^n)'(f_{a,d}(c))| \geq Ce^{\gamma n} \quad \forall n \geq 0. \quad (2.4)$$

We call Eqn (2.4) the CE condition.

This leads to the following Corollary to Thm 2.10.

Corollary 2.11. *For each degree $d \geq 2$, there exists a set of positive Lebesgue measure in D^* such that the corresponding $f_{a,d}$ satisfies the CE condition.*

It also occurs that the first iterate of the critical point c_3 is a repelling fixed point.

Proposition 2.12. *For each $d \geq 3$, there exist parameters $a \in \mathbb{C}^*$ such that $f_{a,d}(c_3)$ is a repelling fixed point and $J(f_{a,d}) = \mathbb{C}_\infty$.*

Proof. For simplicity, write $c = c_3$; note that $a \neq 0, c \neq 0$. It is enough to show the existence of a parameter a such that $c \neq f_{a,d}(c)$ and $f_{a,d}(c)$ is fixed. If so, by the proof of Thm 2.1, it follows that $|a| < 1$ because the forward orbit of c is finite. Then since all critical points are preperiodic and not periodic, the terminating fixed points must be repelling, and therefore $F(f_{a,d}) = \emptyset$.

We calculate $f_{a,d}(c) = a/c^{d-1}$ and

$$f_{a,d}^2(c) = \frac{a^2}{c^{d-1}} \left(1 + \frac{c^{d-1}}{ad}\right)^d.$$

We consider

$$f_{a,d}^2(c) - f_{a,d}(c) = \left(\frac{a^2}{c^{d-1}}\right) \left(1 + \frac{c^{d-1}}{ad}\right)^d - \frac{a}{c^{d-1}},$$

and factor out $\frac{a}{c^{d-1}}$. Therefore we want to find the solutions to

$$s_d(a) = a \left(1 + \frac{c^{d-1}}{ad}\right)^d - 1 = 0. \quad (2.5)$$

Since s_d is a rational map of degree d , (with a pole at $a = 0$), there are d solutions to Eqn (2.5) in \mathbb{C} counting multiplicity. By Eqn (2.3), $f_{a,d}(c) = c$ if and only if $a = c^d$. Therefore $s_d(c^d) = 0$, and we want to compute the order of that zero.

We have

$$\begin{aligned} s'_d(a) &= \left(1 + \frac{c^{d-1}}{ad}\right)^d - \frac{c^{d-1}}{a} \left(1 + \frac{c^{d-1}}{ad}\right)^{d-1} \\ &= \left(\frac{ad + c^{d-1}}{ad}\right)^{d-1} \left(1 + \frac{c^{d-1}}{ad} - \frac{c^{d-1}}{a}\right) \\ &= \left(\frac{ad + c^{d-1}}{ad}\right)^{d-1} \left(1 - \frac{c^d}{a}\right), \end{aligned}$$

since $c = (d-1)/d$. Recall that if $a = -\frac{c^{d-1}}{d}$, then $f_{a,d}(c_3) = c_2$, which is not a fixed point. Thus, we can assume that $ad + c^{d-1} \neq 0$, and therefore $s'_d(a) = 0$ if and only if $a = c^d$, which is true if and only if $f_{a,d}(c) = c$.

Another similar computation shows that $s''_d(c^d) = c^{1-2d} \neq 0$, so we have an order 2 zero at $a = c^d$.

The other $d - 2$ solutions to Eqn (2.5) are simple zeros, $f_{a,d}(c)$ is a fixed point and not equal to c , so every critical point of $f_{a,d}$ is strictly preperiodic. The theorem then holds for these parameters. \square

Remark 2.13. 1. When $d = 3$, choosing $a = -1/27$, the critical point $c_3 = -2/3$ lands on a repelling fixed point at $z = -1/12$. The multiplier is -3 .

2. When $d = 4$, the value $a = \frac{1}{256}(-7 - 4i\sqrt{2})$ gives the orbit:

$$c_3 \mapsto \frac{1}{108}(-7 - 4i\sqrt{2}) \hookrightarrow$$

and the multiplier of the fixed point is $-4 - i\sqrt{2}$.

Lemma 2.14. *Suppose $a_0 \in \mathbb{C}^*$ is a parameter such that $f_{a_0,d}(c_3)$ is a fixed point. Then*

$$\left. \frac{d}{da} (f_{a,d}^2(c_3) - f_{a,d}(c_3)) \right|_{a=a_0} = 0$$

if and only if $f_{a_0,d}(c_3) = c_3$.

Proof. Suppose $a_0 \in \mathbb{C}^*$ is a parameter such that $f_{a_0,d}(c_3)$ is a fixed point. Then a_0 is a solution to Eqn (2.5). Using the notation in the proof of Prop 2.12, we have that

$$\left. \frac{d}{da} (f_{a,d}^2(c_3) - f_{a,d}(c_3)) \right|_{a=a_0} = 0$$

if and only if $s'_d(a_0) = 0$; this holds if and only if $a_0 = c_3^d$, which is true if and only if $f_{a_0,d}(c_3) = c_3$. \square

Theorem 2.15. *For each $d \geq 2$, there exists a positive measure set of parameters $a \in \mathbb{C}^*$ such that $f_{a,d}$ is ergodic with respect to Lebesgue measure and has an invariant probability measure equivalent to Lebesgue.*

Proof. By Prop 2.12 for each $d \geq 2$, there exist parameters in D^* for which the forward orbit of c_3 terminates in a fixed point, which is not a critical point, so the fixed point is necessarily repelling. We then apply Thm A from [15] after verifying the non-degeneracy condition needed for the theorem. This is precisely the condition verified in Lemma 2.14.

The purpose of the non-degeneracy condition given in [15] is to estimate the measure of the set of parameters on which both critical orbits (the orbit of c_3 and the second critical orbit given in Eqn (2.1)), stay far enough away from critical points under forward iteration of $f_{a,d}$. The multiplier of the fixed point at ∞ is $1/a$, so as long as the parameter a stays in the open set D^* , ∞ remains a repelling fixed point and we do not need to check the condition for the critical points $c_2 = -1/d$ and $c_1 = 0$. \square

2.2. Lattès examples. There are some distinguished postcritically finite maps among rational maps with the property that their unique measure of maximal entropy is equivalent to m , the normalized surface area measure on \mathbb{C}_∞ [17]; these are usually called Lattès examples. Assume that the postcritical set of $f_{a,d}$ is $P(f_{a,d}) = \{a_1, a_2, \dots, a_k\}$; if $f_{a,d}$ is not conjugate to $z \mapsto z^d$ it follows that $|P(f_{a,d})| \geq 3$ ([11], Thm 3.4), and, if $f_{a,d}$ is a Lattès example, $|P(f_{a,d})| \leq 4$ [13]. To each a_j we assign the positive integer ν_j which is the least common multiple of the local degrees $\deg(f_{a,d}^k, y)$, for all $k > 0$ and y such that $y \in f_{a,d}^{-k}(x)$, using $x = a_j$. We then have a set of ramification indices, also called a signature of $f_{a,d}$: $\mathcal{N} = \{\nu_1, \nu_2, \dots, \nu_k\}$ with each $\nu_k \geq 2$.

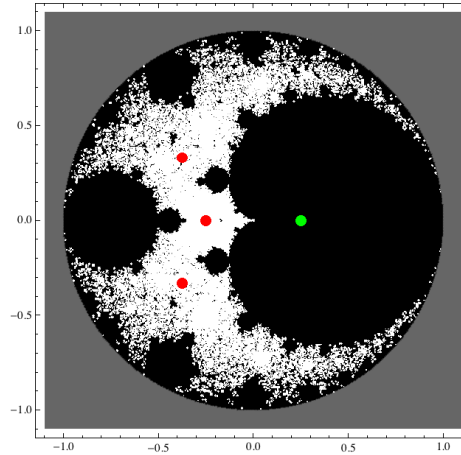


Figure 1. The parameter space for $f_{a,2}$ with parameters corresponding to Lattès maps marked: the Lattès polynomial is green, and the Lattès examples with $J(f_{a,2}) = \mathbb{C}_\infty$ are red. Parameters colored black or gray correspond to maps with attracting cycles.

In the degree 2 family it is known that we have several Lattès examples such that $J(f_{a,d}) = \mathbb{C}_\infty$ [6], as shown in Figure 1; they are located at $a = -1/4$, with signature $\{2, 4, 4\}$ and $a = \frac{-3 \pm \sqrt{7}i}{8}$ with signature $\{2, 2, 2, 2\}$. We also have the polynomial from Prop 2.2 with signature $\{2, 2, \infty\}$. Finally we have that the degree 3 map in Remark 2.13 (1) is Lattès with signature $\{2, 3, 6\}$. In fact, it is conformally conjugate, via the map $\phi(z) = -\frac{2}{3z}$, to the map shown in ([13], Sec. 8.2 (17)).

Since there is a short list of possible signatures for these examples, it is easy to show these maps are the only ones of this type in the families $f_{a,d}$. A map $f_{a,d}$ with a single postcritical orbit has the signature $\{2, 2d, 2d(d-1)\}$, and if the map has two critical orbits, then it has signature $\{2, d, d(d-1)\}$. Therefore our list of examples above exhausts the possible signatures that could yield a Lattès example [13]. For all other parameters such that $J(f_{a,d}) = \mathbb{C}_\infty$, the measure of maximal entropy is completely singular with respect to Lebesgue measure [17]; for the polynomial mapping, the maximal entropy measure is equivalent to one-dimensional Lebesgue measure on the Julia set.

3. Attracting Cycles and Convergence

In this section, we discuss how properties of the rational functions $f_{a,d}$ pass to the limiting family $f_a(z) = aze^{1/z}$. Let $\text{Log}(z) = \log(|z|) + i\text{Arg}(z)$ with $0 < \text{Arg}(z) \leq 2\pi$.

A straightforward computation shows that the fixed points of f_a are located at

$$p_j = \frac{-1}{\operatorname{Log}(a) + 2\pi j i}, \quad (3.1)$$

for all $j \in \mathbb{N} \cup \{0\}$, and

$$f'_a(p_j) = 1 + \operatorname{Log}(a) + 2\pi j i. \quad (3.2)$$

An attracting cycle of period k for $f_{a,d}$ is a zero of the map $h_{a,d} = f_{a,d}^k - \operatorname{Id}$, and $h_{a,d} \rightarrow f_a^k - \operatorname{Id}$ uniformly on compact sets in $\mathbb{C}_\infty \setminus \{0\}$. Therefore, by Hurwitz' Theorem any attracting cycle of f_a is a limit of attracting cycles of $f_{a,d}$. However we can also prove some converse statements. In particular, we describe the convergence of fixed points of $f_{a,d}$ as d increases.

Theorem 3.1. *If $j \in \mathbb{N} \cup \{0\}$ and $d \geq \max\{j + 1, 2\}$, then the fixed point $z_{j,d}$ of $f_{a,d}$ converges to the fixed point p_j of f_a .*

Proof. In Prop 2.4 we characterized the fixed points ($\neq \infty$) of $f_{a,d}$: for $a = re^{i\theta} \in \mathbb{C}^*$, let $b_{j,d} = r^{1/d} e^{i(\theta+2\pi j)/d}$ for each $j = 0, \dots, d-1$. Let $z_{j,d} = b_{j,d}/((1-b_{j,d})d)$ denote the corresponding fixed point of $f_{a,d}$.

Then

$$\begin{aligned} \lim_{d \rightarrow \infty} z_{j,d} &= \lim_{d \rightarrow \infty} \frac{r^{1/d} e^{i(\theta+2\pi j)/d}}{(1 - r^{1/d} e^{i(\theta+2\pi j)/d})d} \\ &= - \lim_{d \rightarrow \infty} r^{1/d} e^{i(\theta+2\pi j)/d} \lim_{d \rightarrow \infty} \frac{1/d}{r^{1/d} e^{i(\theta+2\pi j)/d} - 1} \\ &= \frac{-1}{\operatorname{Log}(a) + 2\pi j i} = p_j, \end{aligned}$$

a fixed point of f_a using Eqn (3.1). □

The multipliers of fixed points of $f_{a,d}$ pass to the limit as well.

Lemma 3.2. *If $j \in \mathbb{N} \cup \{0\}$ and $d \geq \max\{j + 1, 2\}$, then the multiplier of the fixed point $z_{j,d}$ of $f_{a,d}$ converges to the multiplier of the fixed point p_j of f_a .*

Proof. Fix j and $a = re^{i\theta} \in \mathbb{C}^*$. For $d \geq \max\{j, 2\}$, let $z_{j,d} = b_{j,d}/((1-b_{j,d})d)$ denote a fixed point of $f_{a,d}$. Recall that $z_{j,d}$ has multiplier

$$db_{j,d} - (d-1) = \left(dr^{1/d} \cos\left(\frac{\theta+2\pi j}{d}\right) - d + 1 \right) + dr^{1/d} \sin\left(\frac{\theta+2\pi j}{d}\right) i.$$

Since $r \neq 0$, we have the following limits for the real and imaginary parts of the multipliers:

$$\begin{aligned} \lim_{d \rightarrow \infty} \left(dr^{1/d} \cos\left(\frac{\theta+2\pi j}{d}\right) - d + 1 \right) &= 1 + \lim_{d \rightarrow \infty} \frac{r^{1/d} \cos\left(\frac{\theta+2\pi j}{d}\right) - 1}{1/d} \\ &= 1 + \log r \end{aligned}$$

and

$$\begin{aligned} \lim_{d \rightarrow \infty} dr^{1/d} \sin\left(\frac{\theta + 2\pi j}{d}\right) &= \lim_{d \rightarrow \infty} \frac{r^{1/d} \sin\left(\frac{\theta + 2\pi j}{d}\right)}{1/d} \\ &= \theta + 2\pi j, \end{aligned}$$

which agrees with Eqn (3.2). □

We next discuss how the regions Ω_d transform as we take the limit.

Theorem 3.3. *For $a \in D^*$, the center of the attracting fixed point region of $f_{a,d}$ converges to $a = 1/e$, at which f_a has a superattracting fixed point at $p = 1$. The boundary of the attracting fixed point region of $f_{a,d}$ converges to $\alpha(t) = e^{(e^{it}-1)}$.*

Proof. We showed that the center of Ω_d occurs at the parameter $a = (1 - 1/d)^d = c_3^d$ in Eqn (2.3) This center converges to $a = 1/e$ as $d \rightarrow \infty$. Clearly, $f_{1/e}(1) = 1$ and $f'_{1/e}(1) = 0$.

Using Lemma 2.7, the boundary of Ω_d in the a parameter plane is

$$\alpha_d(t) = \left(1 + \frac{e^{it} - 1}{d}\right)^d.$$

Thus, the limiting curve is

$$\alpha(t) = \lim_{d \rightarrow \infty} \alpha_d(t) = e^{(e^{it}-1)}.$$

□

We define $\Omega = \lim_{d \rightarrow \infty} \Omega_d = \bigcap_{d \geq 2} \Omega_d$.

Corollary 3.4. *If $a \in \Omega \setminus \partial\Omega$, then f_a has an attracting fixed point, and $f_{a,d}$ has an attracting fixed point for each d .*

Proof. Suppose $a \in \Omega \setminus \partial\Omega$ and let $b = \text{Log}(a)$. The fixed point of p_0 of f_a is attracting when $|1 + b| < 1$ by Eqn (3.2) using $j = 0$, i.e. when b is in the disk of radius 1 centered at -1 . Let $\beta(t) = e^{it} - 1$ define the boundary of this region. Then, since $b = \text{Log}(a)$, we have the boundary of the attracting fixed point region for f_a is $e^{(e^{it}-1)} = \alpha(t) = \partial\Omega$, as claimed. The second statement follows since $a \in \Omega_d$ for all d . □

In the next lemma, we show that every attracting cycle contains a point that is outside of the disk of radius $1/6$.

Lemma 3.5. *For $d \geq 3$, if z_1, z_2, \dots, z_n is an attracting periodic orbit for $f_{a,d}$, then for some i*

$$|z_i| > \frac{d-2}{2d} \geq \frac{1}{6}.$$

Proof. We use the following expression of the derivative.

$$\begin{aligned} f'_{a,d}(z) &= a \left[\left(1 + \frac{1}{dz}\right)^d - \frac{1}{z} \left(1 + \frac{1}{dz}\right)^{d-1} \right] \\ &= f_{a,d}(z) \frac{(dz + 1 - d)}{z(dz + 1)}. \end{aligned} \tag{3.3}$$

Thus for any point z_i in the cycle,

$$\begin{aligned} \left| \prod_{i=1}^n f'_{a,d}(z_i) \right| &= \left| \prod_{i=1}^n f_{a,d}(z_i) \frac{\prod_{i=1}^n (dz_i + 1 - d)}{\prod_{i=1}^n z_i \prod_{i=1}^n (dz_i + 1)} \right| \\ &= \left| \prod_{i=1}^n z_{i+1} \frac{\prod_{i=1}^n (dz_i + 1 - d)}{\prod_{i=1}^n z_i \prod_{i=1}^n (dz_i + 1)} \right| \\ &= \prod_{i=1}^n \left| 1 - \frac{d}{dz_i + 1} \right| \end{aligned}$$

Since the product is less than one, there must be a term that is less than 1, so for some i ,

$$\left| 1 - \frac{d}{dz_i + 1} \right| < 1.$$

If k is a point for which $|1 - k| < 1$, then we have that $|k| < 2$. Therefore,

$$d < 2|dz_i + 1| < 2d|z_i| + 2,$$

or

$$|z_i| > \frac{d-2}{2d}.$$

□

Lemma 3.5 is the key step in the proof of statement (2) in Thm 3.6.

Theorem 3.6. *Fix $a \in \mathbb{C}^*$.*

- (1) *If f_a has an attracting cycle of period k on \mathbb{C}^* , then there exists $d_0 \in \mathbb{N}$ such that, for all $d \geq d_0$, $f_{a,d}$ has an attracting cycle of period k .*
- (2) *If $f_{a,d}$ has an attracting cycle of period k on \mathbb{C}_∞ for infinitely many $d \in \mathbb{N}$, then f_a has a cycle of period dividing k that is either attracting or neutral.*

Proof. First, if $|a| > 1$, then the point at ∞ is an attracting fixed point for $f_{a,d}$ for all d , and for f_a . So the result is trivial. Therefore we assume that $|a| \leq 1$. (1): This follows from uniform convergence on compact sets and Hurwitz's Theorem.

(2): For $a \in D^*$, suppose that $f_{a,d}$ has an attracting cycle of period k for infinitely many $d \in \mathbb{N}$. By Lemma 3.5, for each such $d \geq 3$, there exists a periodic point z_d of period k such that $|z_d| > (d-2)/2d \geq 1/6$. Since this infinite sequence is contained in a compact subset of the Riemann sphere (namely $\mathbb{C}_\infty \setminus \{z \in \mathbb{C} : |z| \leq 1/6\}$), it has an accumulation point z^* . By the convergence of $f_{a,d}$ to f_a , we have $f_a^k(z^*) = z^*$ and $|(f_a^k)'(z^*)| \leq 1$. □

4. A Comparison of Hyperbolicity and Julia and Fatou Sets

In the last section of this paper we compare some results proved earlier with known results about the limiting maps. We begin with a brief review of the family of limiting maps and their conjugate versions.

The family $g_{\frac{1}{a}}(z) = \frac{z}{a}e^z$, which is conformally conjugate to the limiting map $f_a(z) = aze^{1/z}$, has been well studied (see e.g., [3, 4, 8]). It is also useful to view each map f_a as a map on the punctured plane, $f_a : \mathbb{C}^* \rightarrow \mathbb{C}^*$ with 0 and ∞ as asymptotic values. For example, since f_a only has one critical value and two asymptotic values, it has no Baker domains or wandering domains [7, 9]. From the sources mentioned above we collect some of the properties of the Julia and Fatou sets of f_a .

For each $a \in \mathbb{C}^*$, $f_a : \mathbb{C}^* \rightarrow \mathbb{C}^*$ extends holomorphically to a fixed point at ∞ (i.e., ∞ is a removable singularity). The point at 0 is an asymptotic value, (as is ∞ when the domain is \mathbb{C}^*), and as a map from \mathbb{C}^* to \mathbb{C}_∞ there is one critical point at $c_0 = 1$, with corresponding critical value $v_a \equiv v = ae$. Since the domain of f_a can extend to ∞ , it is often useful to view $J(f_a)$ as a subset of the sphere with the origin removed, which we will denote by \mathbb{C}_∞^* .

We compare some properties of $J(f_a)$ and $J(f_{a,d})$.

Theorem 4.1. *For $a \in \mathbb{C}^*$, and any $d \geq 2$, the maps f_a and $f_{a,d}$ satisfy the following.*

- (1) $J(f_a)$ is not a Cantor set, for any $a \in \mathbb{C}^*$ (i.e., $J(f_a)$ is not totally disconnected); for $|a| > 1$, $J(f_a)$ is not connected while $J(f_{a,d})$ is a Cantor set.
- (2) For $|a| > 1$, $m(J(f_a)) = m(J(f_{a,d})) = 0$ (m is Lebesgue measure on \mathbb{C} .)
- (3) If $a \in D^*$, then $J(f_a) \cup \{0\} \subset \mathbb{C}_\infty$ is connected, and $J(f_{a,d}) \subset \mathbb{C}_\infty$ is connected.
- (4) If $\lim_{n \rightarrow \infty} f_a^n(v) = 0$, then $J(f_a) = \mathbb{C}_\infty^*$; if $\lim_{n \rightarrow \infty} f_{a,d}^n(c_3) = 0$, then $J(f_{a,d}) = \mathbb{C}_\infty$.
- (5) For any $a \in \mathbb{C}^*$ and for any $\varepsilon > 0$, $J(f_a) \cap B_\varepsilon(0) \neq \emptyset$. Further, there exists a $d_0 = d_0(a, \varepsilon)$ such that for all $d \geq d_0$, $J(f_{a,d}) \cap B_\varepsilon(0) \neq \emptyset$.
- (6) For all $a \in \mathbb{C}^*$, $J(f_a)$ has Hausdorff dimension 2. For $d = 2$ and $a = 1/4$, $J(f_{a,d})$ has Hausdorff dimension 1.

Proof. (1): The result that $J(f_a)$ is not Cantor follows from ([3], Cor. 2 to Thm 4.3); that $J(f_a)$ is not connected for $|a| > 1$ follows from the fact that $F(f_a)$ has exactly one component, consisting of the attracting basin at ∞ [8]. The last statement in (1) follows from Thm 2.1.

(2): The first statement is from ([8], Thm 2) and the second statement follows from the hyperbolicity of $f_{a,d}$.

(3): We use from ([8], Thm 1, (2) (b)) the result that every component in $F(f_a)$ is simply connected, and the fact that $\infty \in J(f_a)$. If we add the point at the origin to $J(f_a)$, then it is classical that the complement of a collection of simply connected components of the sphere is connected. The second statement follows from Thm 2.1.

(4) follows from ([3], Cor. to Thm. 4.10) for f_a . To prove (4) for $f_{a,d}$, by Thm 2.1 and its proof the hypothesis implies that $a \in D^*$. Then c_3 is the only critical point not

automatically in $J(f_{a,d})$; if $\lim_{n \rightarrow \infty} f_{a,d}^n(c_3) = 0$, then $c_3 \in J(f_{a,d})$, and the result follows since there cannot be any Fatou component.

(5): The first statement is Thm 4.3 in [3]. The second statement is Thm 2.6.

(6): The first sentence is Thm 6 in [8]. The second sentence follows from Prop 2.2 and [6]. \square

The next proposition gives some properties of the limiting parameter space. This result appears in [8].

Proposition 4.2. *For $a \in \mathbb{C}^*$, the map $f_a(z) = aze^{1/z}$ satisfies the following.*

- i. There are two regions in \mathbb{C}^* for which f_a has an attracting fixed point: $\mathbb{C} \setminus \overline{D}$, and the region Ω described in Cor 3.4.*
- ii. If $a \in \Omega$, then f_a is 2-to-1 on the Fatou component containing the attracting fixed point. Moreover, there are infinitely many Fatou components for f_a , all of which eventually map onto the component containing the attracting fixed point.*
- iii. In D^* there exist attracting periodic regions of all periods, and each attracting periodic region has a component that is tangent to the unit circle.*

Figure 2 shows a picture of the parameter space for $f_{a,4}$; the figure was created using Dynamics Explorer and the following algorithm. Inside the unit disk, the parameters are colored based on the orbit of the free critical point c_3 . If the orbit of the free critical point passes within 10^{-5} of c_2 under a large number of iterates, then the parameter is colored blue. We evaluate $f'_{a,4}$ at $f_{a,4}^k(c_3)$ for each k and take the product of these derivatives. If this product becomes less than 10^{-15} under a large number of iterates, then the parameter is colored red. This condition is called the Buff derivative test and indicates that c_3 is tending toward an attracting cycle. If none of these conditions is satisfied after a large number of iterates, then the parameter is colored black. Parameters outside of the unit disk are colored yellow.

Similarly, Figure 3 is a picture of the parameter space for f_a created with Dynamics Explorer. For this algorithm, if the iterates of 1 under f_a become greater than 10^{15} after a large number of iterates, then the parameter is colored yellow. Again, we evaluate f'_a at each iterate $f_a^k(1)$ and take the product of these derivatives. If this product becomes less than 10^{-15} after a large number of iterates, then the parameter is colored red. If neither of these conditions is satisfied, then the parameter is colored black.

4.1. Hyperbolicity and subhyperbolicity. We denote by f either a rational map $f_{a,d}$ as defined above or a limit map f_a . We define the *postsingular set* of f to be the topological closure of the forward iterates of all singular values (in this case the orbits of all critical values), and denote it by $P(f)$. The notion of hyperbolicity is clear for rational maps, but less so in the setting of f_a . We define an entire map g to be hyperbolic if $P(g)$ is compact in \mathbb{C} and $P(g) \cap J(g) = \emptyset$ ([10], Sec. 6); we define f_a to be hyperbolic if and only if $g_{1/a}$ is. For f rational, when $P(f)$ is disjoint from $J(f)$, we say that f is *hyperbolic*. For all maps $f = f_{a,d}$ or $f = f_a$, f is hyperbolic if and only if every critical point converges to an attracting periodic orbit ([10, 12]). This definition of hyperbolicity

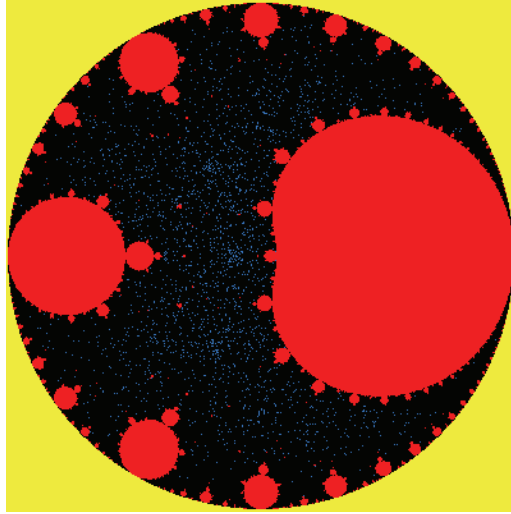


Figure 2. The parameter space for $f_{a,4}$.

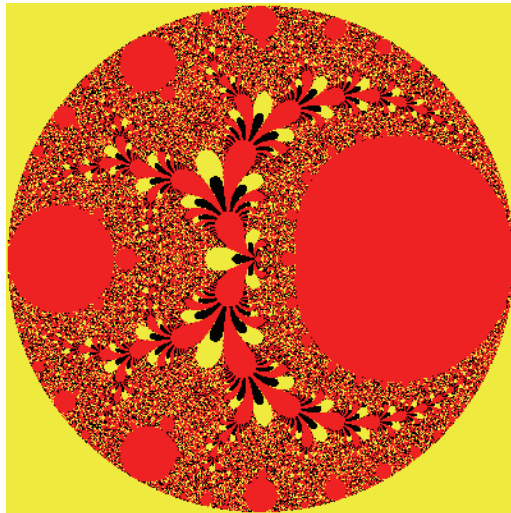


Figure 3. The parameter space for f_a .

is equivalent to uniform expansion of f on the Julia set with respect to a conformal metric defined on a neighborhood of $J(f)$.

For rational maps, there is a related notion of subhyperbolicity; f is *subhyperbolic* if all critical points in the Julia set have finite forward orbits and critical points in the Fatou set converge to an attracting periodic orbit. This notion gives a metric with some singularities with respect to which f is expanding on the Julia set. We have the following result.

Theorem 4.3. *For the maps $f_{a,d}$ and f_a defined above,*

- (1) *if $|a| > 1$, then all maps $f_{a,d}$ and f_a are hyperbolic;*
- (2) *let $a \in D^*$. If there exists an attracting periodic cycle for f_a , then for d large enough, $f_{a,d}$ is subhyperbolic, while f_a is hyperbolic. In particular, the interior of the region Ω from Cor 3.4 is a subhyperbolic region for $f_{a,d}$, and a hyperbolic region for f_a ;*
- (3) *all c_2 -absorbing maps $f_{a,d}$ ($a \in D^*$), are subhyperbolic.*

Proof. In the case of (1), all critical points of $f_{a,d}$ are attracted to the fixed point at ∞ as discussed in the proof of Thm 2.1. For the maps f_a , we do not have critical points corresponding to c_2 and c_1 , so we obtain hyperbolicity whenever an attracting orbit is present. For (2), we apply Thm 3.6(1). For all $a \in D^*$, no map $f_{a,d}$ can be hyperbolic since the critical points $c_2 = -1/d \rightarrow 0$, and $c_1 = 0$ are always in the Julia set. Thus if $f_{a,d}$ has an attracting cycle it must attract c_3 . For f_a , $c_1 = 0$ and ∞ are not in the domain, so f_a is hyperbolic.

(3) holds since c_3 has a finite forward orbit and is in $J(f_{a,d})$. □

References

- [1] M. Aspenberg, The Collet-Eckmann condition for rational functions on the Riemann sphere. *Math. Z.* (3-4) **273** (2013), 935–980.
- [2] C. Bodelon, R. Devaney, M. Hayes, G. Roberts, L. Goldberg, J. Hubbard, Hairs for the complex exponential family, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* (8) **9** (1999), 15171534
- [3] A. Eremenko, and M. Lyubich, The dynamics of analytic transformations. (Russian) *Algebra i Analiz 1* no. 3 (1989), 1–70; translation in *Leningrad Math. J.* (3) **1** (1990), 563–634.
- [4] N. Fagella, Limiting dynamics for the complex standard family. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* (3) **5** (1995), 673–699.
- [5] A. Garijo, X. Jarque, and M. Moreno Rocha, Joining polynomial and exponential combinatorics for some entire maps, *Publ. Mat.* (1) **54** (2010), 113136.
- [6] J. Hawkins, Lebesgue ergodic rational maps in parameter space. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* (6) **13** (2003), 1423–1447.
- [7] J. Kotus, *Iterated Holomorphic Maps on the Punctured Plane*, Dynamical systems (Sopron, 1985), 10–28, Lecture Notes in Econom. and Math. Systems, 287, Springer, Berlin (1987), 10–28.

- [8] D. Kremer, Some investigations on the dynamics of the family $\lambda z \exp(z)$, *Complex Variables, Theory and Application: An International Journal*. (4) **45** (2001), 355–370.
- [9] F. Liping, On the Iteration of Holomorphic Self-Maps of C^* , *Acta Mathematica Sinica, New Series*. (1) **14** (1998), 139–144
- [10] C. McMullen, Area and Hausdorff dimension of Julia sets of entire functions, *Trans. Amer. Math. Soc.* (1) **300** (1987), 329–342.
- [11] C. McMullen, *Complex dynamics and renormalization*, Annals of Mathematics Studies, 135. Princeton University Press, Princeton, NJ, 1994.
- [12] J. Milnor, On Rational Maps with Two Critical Points, *Experiment. Math.* (4) **9** (2000), 481–522.
- [13] J. Milnor, On Lattès maps, Dynamics on the Riemann sphere, 9–43, *Eur. Math. Soc.*, (Zurich 2006).
- [14] F. Peherstorfer and C. Stroh, Connectedness of Julia sets of rational functions, *Comp. Methods and Function Theory 1*, **1** (2001), 61–79.
- [15] M. Rees, Positive measure sets of ergodic rational maps, *Ann. Sci. Ecole Norm. Sup.* (4, no. 3) **19**(1986), 383–407.
- [16] F. Yang, Rational maps without Herman Rings, *Proc. Amer. Math. Soc.* (4) **145** (2017), 1649 – 1659.
- [17] A. Zdunik, Parabolic orbifolds and the dimension of the maximal measure for rational maps, *Invent. Math.* **99** (1990), 627–649.

Joanna Furno, Department of Mathematical Science, Indiana University-Purdue University Indianapolis, 402 N. Blackford, LD 270, Indianapolis, IN 46202

E-mail: jfurno@iupui.edu

Jane Hawkins, Department of Mathematics, University of North Carolina at Chapel Hill, CB #3250, Chapel Hill, North Carolina 27599-3250, USA

E-mail: jmh@math.unc.edu

Lorelei Koss, Department of Mathematics and Computer Science, Dickinson College, P. O. Box 1773, Carlisle, Pennsylvania 17013, USA

E-mail: koss@dickinson.edu