CONFORMAL GEOMETRY AND DYNAMICS
An Electronic Journal of the American Mathematical Society
Volume 18, Pages 85–109 (May 7, 2014)
S 1088-4173(2014)00265-3

JULIA SETS ON $\mathbb{RP}^2$ AND DIANALYTIC DYNAMICS

SUE GOODMAN AND JANE HAWKINS

ABSTRACT. We study analytic maps of the sphere that project to well-defined maps on the nonorientable real surface $\mathbb{RP}^2$. We parametrize all maps with two critical points on the Riemann sphere $\mathbb{C}_\infty$, and study the moduli space associated to these maps. These maps are also called quasi-real maps and are characterized by being conformally conjugate to a complex conjugate version of themselves. We study dynamics and Julia sets on $\mathbb{RP}^2$ of a subset of these maps coming from bicritical analytic maps of the sphere.

1. Introduction

The motivation for this paper comes from multiple directions, with the common theme of studying Julia sets for iterated maps of the real projective plane, which we will denote by $\mathbb{RP}^2$. We iterate maps on $\mathbb{RP}^2$ that lift to analytic maps of the double cover $\mathbb{C}_\infty$; these are called *dianalytic* maps of $\mathbb{RP}^2$. The first mention of the topic is in a paper by Milnor on bicritical rational maps [10], where he devotes a brief section to real bicritical analytic maps of the sphere, defined by having moduli space coordinates that are real. The *moduli space* of a parametrized family of rational maps is the space of conformal conjugacy classes of the maps. In this context, in ([10, Section 6]) there is a paragraph describing antipode-preserving maps along with several illustrative diagrams of parameter space. The second source is a paper by Barza and Ghisa on dianalytic maps on Klein surfaces (nonorientable surfaces with some complex structure) [1]. They give a general form for dianalytic maps on $\mathbb{RP}^2$, which includes the case discussed in [10] and more. Further inspiration for this paper occurred in a talk by Silverman on quasi-real maps of the sphere. These are described in an algebraic context of field extensions, with analytic maps of the sphere giving well-defined maps on a suitable quotient space (in this case $\mathbb{RP}^2$), arising as a basic example. Finally the authors are continuing a study begun in [6] on topological and measure theoretic properties of smooth noninvertible maps of surfaces.

Much of what is contained in this paper is a rigorous melding of the ideas of the three sources mentioned above, with the overarching goal of proving results about dianalytic dynamics and Julia sets on the projective plane by focusing on a class of maps illustrating some of the rich dynamics found there. In particular, we establish

Received by the editors July 10, 2013 and, in revised form, January 5, 2014 and February 5, 2014.

2010 *Mathematics Subject Classification.* Primary 37F45, 37E99, 57M99.

Key words and phrases. Complex dynamics, Julia sets, dianalytic mappings, real projective plane.

This work was completed while the second author was working at and supported by the National Science Foundation.

Any opinion, findings, and conclusions expressed in this paper are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

some basic topological properties of Julia and Fatou sets for bicritical rational maps of $\mathbb{C}_\infty$ that yield dianalytic maps of $\mathbb{RP}^2$; we derive Milnor's normal form for these maps; and we explore in detail the dynamics of degree 3 maps that lie along the boundary of their parameter space. One specific goal was to find a dianalytic map of $\mathbb{RP}^2$ which is topologically transitive, and by finding one with empty Fatou set (Theorem 5.14 of this paper) we accomplish this task.

In Section 2 we give the definitions and ideas behind real and quasi-real maps of the sphere and outline the connection to dianalytic maps of the real projective plane. A rational map is real if it is conformally conjugate to a map that leaves the real line invariant, or equivalently has real coefficients. It is quasi-real if it is not real, but is conformally conjugate to its complex conjugate map. The main results in Section 2 are to give examples of real and quasi-real rational maps of $\mathbb{C}_\infty$, and to give a normal form for quasi-real maps that are bicritical, which we do in Theorem 2.6. We then show in Corollary 2.8 that each of these quasi-real maps induces a dianalytic map on $\mathbb{RP}^2$ by showing it commutes with the antipodal map.

In Section 3 we connect Milnor's normal form of bicritical quasi-real maps with real moduli space [10], to a normal form of dianalytic maps given by Barza and Ghisa [1], and to the form derived in Section 2. For any odd degree $n \geq 3$, we give a reduced form for bicritical quasi-real maps, and describe the closure of their moduli space.

Section 4 turns to a study of Julia sets of these maps, and in Theorem 4.5 we describe what types of nonrepelling cycles can occur. In Proposition 4.7 we show that Julia sets of bicritical dianalytic maps on $\mathbb{RP}^2$ and their lifts on $\mathbb{C}_\infty$ are always connected sets. However some interesting topological changes occur under the projection from the sphere to the real projective plane obtained by identifying antipodal points as is described in Proposition 4.8 and its corollary.

In Section 5 we show the existence of dianalytic maps of $\mathbb{RP}^2$ with full Julia set and also prove results that describe in some detail moduli space, where each point represents a conformal conjugacy class of a map, using techniques from circle homeomorphisms that apply in this setting. Finally in Section 6 we show that minor changes in the form of the maps studied in this paper still induce dianalytic maps of degree 3, but by dropping the bicritical condition, more complicated behavior can occur. We support these claims with numerical and graphical evidence.

2. Real and quasi-real rational maps

The main goal of this section is to show that every bicritical quasi-real map induces a unicritical dianalytic map on $\mathbb{RP}^2$, by first conjugating each to a normal form. Theorem 2.6 and its corollary are the main results that accomplish this. We begin with a brief discussion about quasi-real maps. If R is a rational map of the Riemann sphere, denoted $\mathbb{C}_\infty$, then we write $R(z) = \frac{p(z)}{q(z)}$ with no common factors between the polynomials p and q. Whenever we say a rational map R is *conformally conjugate* to another rational map f, we mean that there is a Mobius transformation $M = \frac{az+b}{cz+d}$, $ad - bc \neq 0$, such that $R = M \circ f \circ M^{-1}$. We use the terminology conformally conjugate throughout, since we use another meaning of conjugate (meaning complex conjugation) regularly as well. A rational map f is *defined over* $\mathbb{R}$, or simply *real* if either f has real coefficients or f is conformally conjugate to a rational map R such that R has real coefficients.

We characterize real numbers in $\mathbb{C}$ by this property: $z \in \mathbb{R} \Leftrightarrow \overline{z} = z$, where $\overline{z}$ denotes complex conjugation. In order to transfer some interesting complex dynamics to the real projective plane $\mathbb{RP}^2$, we perform an operation, analogous to complex conjugation, on rational maps.

Given a rational map R, we define the *complex conjugate of* R by

$$\overline{R}(z) = \overline{R(\overline{z})}.$$

Our first observation is that if $\gamma(z) = \overline{z}$, then γ is an involution and $\overline{R} = \gamma \circ R \circ \gamma$. We note that if $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, and $q(z) = b_0 + b_1 z + \cdots + b_m z^m$, then using $\gamma(az) = \gamma(a)\gamma(z)$, and $\gamma \circ \gamma(z) = z$,

$$\overline{R}(z) = \frac{\overline{a_0} + \overline{a_1} z + \cdots + \overline{a_n} z^n}{\overline{b_0} + \overline{b_1} z + \cdots + \overline{b_m} z^m}. \tag{2.1}$$

In other words $\overline{R}$ just conjugates the coefficients of R; however the map γ is anti-holomorphic, so $\overline{R}$ is not necessarily conformally conjugate to R. For example, given a quadratic polynomial of the form $p(z) = z^2 + c$, it is well-known that if $c \notin \mathbb{R}$, then $z^2 + c$ is not conformally conjugate to $z^2 + \overline{c}$. (see, e.g., [11]) This is the jumping off point of our study. We first take a look at some real rational maps.

If R has real coefficients, then clearly R leaves the real line invariant; however a real rational map could leave a circle invariant instead since it is only conformally conjugate to one with real coefficients. We have the following class of examples of real maps.

Lemma 2.1. *If $f(z) = az^n$ or $f(z) = a/z^n$, $a \in \mathbb{C} \setminus \{0\}$, $n \geq 2$ an integer, then f is defined over $\mathbb{R}$.*

Proof. If $f(z) = az^n$, then $M(z) = \zeta z$, with $\zeta = a^{-1/(n-1)}$ satisfies $M^{-1} \circ f \circ M = z^n$. If $f(z) = a/z^n$, then $M(z) = \zeta z$, with $\zeta = a^{1/(n+1)}$ satisfies $M^{-1} \circ f \circ M = 1/z^n$. □

The following terminology, which we adopt for this study, was used by Silverman in a presentation [14].

Definition 2.2. Assume that f is a rational map of $\mathbb{C}_\infty$ of degree $n \geq 2$. We define f to be *quasi-real* if f is conformally conjugate to $\overline{f}$ but f is not defined over $\mathbb{R}$.

Example 2.3. We consider the map $R(z) = -i\dfrac{(z+1)^3}{(z-1)^3}$, with $\overline{R}(z) = i\dfrac{(z+1)^3}{(z-1)^3}$.

We can easily verify that

$$\overline{R}(z) = \frac{-1}{R(-1/z)} = \frac{-1}{i}\frac{(1+z)^3}{(-1+z)^3} = i\frac{(z+1)^3}{(z-1)^3}. \tag{2.2}$$

Remark 2.4. It is useful to think of $\mathbb{C}_\infty$ as an orientable double cover of $\mathbb{RP}^2$, via the anti-holomorphic involution $\phi(z) = -1/\overline{z}$, which is the antipodal map on $\mathbb{C}_\infty$. We note that ϕ has no fixed points so induces an equivalence relation on $\mathbb{C}_\infty$ with each equivalence class containing exactly 2 points. Taking the quotient by the relation $\sim_\phi$ gives $\mathbb{RP}^2$ the structure of a nonorientable Klein surface [1], and we use $\mathfrak{p}$ to denote the quotient map from $\mathbb{C}_\infty$ to $\mathbb{RP}^2$ using $\sim_\phi$.

The anti-holomorphic conjugacy of equation (2.2) implies that R induces a well-defined dianalytic map on $\mathbb{RP}^2$, which we will denote by $\hat{R}$. To see this, we note that antipodal points z^* and $-z^*$ on the sphere in $\mathbb{R}^3$ correspond via stereographic projection to points in the plane of the form z and $-1/\overline{z}$, respectively. For a rational map R on $\mathbb{C}_\infty$ to be well-defined on $\mathbb{RP}^2$, therefore, R must take antipodal points to antipodal points. This means that for all $z \in \mathbb{C}_\infty$,

$$-1/\overline{R(z)} = R\left(\frac{-1}{\overline{z}}\right) \tag{2.3}$$

for all $z \in \mathbb{C}$. Taking the complex conjugate of both sides of (2.3) and substituting $w = -1/z$ gives the leftmost equality of (2.2).

A result of Borsuk from 1933 says that every continuous map of $\mathbb{C}_\infty$ that commutes with the antipodal map must have odd degree [3], (see also [7, Chapter 2, §6]). A special case of this result is also proved in Theorem 2.6 below, but first we look at some quasi-real maps. While the next result is true in greater generality, we turn to the class of mappings of interest in this paper to show that they are quasi-real or real.

Proposition 2.5. *Let $n \geq 3$ be an odd integer. For any $a, b \in \mathbb{C}$, not both zero,*

$$f_{(a,b)}(z) = \frac{az^n + b}{-\overline{b}z^n + \overline{a}} \tag{2.4}$$

is conformally conjugate to $\overline{f_{(a,b)}} = f_{(\overline{a},\overline{b})}$. That is, each map satisfying (2.4) is quasi-real or real. Moreover, $f_{(a,b)}$ commutes with the antipodal map ϕ.

Proof. We define the involution $M_\iota(z) = -1/z$, and conjugate $f_{(a,b)}$ by M_ι. Writing $f = f_{(a,b)}$ we compute that

$$f\left(-\frac{1}{z}\right) = \frac{bz^n - a}{\overline{a}z^n + \overline{b}} = -\frac{1}{\overline{f}(z)},$$

so $f \circ M_\iota = M_\iota \circ \overline{f}$.

Since $\phi = M_\iota \circ \gamma = \gamma \circ M_\iota$,

$$\phi \circ f \circ \phi = M_\iota \circ (\gamma \circ f \circ \gamma) \circ M_\iota = M_\iota \circ \overline{f} \circ M_\iota = f, \tag{2.5}$$

so f commutes with ϕ. □

For bicritical quasi-real maps, we prove a converse to Proposition 2.5. In [10] a short proof is given to show that any analytic map $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$, of degree $n \geq 2$ with exactly two distinct critical points is, up to conjugation by a Mobius transformation, of the form

$$R(z) = \frac{az^n + b}{cz^n + d}, \quad a, b, c, d, \in \mathbb{C},\ ad - bc \neq 0. \tag{2.6}$$

The two critical points are at 0 and ∞, and each is of order $n - 1$.

Theorem 2.6. *If f is a quasi-real (and not real), bicritical, rational map of degree $n \geq 2$, then n is odd and f is conformally conjugate to a rational map of the form (2.4).*

Moreover, the Mobius map $M_\iota(z) = -1/z$ satisfies $M_\iota \circ f = \overline{f} \circ M_\iota$.

Proof. We start with a bicritical rational map f of the form (2.6), and assume that f is conformally conjugate to $\overline{f}$. We also assume that f is not real. The conjugating Mobius map $M(z)$ must either fix the critical points 0 and ∞ or switch them. We consider first the case when M fixes them, in which case $M(z) = \zeta z$

for some $\zeta \in \mathbb{C}$. Assume for now that none of a, b, c or d is 0. Then calculating $M^{-1}(f(M(z))) = \overline{f}(z)$ gives

$$\zeta^{-1} f(\zeta z) = \frac{a\zeta^{n-1} z^n + \zeta^{-1} b}{c\zeta^n z^n + d} = \frac{\overline{a} z^n + \overline{b}}{\overline{c} z^n + \overline{d}}\,;$$

this implies

$$d = \overline{d},\ \zeta^{-1} b = \overline{b},\ a\zeta^{n-1} = \overline{a},\ c\zeta^n = \overline{c}\,. \tag{2.7}$$

Therefore d must be real. We label the critical values $v_0 = f(0) = b/d$ and $v_\infty = f(\infty) = a/c$; since M must also map the critical values of f to the critical values of $\overline{f}$, $M(b/d) = \zeta b/d = \overline{(b/d)}$ so $\zeta = \zeta^{-1} = 1$, using the second equality in (2.7). Hence M is the identity and f is a real map, contrary to our assumption.

If $d = 0$, then $a, b, c \neq 0$ by Lemma 2.1, and we have $|\zeta| = 1$; under this hypothesis $f(0) = \infty = \overline{f}(0)$. Then M maps v_∞ to the corresponding critical value for $\overline{f}$, namely $\overline{v_\infty}$, so $M(a/c) = \zeta a/c = \overline{(a/c)}$. Using substitutions from equation (2.7), we have that $\zeta = \zeta^{-1} = 1$. Again we reach a contradiction to f not being real. We argue similarly if any other one of a, b, c is 0.

Therefore $M(0) = \infty$ and $M(\infty) = 0$; i.e., $M(z) = \zeta/z$. For similar reasons to those given above and using Lemma 2.1, we can assume $a, b, c, d \neq 0$. Expanding $f \circ M = M \circ \overline{f}$, we obtain the equation

$$f(\zeta/z) = \frac{bz^n + \zeta^n a}{dz^n + c\zeta^n} = \frac{\zeta}{\overline{f}(z)} = \frac{\zeta\overline{c} z^n + \overline{d}\zeta}{\overline{a} z^n + \overline{b}}\,, \tag{2.8}$$

which implies the following:

$$\overline{a} = d,\ \zeta\overline{c} = b,\ \zeta^n c = \overline{b},\ \zeta^{n-1} a = \overline{d}\,. \tag{2.9}$$

The first and last equations in (2.9) imply that

$$\zeta^{n-1} = 1; \tag{2.10}$$

so, in particular, $|\zeta| = 1$ and

$$\overline{\zeta} = \zeta^{-1}. \tag{2.11}$$

Then the two middle equations in (2.9) with (2.11) imply that

$$\zeta^{n+1} = 1. \tag{2.12}$$

It is clear now that equations (2.10)–(2.12) imply that $\zeta^2 = 1$, and the first possibility is that $\zeta = -1$ and so $M(z) = -1/z$, which implies that $d = \overline{a}$ and $-\overline{b} = c$ and n is odd (using all the equations of 2.9), and f is conjugate to a map of the form (2.4).

The remaining case is that $\zeta = 1$ and $M(z) = 1/z$; if so, then equations (2.9) give

$$f(z) = \frac{az^n + b}{\overline{b} z^n + \overline{a}}. \tag{2.13}$$

We assume that both a and b are nonzero because those cases were covered earlier, and that $b \neq e^{is} a$ for some real s, because this would make f a constant function. Equation (2.13) is equivalent to

$$f(z) = e^{-it} \frac{z^n + b/a}{\overline{(b/a)}\ z^n + 1}, \quad e^{it} = \overline{a}/a, \tag{2.14}$$

which maps the unit circle into itself since f can be written as a Blaschke product. (There is a good discussion of Blaschke products in [11, Chapter 15].) By conjugating f by a Mobius map that takes the unit circle to the real line, we see that f is defined over $\mathbb{R}$; the conjugated map would leave $\mathbb{R}$ invariant, and hence has real coefficients. Therefore the theorem is proved. □

Remark 2.7. To express equation (2.14) as a Blaschke product, we set $c = -b/a$ and let $c_1, c_2, \ldots, c_n$ be the distinct roots of the equation $z^n - c = 0$. None of the c_j lie on the unit circle, so the product can be written as

$$f(z) = e^{i\theta} \prod_{j=0}^{n} \frac{z - c_j}{-\overline{c_j} z + 1}, \text{ for some } \theta \in [0, 2\pi).$$

We obtain the following corollary which helps us analyze the complex and dynamical structure of quasi-real maps more closely in Section 3. A similar result to Corollary 2.8 (2) appears in [1].

Corollary 2.8. (1) *Every bicritical quasi-real analytic map f of degree $n \geq 2$ on $\mathbb{C}_\infty$ induces a degree n dianalytic map of $\mathbb{RP}^2$ with one critical point.*

(2) *Let $n \geq 3$ be an odd integer. For any $a, b \in \mathbb{C}$, not both zero, suppose $f_{(a,b)}(z) = \dfrac{az^n + b}{-\overline{b}z^n + \overline{a}}$, as in equation (2.4). Then $f(z)$ induces a degree n dianalytic map on $\mathbb{RP}^2$ with one critical point.*

Proof. Under the hypotheses given by Theorem 2.6, we can assume f is of the form (2.4), so it is enough to prove (2). If f satisfies (2.4) then by Proposition 2.5 f commutes with the antipodal map and therefore defines a dianalytic map on $\mathbb{RP}^2$.

The map f is bicritical with critical points of order $n - 1$ at 0 and ∞. Since $\phi(0) = \phi(\infty)$, these critical points are identified on $\mathbb{RP}^2$. Similarly each point $\omega \in \mathbb{C}_\infty$ has n preimages, $S = \{\omega_1, \ldots, \omega_n\}$ counted with multiplicity, and these are the same preimages for the induced map on $\mathbb{RP}^2$, after identifying S with $\phi(S)$. □

3. Normal forms of bicritical quasi-real maps and Milnor's coordinates

We are interested in studying the dynamics of unicritical dianalytic maps of $\mathbb{RP}^2$ along with their Julia sets. Typically we study these maps only up to conformal conjugacy so we consider moduli space. Recall that a point in moduli space corresponds to an equivalence class of conformally conjugate maps.

Most of the analysis is done through studying the lifts of the maps on the Riemann sphere, where they are bicritical rational maps. It is helpful to start with an appropriate normal form for our maps. The goal of this section is to prove that the moduli space for degree n unicritical dianalytic maps of $\mathbb{RP}^2$, with n any odd integer greater than 3, is naturally viewed as a sector of the complex plane whose angle depends only on n. We also compare our result with Milnor's presentation of the same moduli space given in [10].

In the previous section, we showed that a quasi-real bicritical map is conformally conjugate to one of the form (2.4) with n odd, and further that such a map commutes with the antipodal map ϕ. Combining results of Milnor ([10, Lemma 1.1]) and Barza-Ghisa ([10, Theorem 3.1]), an antipode-preserving map with critical points only at 0 and ∞ must be of the form

$$f(z) = e^{i\theta}\frac{az^n + b}{-\bar{b}z^n + \bar{a}} \tag{3.1}$$

for some odd $n \in \mathbb{N}$ and with $|a|^2 + |b|^2 \neq 0$. The critical points at 0 and ∞ both have order $n - 1$.

We can represent each map f of the form in (2.4) in matrix form:

$$A_f = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}.$$

We need $\det A_f \neq 0$ (or we do not get a rational map of degree n), but clearly A_f is not unique. Multiplying every coefficient by the same constant we leave f unchanged while changing A_f, so by multiplying each entry of A_f by $(\det A_f)^{-1/2}$, we can assume without loss of generality that $\det A_f = 1$. Moreover, the matrix $-A_f$ represents the same map f.

If f is of the form given in (3.1), using the same matrix notation,

$$A_{(f,\theta)} = \begin{bmatrix} e^{i\theta}a & e^{i\theta}b \\ -\bar{b} & \bar{a} \end{bmatrix}.$$

We remark that $\det A_{(f,\theta)} = e^{i\theta}\det A_f$. By multiplying each entry by $e^{-i\theta/2} = \left(\det A_{(f,\theta)}\right)^{-1/2}$, and noting that $\overline{e^{i\theta/2}a} = e^{-i\theta/2}\bar{a}$, we now see that any bicritical dianalytic map of degree $n \geq 3$, hence a map of the form (3.1) for some odd $n \geq 3$, can be written as

$$f(z) = \frac{az^n + b}{-\bar{b}z^n + \bar{a}}, \ |a|^2 + |b|^2 = 1. \tag{3.2}$$

The group of matrices A satisfying

$$\det A = \det \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} = |a|^2 + |b|^2 = 1, \tag{3.3}$$

is isomorphic to the group $SU(2)$; and since the matrix is defined only up to sign, the group of dianalytic maps on $\mathbb{RP}^2$ is isomorphic to $SO(3)$, which has 3 real dimensions. Consequently we can identify the group of transformations of the form (3.1) with the group $SO(3)$, which is doubly covered by $SU(2)$, the group of 2×2 unitary matrices (with complex entries) with determinant 1. The group $SO(3)$ is 3 (real) dimensional, but we will make identifications on the maps that reduce the dimension further.

In the dianalytic setting, conjugating two maps that commute with the antipodal map on $\mathbb{C}_\infty$ imposes constraints on the conjugating Mobius map $M(z) = \frac{az+b}{cz+d}$. In particular, using the same matrix shorthand to represent the map M, A_M needs to satisfy the condition (3.3), and M must fix or interchange the critical points. Therefore conjugating maps must have the form $M(z) = e^{i\theta}z$ or $M(z) = e^{i\theta}/z$.

In Theorem 3.1 we present a convenient normalization of the form: $f_\alpha(z) = \frac{z^n+\alpha}{-\bar{\alpha}z^n+1}$, $\alpha \in \mathbb{C}$, and parametrize it by the complex number $\alpha = u + iv$, $u, v \in \mathbb{R}$; therefore moduli space has 2 real dimensions.

Theorem 3.1. *Every bicritical rational map of $\mathbb{C}_\infty$ of (odd) degree $n \geq 3$ inducing a dianalytic map of $\mathbb{RP}^2$ is conformally conjugate to exactly one of the form*

$$f_\alpha(z) = \frac{z^n + \alpha}{-\overline{\alpha} z^n + 1}, \quad \arg(\alpha) \in [0, \pi/(n-1)]. \tag{3.4}$$

Proof. We first assume that f is bicritical of odd degree $n \geq 3$, has critical points at 0 and ∞, and induces a dianalytic map on $\mathbb{RP}^2$, so it is of the form

$$f(z) = \frac{az^n + b}{-\overline{b}z^n + \overline{a}}$$

for some odd n; writing $a = re^{i\theta}$, we can assume $r^2 + |b|^2 = 1$.

We consider the Mobius map $M(z) = e^{2i\theta/(n-1)}z$ and compute the conjugate map $g = M \circ f \circ M^{-1}$. We have

$$g(z) = e^{2i\theta/(n-1)} \frac{re^{i\theta}e^{-i2\theta(n/n-1)}z^n + b}{-\overline{be^{2i\theta(n/n-1)}}z^n + re^{-i\theta}},$$

and simplifying, and multiplying each coefficient by $e^{i\theta}$ gives

$$g(z) = \frac{rz^n + be^{i\theta(n+1)/(n-1)}}{-\overline{be^{i\theta(n+1)/(n-1)}}z^n + r}.$$

We note that we have not changed the determinant; i.e., $\det A_g = r^2 + |b|^2 = 1$. To obtain the normal form stated in the theorem, we divide each entry by r. We then set $\alpha = (b/r)e^{i[\theta(\frac{n+1}{n-1})]} = (b/r)\left(e^{i[(n+1)\theta]}\right)^{\frac{1}{n-1}}$.

We remark that α is only defined up to a choice of $(n-1)^{\text{th}}$ roots. Therefore we can find precisely one value of α in the sector $\arg(\alpha) \in (-\pi/(n-1), \pi/(n-1)]$.

If $\Im(\alpha) \geq 0$, the result is proved. If $\Im(\alpha) < 0$, by Proposition 2.5, we have that $f_{(1,\alpha)}$ is conformally conjugate to $\overline{f_{(1,\alpha)}} = f_{(1,\overline{\alpha})}$, so we choose $\overline{\alpha}$ to be the unique parameter that works. □

Milnor's Coordinates. For $n = 3$ the parametrization in Theorem 3.1 is equivalent to that in [10], where it is shown that every conformal equivalence class can be represented by a pair of real coordinates, X and Y, given by $X = -\frac{|\alpha|^2}{1+|\alpha|^2}$, and $Y = \frac{2\Re(\alpha^2)}{(1+|\alpha|^2)^3}$.

We show in Figure 1 the image of the first quadrant mapped onto the moduli space in [10] given as follows:

Writing each $\alpha = u + iv, u, v \in \mathbb{R}^+$, we define the map

$$(X, Y) = G(u, v) = \left(\frac{-(u^2 + v^2)}{1 + u^2 + v^2}, \frac{2(u^2 - v^2)}{(1 + u^2 + v^2)^3} \right).$$

Using polar coordinates and writing $\alpha = (r, \theta)$, then

$$(X, Y) = G(u(r, \theta), v(r, \theta)) = \left(-\frac{r^2}{1 + r^2}, \frac{2r^2 \cos(2\theta)}{(1 + r^2)^3} \right).$$

Under the assumption that $u, v \geq 0$, the closed first quadrant is mapped injectively onto the region in $\mathbb{R}^2$ with $X \in (-1, 0)$, and Y bounded by the curves $Y = \pm 2X(X + 1)^2$. We show the regions for $\alpha \in \mathbb{C}$, with $\Im(\alpha), \Re(\alpha) \geq 0$ and its homeomorphic image as (X, Y) in Figure 1. Conversely, given a pair of coordinates

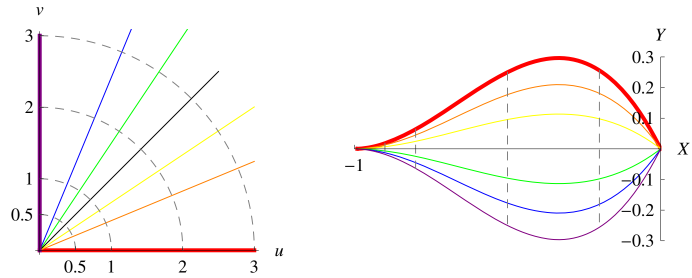

FIGURE 1. The image of the first quadrant mapped by G onto the Milnor (X, Y) parametrization. The ray $u = v$ is mapped to $Y = 0$. The (red) ray $v = 0$ is mapped to $Y = 2\sqrt{X^2(X+1)^4}$

(X, Y) (with the bounds given above), we can find a unique $\alpha = re^{i\theta}$ with $\theta \in [0, \frac{\pi}{2}]$ using

$$r = \sqrt{\frac{-X}{X+1}},\ \theta = \frac{1}{2}\arccos\left(\frac{-Y}{2X(1+X)^2}\right).$$

For $n > 3$ and odd, coordinates are defined analogously in [10]. As before, $X = -|\alpha|/(1+|\alpha|^2)$, but $Y = 2\Re(\alpha^{n-1})/(1+|\alpha|^2)^n$, which has easily computed rectangular and polar forms.

We continue our study of moduli space in Section 5, but first we turn to a discussion of Julia sets.

4. JULIA SETS ON $\mathbb{RP}^2$

For any rational map $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ of degree $n \geq 2$, the *Fatou set*, $F(R)$, is the maximal open set in $\mathbb{C}_\infty$ on which the family of analytic maps $\{R^n\}$ is normal, where $R^n = R \circ R \circ \cdots \circ R$ denotes the n-fold composition. The *Julia set*, $J(R)$, is its complement.

A point $z_0 \in \mathbb{C}_\infty$ is a *periodic point* of R of period k if $R^k(z_0) = z_0$ and $k \in \mathbb{N}$ is minimal. When $k = 1$, z_0 is a *fixed point*. A periodic point z_0 of period k lies in a k-cycle, namely $\{z_0, R(z_0), \ldots, R^{k-1}(z_0)\}$, and each point in the cycle is a fixed point of R^k. It is a standard exercise that $F(R) = F(R^k)$ for each $k \geq 2$, from which it follows that $J(R) = J(R^k)$.

For each fixed point of R, we define the *multiplier* of z_0 by $R'(z_0)$ if $z_0 \in \mathbb{C}$; if $z_0 = \infty$, we define it to be $\lim_{z\to 0} 1/R'(1/z)$. Using a slight abuse of notation, we denote the multiplier by $R'(z_0)$ for all $z_0 \in \mathbb{C}_\infty$. The multiplier of a k-cycle containing z_0 is the multiplier of z_0 as a fixed point of R^k. We recall the classification of periodic points for a rational map R of degree $n \geq 2$. If z_0 is a periodic point of period $k \geq 1$, then:

(1) z_0 is *attracting* (*superattracting*) if $|(R^k)'(z_0)| < 1 (= 0)$;
(2) z_0 is *repelling* if $|(R^k)'(z_0)| > 1$;
(3) z_0 is *neutral* if $|(R^k)'(z_0)| = 1$;
   (a) a neutral periodic point is *rationally neutral* or *parabolic* if $(R^k)'(z_0)$ is an mth root of unity.
   (b) a neutral periodic point is *irrationally neutral* if it is not parabolic.

We refer to books by Beardon [2], Carleson and Gamelin [5], and Milnor [11] for statements and proofs of the fundamental results in complex dynamics connecting periodic points and Julia and Fatou sets. We summarize a few results that are useful in our setting.

Theorem 4.1. *Assume f is a rational map of the form* (3.4) *of odd degree $n \geq 3$ and U is a connected component of the Fatou set $F(f)$. Then some forward iterate of U is periodic, i.e., there exists $m \in \mathbb{N}$ such that $V = f^m(U)$ is periodic, and V is one of the following types:*

(1) *an* attracting component *and V contains an attracting (superattracting) periodic point z_0;*
(2) *a* parabolic component *and ∂V contains a parabolic periodic point z_0;*
(3) *if V contains an irrationally neutral periodic point z_0, then V is a* Siegel disk.

Proof. This follows from the Sullivan Nonwandering Theorem (see, e.g., Theorem 1.3 in [5]) and the classification of forward invariant Fatou components for rational maps (see, e.g., Theorem 2.1 in [5]). For bicritical maps, Herman rings cannot occur, and a proof is given in ([10, Theorem A.1]). □

Assume that f_α is of the form given in (3.4). Since the spherical metric on $\mathbb{C}_\infty$ is ϕ invariant, the notions of equicontinuity and normality of the family of maps $\{f_\alpha^n\}_{n\geq 1}$ pass from $\mathbb{C}_\infty$ to $\mathbb{RP}^2$, so the Julia set induced by f_α on $\mathbb{RP}^2$ is well-defined and is just the projection of $J(f_\alpha)$ on $\mathbb{C}_\infty$. The map on $\mathbb{RP}^2$ given by $\hat{f}_\alpha([z]) := \mathfrak{p} \circ f_\alpha \circ \mathfrak{p}^{-1}([z])$, with $[z]$ the equivalence class of $z \in \mathbb{C}_\infty$ under ϕ, is well-defined as are its iterates. We denote by $J(\hat{f}_\alpha)$ the Julia set on $\mathbb{RP}^2$ and define it to be the projection by the map $\mathfrak{p}$ of $J(f_\alpha)$; equivalently, we define $F(\hat{f}_\alpha)$ directly on $\mathbb{RP}^2$ in the obvious way. Every map f_α induces $\hat{f}_\alpha$ on $\mathbb{RP}^2$; then $F(\hat{f}_\alpha)$ is the domain of normality of the family of iterates $\{\hat{f}_\alpha^{\,n}\}$, and $J(\hat{f}_\alpha)$ is its complement in $\mathbb{RP}^2$.

The antipodal symmetry of the Julia set of f_α is evident on the sphere as can be seen from the next result (see also Figure 2).

Proposition 4.2. *If f_α is a rational map of degree $n \geq 3$ of the form* (3.4), *then $J(f_\alpha)$ and $F(f_\alpha)$ have the following symmetries:*

(1) *$\omega^j J(f_\alpha) = J(f_\alpha)$, for $\omega = e^{\frac{2\pi i}{n}}$, $j = 0, \ldots, n-1$.*
(2) *$\omega^j F(f_\alpha) = F(f_\alpha)$, for $\omega = e^{\frac{2\pi i}{n}}$, $j = 0, \ldots, n-1$.*
(3) *$\phi(J(f_\alpha)) = J(f_\alpha)$ and $\phi(F(f_\alpha)) = F(f_\alpha)$.*

Proof. Since for each $z \in \mathbb{C}_\infty$, $f_\alpha^k(z) = f_\alpha^k(\omega^j z)$ for all $k \in \mathbb{N}$, and for each $j = 0, 1, \ldots, n-1$, it follows that $z_0 \in F(f_\alpha) \Leftrightarrow \omega^j z_0 \in F(f_\alpha)$. The third statement holds since ϕ is an isometry on $\mathbb{C}_\infty$ with respect to the spherical metric, so $z_0 \in F(f_\alpha) \Leftrightarrow \phi(z_0) \in F(f_\alpha)$. □

Lemma 4.3. *Assume $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ commutes with the map $\phi(z) = -1/\overline{z}$. Then z_0 is a fixed point for f if and only if $w_0 = -1/\overline{z_0}$ is a fixed point for f.*

Proof. If $f(z_0) = z_0$, then the hypothesis implies that $f(-1/\overline{z_0}) = -1/\overline{z_0}$. □

Proposition 4.4. *Suppose f is analytic on $\mathbb{C}_\infty$ and commutes with ϕ, and z_0 is a fixed point for f. Then $w_0 = -1/\overline{z_0}$ is also fixed for f and $f'(w_0) = \overline{f'(z_0)}$.*

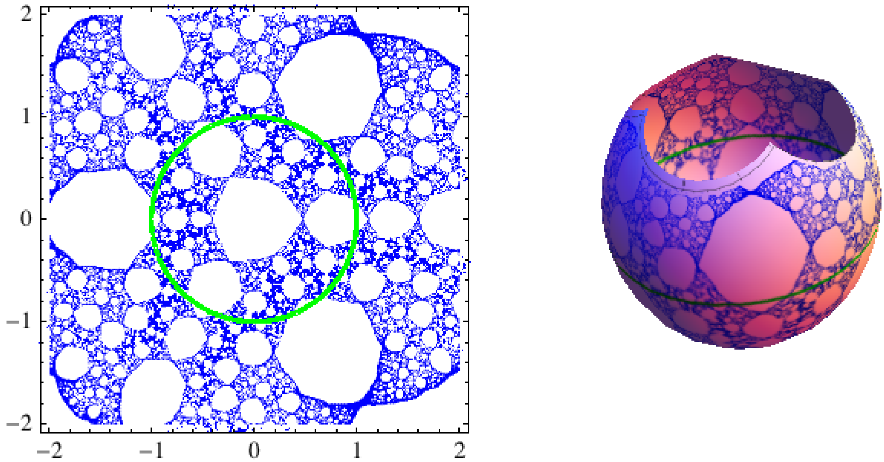

FIGURE 2. On the left, $J(f_{1.86})$ has a single attracting 8-cycle. On the right, the antipodal symmetry of the Julia set is shown with the unit circle in green in both pictures.

Proof. By Lemma 4.3 it is enough to prove the statement about the multipliers of the fixed points. We assume z_0 and $w_0 = -1/\overline{z_0}$ are fixed under f. If we regard the sphere as a smooth manifold, we consider the derivatives of the maps on $\mathbb{C}_\infty$, $f \circ \phi = \phi \circ f$, i.e.,

$$D(f \circ \phi)(z) = D(\phi \circ f)(z), \ \forall z \in \mathbb{C}.$$

Since $f(z_0) = z_0$, then

$$Df(\phi(z_0))D\phi(z_0) = D\phi(z_0)Df(z_0).$$

Therefore, since ϕ is anti-holomorphic, $D\phi(z_0)$ is nonsingular and thus the two linear transformations $Df(w_0)$ and $Df(z_0)$ have the same spectrum, namely the set of eigenvalues: $\{f'(z_0), \overline{f'(z_0)}\}$ (possibly repeated). Therefore $f'(w_0) = \overline{f'(z_0)}$. □

We recall that if z_0 is an attracting fixed point of f, then $z_0 \in F(f)$ and the *immediate basin of attraction* is the connected component of $F(f)$ containing z_0. If $\{z_0, f(z_0), \ldots, f^{k-1}(z_0)\}$ is an attracting cycle of period k, then each $z_j = f^j(z_0)$ is an attracting fixed point for f^k and the *immediate basin of attraction for the k-cycle* is the union of the immediate basins of the k fixed points z_j of f^k. The *basin of attraction* of the fixed point z_0 is the open set $V \subset F(f)$ consisting of all points $z \in \mathbb{C}_\infty$ such that $\lim_{n\to\infty} f^n(z) = z_0$.

Theorem 4.5. *Suppose for a map f_α of the form* (3.4) *there exists a periodic connected component $U \subset F(f_\alpha)$, associated with the nonrepelling k-cycle $B = \{z_0, f_\alpha(z_0), \ldots, f_\alpha^{k-1}(z_0)\}$. Then exactly one of the following holds:*

1. *$B = \phi(B)$; in this case B is the only nonrepelling cycle for f_α. This cannot occur for $k = 1$ (or any odd period).*
2. *$B \cap \phi(B) = \emptyset$; in this case, there exist exactly two nonrepelling cycles for f_α and they are antipodal to each other on $\mathbb{C}_\infty$.*

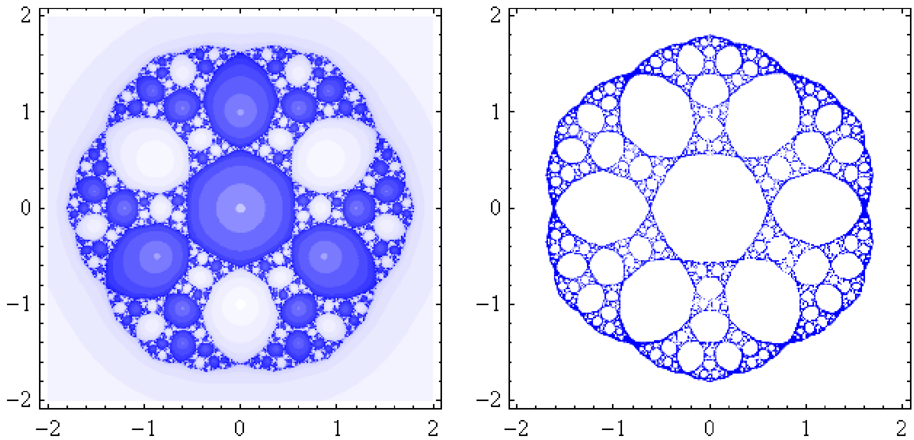

FIGURE 3. On the left, $J(f_i)$ has two attracting 2-cycles $(0, i)$ and $(\infty, -i)$. On the right, $J(f_1)$ has one attracting 4-cycle at $(0, 1, \infty, -1)$. Both collapse to a single attracting 2-cycle on $\mathbb{RP}^2$.

Proof. Assume U is a periodic component of $F(f_\alpha)$. Then by Theorem 4.1, U has a nonrepelling k-cycle associated to it, and, in turn at least one critical point associated to it. More precisely there is a critical point in the immediate attracting basin for cases (1) and (2) in Theorem 4.1, or with an infinite forward orbit closure containing the boundary of the Siegel disk cycle in case (3) [2]. If there are two nonrepelling cycles for f_α, then one of the cycles, say B, is associated to 0, while ∞ corresponds to the other one. By Lemma 4.3 the second cycle must be $\phi(B)$. Therefore $B \cap \phi(B) = \emptyset$.

If there is a pair of points $z_0, \phi(z_0) \in B$, then $\phi(z_0) = f_\alpha^m(z_0)$, for some $1 \le m \le k-1$, so since $\phi \circ f_\alpha^m = f_\alpha^m \circ \phi$, for all $m \in \mathbb{N}$, $B = \phi(B)$. In this case both critical points $c_1 = 0$ and $c_2 = \infty = \phi(c_1)$ must be associated to the same cycle B so no other nonrepelling cycles can occur. Since ϕ has no fixed points, B must contain z_0 and $\phi(z_0)$ for each $z_0 \in B$, so has an even number of points and hence an even period k. □

Remark 4.6. Figure 3 illustrates the result of Theorem 4.5. A cycle satisfying (1) of Theorem 4.5 must have an even period, say $2k$, and is referred to in [10] as a $k+k$ cycle since under the quotient map $\mathfrak{p}$ it becomes a k-cycle on $\mathbb{RP}^2$ (see Remark (2.4)). We refer to it here as a *collapsing cycle*. Each cycle of type (2) is called a k-cycle or a *noncollapsing* k-cycle.

Proposition 4.7. *$J(f_\alpha)$ and $J(\hat{f}_\alpha)$ are connected.*

Proof. If $f_\alpha(z_0) = z_0$, then by Lemma 4.3 $w_0 = -1/\overline{z_0} \neq z_0$ is also fixed. For a bicritical rational map R, there is a dichotomy stating that the Julia set is either connected or totally disconnected (homeomorphic to a Cantor set); this is Theorem 3.1 in [10]. In the totally disconnected case there is exactly one nonrepelling fixed point, and both critical points lie in the immediate attracting basin of that point (see, e.g., [10]). Since in our setting there cannot be exactly one nonrepelling fixed

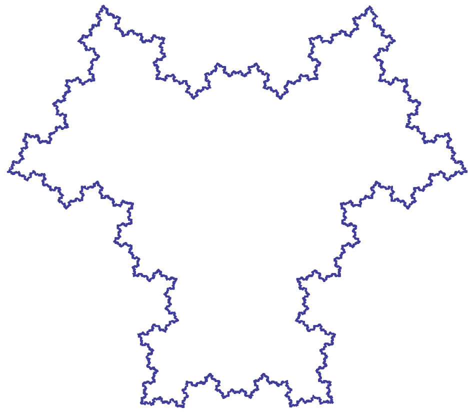

FIGURE 4. For many parameters, (extremely small and extremely large), $J(f_\alpha)$ is a quasicircle.

point by Proposition 4.4, the disconnected case is impossible. Therefore $J(f_\alpha)$ is connected on $\mathbb{C}_\infty$, and since $J(\hat{f}_\alpha)$ is the image of a connected set under a continuous mapping, it is connected as well. □

A Jordan curve $\mathcal{C}$ is a *quasicircle* if $\mathcal{C}$ is the image of a circle under a quasiconformal homeomorphism of the sphere.

Proposition 4.8. *Assume f_α is of odd degree $n \geq 3$.*

(1) *If f_α has exactly one attracting period 2 orbit, then it is collapsing, and $J(f_\alpha)$ is a quasicircle. Moreover, the map $\hat{f}_\alpha$ on $\mathbb{RP}^2$ has a single attracting fixed point and no other nonrepelling cycles.*
(2) *If f_α has two distinct attracting fixed points, p and q, then $J(f_\alpha)$ is a quasicircle and $\hat{f}_\alpha$ has one attracting fixed point on $\mathbb{RP}^2$.*

Proof. If $f = f_\alpha$ has exactly one attracting period 2 orbit, $\{p, q\}$, then $q = -1/\overline{p}$. If not, then let $\tilde{p} = -1/\overline{p}$, and we have that $\{\tilde{p}, \tilde{q}\}$ is also an attracting period 2 orbit. By hypothesis, we have $\tilde{p} = q$, since it cannot be equal to p. We must have 0 lying in the same Fatou component as either p or q, so assume it is in the component containing p, call it F_p, hence the other critical point ∞ lies in the same Fatou component as q, namely $F_q = \phi(F_p)$. Consider $f^{-1}(F_p)$: it must contain F_q since $f(q) = p$, and so contains the order $n-1$ critical point at ∞. This implies that F_q is an n-fold covering of F_p by the map f, hence contains all the preimages of F_p. By symmetry the same is true for F_p covering F_q. Therefore there cannot be any other Fatou components. It has been proved that if the Fatou set has exactly two components and f is hyperbolic on $J(f)$, then $J(f)$ is a quasicircle (see [5, Theorem VI.2.1]). The points p and q are identified under ϕ so there is a single attracting fixed point for $\hat{f}$ on $\mathbb{RP}^2$.

If f has exactly 2 attracting fixed points, p and q, then $q = -1/\overline{p}$. The rest of the argument is the same. □

Recall that a Jordan curve $\mathcal{C}$ lying on a surface S is separating if $S \setminus \mathcal{C}$ consists of 2 connected components; otherwise $\mathcal{C}$ is called nonseparating. Every quasicircle on $\mathbb{C}_\infty$ is separating. Before giving a corollary of Proposition 4.8, we prove a topological lemma which is of general interest. To avoid confusion with complex conjugation, we let $\mathrm{cl}(U)$ denote the topological closure of a set $U \subset \mathbb{C}_\infty$.

Lemma 4.9. *If $\mathcal{C}$ is a Jordan curve on $\mathbb{C}_\infty$, such that $\phi(\mathcal{C}) = \mathcal{C}$, then $\mathfrak{p}(\mathcal{C})$ is a nonseparating Jordan curve on $\mathbb{RP}^2$.*

Proof. By the Jordan curve theorem and the Schoenflies theorem, if U_1 and U_2 are the two components of $\mathbb{C}_\infty \setminus \mathcal{C}$, then $\mathrm{cl}(U_1)$ and $\mathrm{cl}(U_2)$ are each homeomorphic to the closed unit disk in $\mathbb{C}$ [13]. Then either $\phi(U_1) = U_1$ or $\phi(U_1) = U_2$. In the first case, by continuity, $\phi(\mathrm{cl}(U_1)) \subseteq \mathrm{cl}(U_1)$ and the Brouwer fixed point theorem implies that ϕ has a fixed point in $\mathrm{cl}(U_1)$. This is a contradiction, so $\phi(U_1) = U_2$ and because it is a homeomorphism, $\phi(U_2) = U_1$.

As before let $\mathfrak{p}$ denote the projection map from $\mathbb{C}_\infty$ to $\mathbb{RP}^2$ induced by equivalence under ϕ. The curve $\mathcal{C}$ projects to a homeomorphic image of a circle on $\mathbb{RP}^2$, since $\mathfrak{p}$ is a covering map and hence a local homeomorphism. Therefore the image of $\mathcal{C}$ is a compact one-dimensional manifold without boundary. Given any point $\omega \in \mathbb{RP}^2$, if $\omega \notin \mathfrak{p}(\mathcal{C})$, then $\mathfrak{p}^{-1}(\omega)$ consists of one point in $\omega_1 \in U_1$ and the point $\phi(\omega_1) \in U_2$. Now choose any two points $\omega \neq z \in \mathbb{RP}^2 \setminus \mathfrak{p}(\mathcal{C})$; we can find inverse images of each under $\mathfrak{p}$, $\omega_1, z_1 \in U_1$. Take any path from ω_1 to z_1 lying in U_1; then the projection of the path under $\mathfrak{p}$ cannot intersect $\mathfrak{p}(\mathcal{C})$. Therefore $\mathfrak{p}(\mathcal{C})$ is a nonseparating Jordan curve. □

Corollary 4.10. *Under the hypotheses of Proposition* 4.8 *$J(\hat{f}_\alpha)$ is a non-separating Jordan curve on $\mathbb{RP}^2$*

Proof. By Proposition 4.8 $J(f_\alpha)$ is a quasicircle which is invariant with respect to ϕ, and by Lemma 4.9 $J(\hat{f}_\alpha)$ is therefore a non-separating Jordan curve. □

5. Moduli space

In this section we turn to a study of moduli space of bicritical quasi-real maps, the space of conformal conjugacy classes of each such rational map. Equivalently we can frame the same idea using unicritical dianalytic maps of $\mathbb{RP}^2$. We have shown the following.

Theorem 5.1. *Moduli space for degree $n \geq 3$ (n odd) dianalytic maps of $\mathbb{RP}^2$ with one critical point, is homeomorphic to the closed sector of the complex plane given by $\{\alpha \in \mathbb{C} : \alpha = re^{i\theta}, \theta \in [0, \frac{\pi}{n-1}],\ r \geq 0\}$. Equivalently, each map of the form* (3.2) *is conformally conjugate to precisely one of the form* (3.4).

Proof. This follows from Theorem 3.1 and its proof. If we say two dianalytic maps of $\mathbb{RP}^2$ are (dianalytically) conjugate if and only they are conjugate via a dianaltyic automorphism $\hat{M}$, then $\hat{M}$ must lift to a Mobius transformation M on $\mathbb{C}_\infty$ that commutes with ϕ. Every dianalytic map $\hat{f}$ of $\mathbb{RP}^2$ lifts to a map of the form given in equation (3.2), which in turn by Theorem 3.1 is conformally conjugate to exactly one map of the form (3.4), with α in the sector given above. Moreover, the conjugating maps commute with ϕ. □

While properties of this space were described in [10], we make some rigorous statements here that explain Figure 9 in [10], focusing on degree 3 maps. By Definition 2.2 the positive axes, which form the boundary of moduli space, do not correspond to quasi-real maps even though they define dianalytic maps of $\mathbb{RP}^2$. For

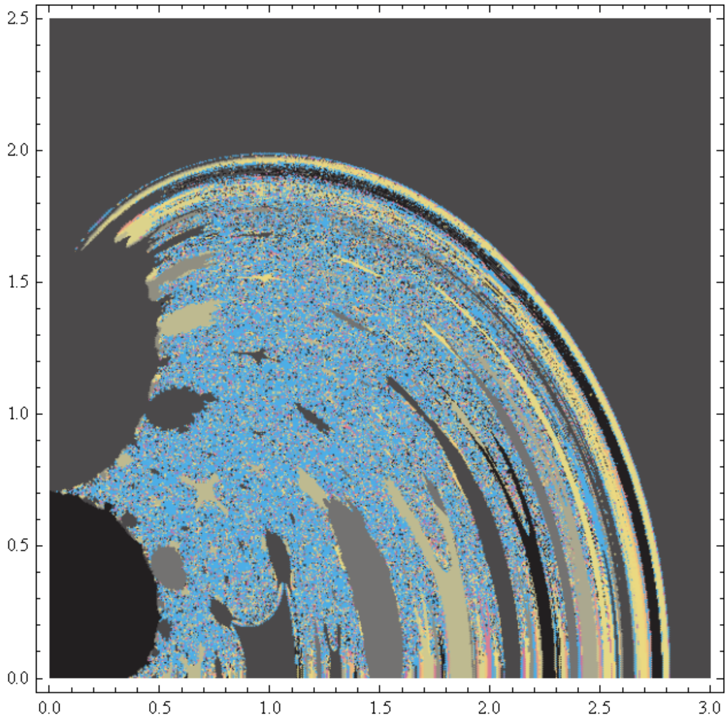

FIGURE 5. This is parameter space for f_α, parametrized by α, when $n = 3$; it is homeomorphic to Figure 9 in [10].

$b > 0$, the map f_b is obviously real, and the map f_{ib} is conformally conjugate to $g_b(z) = \frac{z^3+b}{bz^3-1}$. However it is clear from the parameter space shown in Figure 5 that the dynamics occurring along the axes is significant. We show that for maps f_α with α on the boundary, a C^∞ homeomorphism is induced on a great circle of $\mathbb{C}_\infty$ that contains both critical points. The fact that along the imaginary axis the circle homeomorphism is orientation-reversing accounts for the simple bifurcations we see there, and that the one along the real axis is orientation-preserving gives us the rich dynamics that result from the resulting rotation number changes. We first look at what happens along the imaginary axis, when $\alpha = ib$, $b \in \mathbb{R}$.

Proposition 5.2. *Let $\mathbb{I}$ denote the imaginary axis, and $\bar{\mathbb{I}}$ its compact closure in $\mathbb{C}_\infty$ (a great circle). The map f_{ib}, $b \in \mathbb{R}$, maps $\bar{\mathbb{I}}$ onto $\bar{\mathbb{I}}$, inducing an orientation-reversing homeomorphism of the circle.*

Proof. Assume $f_{ib}(z) = \frac{z^3+ib}{ibz^3+1}$, $b \in \mathbb{R}$; using $h(z) = iz$ we consider the map

$$g_b(z) = \frac{z^3+b}{bz^3-1} = h \circ f_{ib} \circ h^{-1}(z), \tag{5.1}$$

with a pole at $p_0 = b^{-1/3} > 0$ and derivative

$$g_b'(z) = \frac{-3z^2(1+b^2)}{(bz^3-1)^2} \tag{5.2}$$

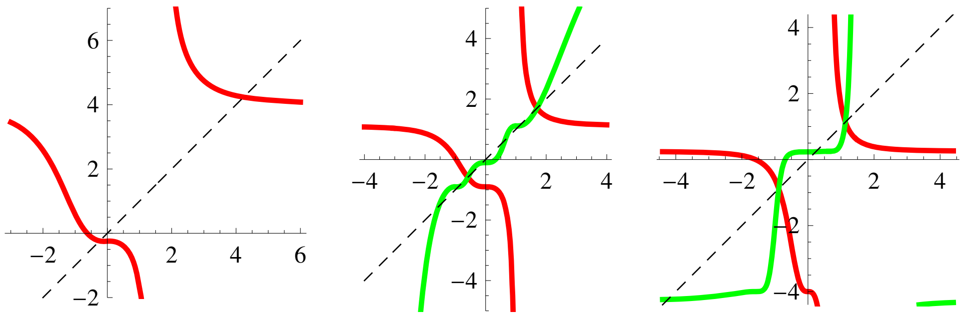

FIGURE 6. The orientation-reversing circle homeomorphism (red graph) induced by f_{ib} on the imaginary axis. From left to right, $b < 1/\sqrt{2}$, $1/\sqrt{2} < b < \sqrt{2}$, and $b > \sqrt{2}$. The green graph is $g_b^2(x)$ (orientation-preserving), and $x = y$ is the dotted black line.

with $g_b'(x) < 0$ for all $x \in \mathbb{R} \setminus \{p_0\}$, and so $g_b|_{\mathbb{R}}$ is decreasing. Let $\overline{\mathbb{R}}$ denote the compact closure of the reals in $\mathbb{C}_\infty$. We next show that $g_b|_{\overline{\mathbb{R}}}$ induces an orientation-reversing homeomorphism of the circle, viewed on the sphere passing through 0 and ∞, from which the proposition follows. It is clear that $g_b(x) \in \overline{\mathbb{R}}$ if $x \in \mathbb{R}$, $g_b(\infty) = 1/b$, and $g_b(p_0) = \infty$, which defines g_b as a smooth map on $\overline{\mathbb{R}}$. Setting $\omega = e^{2\pi i/3}$ we see that $g_b(\omega^j z) = g_b(z)$, $j = 1, 2$ for all z. Then for each real w that is not a critical value, $g_b^{-1}(w)$ consists of one real point w_1, and the 2 points $w_j = \omega^j w_1, j = 1, 2$. Clearly the critical values and ∞ each have a preimage in $\overline{\mathbb{R}}$, so g_b is surjective. Equation (5.2) shows it is injective; we note that the critical value $-b$ has a triple preimage at the critical point 0, and its antipode, $1/b$ has a triple preimage at ∞. It is also clear (by mapping the upper half-plane conformally to the unit disc and because $g_b|_{\mathbb{R}}$ is decreasing), that $g_b|_{\overline{\mathbb{R}}}$ is conjugate to an orientation-reversing homeomorphism of the circle. □

Corollary 5.3. *Assume* $f_{ib}(z) = \frac{z^3+ib}{ibz^3+1}$, $b \in \mathbb{R}$; *then* $f_{ib}|_{\mathbb{I}}$ *has exactly two fixed points in* $\mathbb{I}$. *Every point* $z \in \overline{\mathbb{I}}$ *is either asymptotic under* f_{ib}^n *to one of the fixed points or to a periodic point of period* 2 *in* $\overline{\mathbb{I}}$.

Proof. It is a basic property of orientation-reversing circle homeomorphisms that each one has exactly 2 fixed points and that each point in the circle is attracted to a fixed point, or, attracted to a period 2 cycle. This result and proof can be found for example in [12, Chapter 1.1]. Since $f_{ib}(0) = ib$, and $f_{ib}(\infty) = -i/b$, the fixed points are in $\mathbb{I}$. □

We see immediately that this basic result on circle homeomorphisms, often found as an exercise in books, completely determines the dynamics for parameters along the imaginary axis. Since Corollary 5.3 implies that the critical points for these maps are attracted to fixed points or attracting period 2 orbits, which is stable behavior in the holomorphic family f_α, or are contained in an attracting period 2 orbit, this explains some of the dynamics off the axis as well. Most of the following statements are easily verified but we give short proofs of the parts that are not immediately verifiable. Figure 6 illustrates the trichotomy given by (4), (6), and (8) in the result below. This can also be seen by looking along the imaginary axis in parameter space in Figure 5. The dark bulb near the origin contains parameters associated with attracting fixed points for f_α; moving up the imaginary axis there are two antipodal period 2 orbits that project onto one attracting period 2 orbit, and the large region outside corresponds to a single attracting period 2 orbit on $\mathbb{C}_\infty$ giving an attracting fixed point on $\mathbb{RP}^2$.

Proposition 5.4. *Assume* $f_{ib}(z) = \frac{z^3+ib}{ibz^3+1}$, $b \in \mathbb{R}$.

(1) *The map* $S(z) = 1/z$ *commutes with* f_{ib}.
(2) *The fixed points of* f_{ib} *are:* $z_\pm = \pm 1$, *and* $w_\pm = \dfrac{-i}{2b}\left(1 \pm \sqrt{1+4b^2}\right) \in \mathbb{I}$.
(3) *We have* $|f'_{ib}(\pm 1)| = 3$, *so* $z_\pm$ *are repelling fixed points for all* $b \in \mathbb{R}$, *and* $f'_{ib}(w_\pm) = \frac{-3b^2}{1+b^2} < 0$.
(4) *There exists an antipodal pair of attracting fixed points in* $\mathbb{I}$ *if and only if* $b < 1/\sqrt{2}$. *At* $b = 1/\sqrt{2}$, f_{ib} *has 2 neutral imaginary fixed points.*
(5) *At* $b = 1$, *the map* f_i *has two super-attracting period two orbits in* $\overline{\mathbb{I}}$, *one at* $\{0, i\}$, *and the other is its antipodal pair* $\{\infty, -i\}$. *These two orbits are identified to form a single super-attracting period 2 orbit for the induced map on* $\mathbb{RP}^2$.
(6) *For every* $b \in \mathbb{R}$ *there exist period 2 orbits in* $\mathbb{I}$ *of the form:* $\{p_1, p_2\}$, *with* $p_2 = 1/p_1 = -1/\overline{p_1}$. *We have*

$$p_1 = i\left(\frac{b}{2} - \sqrt{1+\left(\frac{b}{2}\right)^2}\right)$$

and

$$p_2 = i\left(\frac{b}{2} + \sqrt{1+\left(\frac{b}{2}\right)^2}\right).$$

For $b > \sqrt{2}$, $\{p_1, p_2\}$ *is an attracting period 2 orbit that collapses to an attracting fixed point on* $\mathbb{RP}^2$.
(7) *At* $b = \sqrt{2}$ *the period 2 orbit* $\{p_1, p_2\}$ *given above is neutral.*
(8) *For all* $b \in (\dfrac{1}{\sqrt{2}}, \sqrt{2})$, *there exist two antipodal noncollapsing period 2 attracting cycles in* $\overline{\mathbb{I}}$.

Proof. (2): A point z_0 is fixed for f_{ib} if and only if z_0 is a root of the quartic polynomial

$$q(z) = ibz^4 - z^3 + z - ib.$$

It is easy to check that 1 and -1 are roots of $q(z)$; we can factor out the polynomial $z^2 - 1$ and obtain the quadratic polynomial

$$Q(z) = i(bz^2 + iz + b).$$

The roots of $Q(z)/i$ are $\frac{-i \pm \sqrt{-1-4b^2}}{2b}$, the remaining two fixed points.

To show (6), we start with the observation that if $z = iy$, $y \in \mathbb{R}$, then $-\overline{z} = z$. We next compute that for any $b \in \mathbb{R}$,

$$(ib/2 - i\sqrt{1+(b/2)^2})(ib/2 + i\sqrt{1+(b/2)^2}) = -b^2/4 + 1 + b^2/4 = 1,$$

so if $p_1 = (ib/2 - i\sqrt{1+(b/2)^2}) = i(b/2 - \sqrt{1+(b/2)^2})$ and $p_2 = i(b/2 + \sqrt{1+(b/2)^2})$, then they satisfy $p_2 = 1/p_1 = -1/\overline{p_1}$. We now solve $f_{ib}(p) = 1/p$. Then

$$f_{ib}(p) = \frac{p^3 + ib}{ibp^3 + 1} = 1/p,$$

or equivalently,

$$p^4 - ibp^3 + ibp - 1 = 0. \tag{5.3}$$

Clearly 1 and -1 are roots of equation (5.3), so we factor out $p^2 - 1$ from (5.3) to obtain the quadratic polynomial

$$p^2 - ibp + 1 = 0,$$

which in turn has the easily computable roots p_1, p_2 given in (6). Since $f_{ib}(p_1) = 1/p_1 = p_2$, and $f_{ib}(p_2) = p_1$ we have a period 2 orbit.

In order to compute the multiplier of the period 2 cycle, we note that for $z = iy$, we have the negative real derivative $f'_{ib}(iy) = \frac{-3y^2(1+b^2)}{(1+by^3)^2}$, and a bit of computation and cancellation gives us

$$f'_{ib}(p_1)f'_{ib}(p_2) = \frac{9}{(1+b^2)^2}. \tag{5.4}$$

From equation (5.4) we see that whenever $b > \sqrt{2}$, we have an attracting period 2 cycle for f_{ib} that collapses to an attracting fixed point on $\mathbb{RP}^2$. When $b = \sqrt{2}$ the 2-cycle is neutral, and otherwise it is repelling.

For the remaining case we assume that $b \in (1/\sqrt{2}, \sqrt{2})$. Looking at the ten possible fixed points for the degree 9 map f_{ib}^2, by the above cases we have four repelling fixed points and a repelling 2-cycle in $\overline{\mathbb{I}}$. However, by Corollary 5.3 there must be an attracting period 2 orbit, and since it cannot be of the form in (6), there must be two antipodal attracting 2-cycles, which proves (8). □

Proposition 5.4 shows that the behavior of the map f_α is fairly simple when α lies on the imaginary axis. However, there are many bifurcations in parameter space if we look elsewhere. Some analogous results for real parameters show that the bifurcations are much more complicated.

We begin by showing that for maps f_α with $\alpha \in \mathbb{R}$, $f_\alpha|_{\overline{\mathbb{R}}}$ is an orientation-preserving homeomorphism of the circle (see Figure 7). The arguments are similar to those given in Proposition 5.2.

Proposition 5.5. *Consider the compact closure of the reals, $\overline{\mathbb{R}} \subset \mathbb{C}_\infty$, as a great circle. The map f_α, $\alpha > 0$ maps $\overline{\mathbb{R}}$ onto $\overline{\mathbb{R}}$, inducing an orientation-preserving homeomorphism of the circle.*

Proof. Assume $f_\alpha(z) = \frac{z^3+\alpha}{-\alpha z^3+1}$, $\alpha > 0$; f_α has a pole at $p_0 = \alpha^{-1/3} > 0$ and derivative

$$f'_\alpha(z) = \frac{3z^2(1+\alpha^2)}{(-1+\alpha z^3)^2} > 0 \tag{5.5}$$

on $\mathbb{R} \setminus \{p_0\}$. Also $f'_\alpha(\infty) = -1/\alpha$. We show that $f_\alpha|_{\overline{\mathbb{R}}}$ induces an orientation-preserving homeomorphism map of a circle on the sphere passing through 0 and ∞, from which the proposition follows. Since $f_\alpha(x) \in \mathbb{R}$ if $x \in \mathbb{R}$, $f_\alpha(\infty) = -1/\alpha$, $f_\alpha(p_0) = \infty$ defines f_α as a smooth map on $\mathbb{R} \cup \infty$. With $\omega = e^{2\pi i/3}$, $f_\alpha(\omega^j z) = f_\alpha(z)$, $j = 1, 2$ for all z. Then for each real w that is not a critical value,

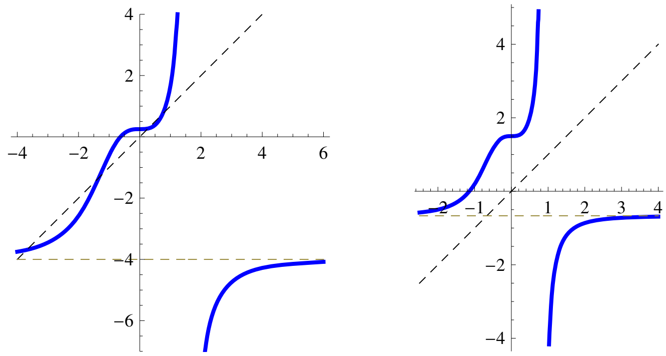

FIGURE 7. The orientation-preserving circle homeomorphism (blue graph) induced by f_α on the real axis. From left to right, $\alpha < 1/(2\sqrt{2})$, and $\alpha = 3/2$, for which there are no real fixed points. The dashed graphs are $x = y$ and $x = -1/\alpha$, the value of $f_\alpha(\infty)$.

$f_\alpha^{-1}(w)$ consists of one real point w_1, and the two points $w_j = \omega^j w_1, j = 1, 2$. The critical values and ∞ each have a preimage in $\overline{\mathbb{R}}$, so f_α is surjective. Equation (5.5) shows it is injective; we note that the critical value α has a triple preimage at the critical point 0, and its antipode, $-1/\alpha$ has a triple preimage at ∞ so it is not a diffeomorphism. However, $f_\alpha|_{\overline{\mathbb{R}}}$ is (topologically) conjugate to an orientation-preserving homeomorphism. □

Rotation numbers of orientation-preserving homeomorphisms of the circle. It is a classical result that every orientation-preserving homeomorphism has a rotation number. A detailed treatment of the topic can be found for example in [8] or [12]. We recall that viewing $S^1 = \mathbb{R}/\mathbb{Z}$, there is a natural projection map $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$, and each orientation-preserving homeomorphism $f : S^1 \to S^1$ lifts to a continuous map $\tilde{f} : \mathbb{R} \to \mathbb{R}$ such that $f \circ \pi = \tilde{f} \circ \pi$. The lift is unique up to addition by an integer constant, but $1 = \deg(f) := \tilde{f}(x+1) - \tilde{f}(x)$. From now on we choose the unique lift $\tilde{f}$ such that $0 \le \tilde{f}(0) < 1$.

We define the rotation number of f via the classical proposition found in many books such as [12]. This serves as a source of most of the general results we use here.

Proposition 5.6. *If $f : S^1 \to S^1$ is an orientation-preserving homeomorphism, with a lift $\tilde{f}$, then*

$$\rho(\tilde{f}) = \lim_{n\to\infty} \frac{\tilde{f}^n(x) - x}{n} \tag{5.6}$$

exists for all $x \in \mathbb{R}$, and is independent of x.

If $\tilde{g}$ is another lift of f, $\rho(\tilde{f}) - \rho(\tilde{g}) = \tilde{f}(x) - \tilde{g}(x) \in \mathbb{Z}\, \forall x \in \mathbb{R}$.

The value $\rho(f) := \rho(\tilde{f}) \in [0, 1)$ is called the *rotation number* of the map $f : S^1 \to S^1$. Moreover, $\rho(f)$ is invariant under topological conjugacy; i.e., $\rho(h \circ f \circ h^{-1}) = \rho(f)$ for any orientation-preserving homeomorphsim $h : S^1 \to S^1$.

We state a few properties of interest to us in this setting about $\rho(f)$, when $f = f_\alpha|_{\overline{\mathbb{R}}}$. We begin with the case of rational rotation numbers.

Proposition 5.7. *Assume f_α is of the form* (3.4)*, with $\alpha > 0$ and set $f = f_\alpha|_{\overline{\mathbb{R}}}$.*

(1) *$\rho(f) \in \mathbb{Q}$ if and only if f has a periodic point. We write $\rho(f) = p/q$, with p/q in reduced form.*
(2) *$\rho(f) = p/q$ if and only if there exists a periodic orbit of period q. Moreover, all periodic orbits of f have period q in this case.*
(3) *If there exists a periodic orbit for f, $B = \{x_0, f(x_0) \dots, f^{q-1}(x_0)\}$, then the circular ordering of points in B on S^1 is determined completely by p and q.*
(4) *Under the assumptions that: (i) $\rho(f) = p/q$, (ii) f has exactly one periodic orbit, B, and (iii) if x is not periodic, we have*

$$\lim_{m\to\infty} f^{-mq}(x) = a, \lim_{m\to\infty} f^{mq}(x) = b, \quad a, b \in B, \tag{5.7}$$

with $a = b$ if and only if $q = 1$ (so $\rho(f) = 0$).
(5) *Under the assumptions in (4), replacing (ii) with: f has two periodic orbits, then equation* (5.7) *holds with a and b lying on different periodic orbits.*

It is a classical theorem that if $\rho(f)$ is irrational, then f is semiconjugate or conjugate to the irrational rotation $T_{\rho(f)}(x) = x + \rho(f) \bmod 1$. The theory for smooth homeomorphisms with critical points is complex and interesting but Yoccoz showed that when the circle map is induced by an analytic map of $\mathbb{C}_\infty$, then the result is the best possible [15].

A critical point x_0 of a smooth circle map f is *nonflat* if some k^{th} derivative is nonzero, i.e., $f^{(k)}(x_0) \neq 0$ for some $k \in \mathbb{N}$.

Proposition 5.8. *Assume $f_\alpha(z) = \frac{z^3+\alpha}{-\alpha z^3+1}$, $\alpha \in \mathbb{R}$. We write $f \equiv f_\alpha|_{\overline{\mathbb{R}}}$. If $\rho(f)$ is irrational then the following hold:*

(1) *There exists a homeomorphism $h : \overline{\mathbb{R}} \to S^1$ conjugating f to $T_{\rho(f)}$, irrational rotation by $\rho(f)$ on S^1.*
(2) *$J(f_\alpha) = \mathbb{C}_\infty$ and $J(\hat{f_\alpha}) = \mathbb{RP}^2$.*

Proof. (1) is the result of Yoccoz which states that if there is a nonflat critical point for an analytic circle homeomorphism with an irrational rotation number, then f is topologically conjugate to $T_{\rho(f)}$. In the family f_α we have two order 3 inflection points at 0 and ∞ so the theorem of Yoccoz applies. To prove (2) we assume that there is some nonempty connected component of $F(f_\alpha)$. Then Theorem 4.5 implies the existence of a nonrepelling periodic orbit or a Siegel disk cycle. The nonrepelling orbit is impossible by Proposition 5.7 and using the fact that the critical orbits remain real; i.e., if a nonrepelling orbit exists it must be real. By conjugacy to $T_{\rho(f)}$, we know that for any $x \in \mathbb{R}$, $\{f^k(x)\}_{k\geq 1}$ is dense in $\overline{\mathbb{R}}$. The Siegel disk cannot occur because its boundary would be the equator ($\overline{\mathbb{R}}$), which along with the antipodal symmetry of f_α, would imply that f_α is itself a homeomorphism, contradicting the assumption that $\deg(f_\alpha) = 3$. Therefore there can be no nonempty component of $F(f_\alpha)$. □

It remains to explore what actually occurs in the family of maps we study in this paper. First we show that a variety of rational rotation numbers for f_α occurs along the positive real axis.

The next results show that the rotation number is 0 for values near the origin and 1/2 if α is large enough.

Proposition 5.9. *Assume* $f_\alpha(z) = \frac{z^3+\alpha}{-\alpha z^3+1}$, $\alpha > 0$. *We write* $f \equiv f_\alpha|_{\overline{\mathbb{R}}}$.

(1) *For each* $\alpha > 0$ *the fixed points are:*

$$p_1 = \frac{-1-\sqrt{1-8\alpha^2}+\sqrt{2}\sqrt{1+4\alpha^2+\sqrt{1-8\alpha^2}}}{4\alpha},\ p_2 = -1/\overline{p_1}$$

and

$$p_3 = \frac{-1+\sqrt{1-8\alpha^2}-\sqrt{2}\sqrt{1+4\alpha^2-\sqrt{1-8\alpha^2}}}{4\alpha},\ p_4 = -1/\overline{p_3}.$$

(2) *If* $0 < \alpha < \dfrac{1}{2\sqrt{2}}$, *all fixed points are real and* p_1 *and* p_2 *are attracting fixed points for* f_α.

(3) *At the parameter value* $\alpha = \sqrt{2}/5$, *the four real fixed points consist of two attracting with multiplier* $1/3$ *at* $p_1 = -\sqrt{2}+\sqrt{3}$ *and* $p_2 = -\sqrt{2}-\sqrt{3} = -1/p_1$, *and two repelling at* $p_3 = -\sqrt{2}$ *and* $p_4 = -1/p_3$, *with multiplier* 2.

(4) *If* $\alpha = \dfrac{1}{2\sqrt{2}}$, *there are two real neutral fixed points at* $p = \dfrac{-\sqrt{2}\pm\sqrt{6}}{2}$ *(with multiplier* 1*), and each has multiplicity* 2.

(5) *When* $\alpha \le 1/(2\sqrt{2})$, $\rho(f) = 0/1 = 0$.

Proof. To prove (1) we use the quartic formula on the polynomial resulting from setting $f_\alpha(x) = x$; namely $p(x) = \alpha x^4 + x^3 - x + \alpha$, to obtain the roots given in (1).

For any $x \in \mathbb{R}$, $x \neq 0$ we have

$$f'(x) = \frac{3(1+\alpha^2)x^2}{(\alpha x^3-1)^2} > 0. \tag{5.8}$$

For (2)–(4) we note that plugging p_1 in (1) into equation (5.8) gives

$$f'_\alpha(p_1) = \frac{3(1-2\alpha^2-\sqrt{1-8\alpha^2})}{2(1+\alpha^2)}, \tag{5.9}$$

and by Proposition 4.4 the multiplier is the same for p_2. The fixed points are real (and then so are their derivatives by equation (5.8)) if and only if $\alpha \le 1/(2\sqrt{2})$, and if this holds then $f'(p_1) \le 1$, with $f'(p_1) < 1$ when $\alpha < 1/(2\sqrt{2})$.

To show (3) we simply substitute $\alpha = \frac{\sqrt{2}}{5} < \frac{1}{2\sqrt{2}}$ into the formulas.

When $\alpha = 1/(2\sqrt{2})$ we see that $p_3 = -1/p_1 = p_2$, and $f'(p_j) = 1$ for all j. (5) follows from Proposition 5.7. □

Similarly, we show that when α is large enough and real, $\rho(f_\alpha|_{\overline{\mathbb{R}}}) = 1/2$. For this we use the following lemma.

Lemma 5.10. *For* $f_\alpha(z) = \frac{z^3+\alpha}{-\alpha z^3+1}$, $\alpha \in \mathbb{R}$, *we have*

$$f_\alpha(-1/z) = f_{-1/\alpha}(z) \tag{5.10}$$

for all z.

Proof. Both sides of equation (5.10) are $\dfrac{\alpha z^3-1}{z^3+\alpha}$ for every $z \in \mathbb{C}_\infty$. □

Proposition 5.11. *If* p *is a fixed point for* $f_\alpha(z) = \frac{z^3+\alpha}{-\alpha z^3+1}$, $\alpha \in \mathbb{R}$, *then* $-p$ *is a periodic point for* $f_{1/\alpha}$ *of period 2.*

Proof. We first observe that

$$f_\alpha(p) = p \Leftrightarrow f_{-\alpha}(-p) = -p,$$

because $f_{-\alpha}(-z) = -f_\alpha(z)$ for all z. Therefore assume p_1 is real and fixed for f_α, then so is $p_2 = -1/p_1$ by Lemma 4.3. Then by the observation, $f_{-\alpha}$ has fixed points $q_1 = -p_1$ and $q_2 = -p_2 = 1/p_1$. By Lemma 5.10, $f_{1/\alpha}(q_2) = f_{-\alpha}(-1/q_2) = p_1$ and $f_{1/\alpha}(p_1) = f_{-\alpha}(q_2) = q_2$. Therefore $f^2_{1/\alpha}(q_2) = q_2$, and since $p_1 \neq q_2$, it is not fixed. □

Proposition 5.12. *Assume* $f_\alpha(z) = \frac{z^3+\alpha}{-\alpha z^3+1}$, $\alpha \in \mathbb{R}$.

(1) *When* $\alpha = 2\sqrt{2}$, *there exists a neutral real period 2 orbit.*
(2) *When* $\alpha > 2\sqrt{2}$, *there exists an attracting real 2-cycle.*
(3) *When* $\alpha \geq 2\sqrt{2}$, $\rho(f_\alpha|_{\overline{\mathbb{R}}}) = 1/2$.

Proof. From Proposition 5.11 we see that we have a real 2-cycle whenever $\alpha \geq 2\sqrt{2}$; if $\{p_1, p_2\}$ are the attracting fixed points for f_α, then $\{-p_1, -p_2\}$ form an attracting 2-cycle for $f_{1/\alpha}$. We can use the chain rule to calculate directly that the cycle is attracting (or neutral), or we can apply Proposition 5.7 to see that the cycle must attract all real points, hence must be a nonrepelling cycle on $\mathbb{C}_\infty$. □

We write $f \equiv f_\alpha|_{\overline{\mathbb{R}}}$. Finally we show that since the rotation number $\rho(f)$ varies continuously with f using the uniform topology on the space of homeomorphisms of S^1, together with the holomorphic dependence of f on α, we obtain irrational rotation numbers. In particular, every irrational number between 0 and 1/2 occurs as a rotation number of f_α for some real $\alpha > 0$, so Proposition 5.8 holds for many maps.

Proposition 5.13. *Assume* $f_\alpha(z) = \frac{z^3+\alpha}{-\alpha z^3+1}$, $\alpha > 0$. *Then as* α *increases from* 0 *to* $2\sqrt{2}$, $\rho(f_\alpha|_{\overline{\mathbb{R}}})$ *increases continuously from* 0 *to* 1/2.

Proof. By Proposition 5.9 we have that $\rho(f_\alpha) = 0$ for $\alpha \in [0, \frac{1}{2\sqrt{2}}]$. By Propostion 5.12 we have that $\rho(f_\alpha) = 1/2$ for $\alpha > 2\sqrt{2}$. Therefore it is enough to show that, when viewed on the circle, the maps increase (angularly) monotonically and continuously in α. Then the classical rotation number theory shows that the rotation number behaves in the same way.

We conjugate f_α by a Mobius transformation that maps the upper half-plane to the unit disk, and hence the real axis to the unit circle, $M(z) = \dfrac{z-i}{z+i}$. Then we consider the map

$$h_\alpha(z) := M \circ f_\alpha \circ M^{-1}(z) = \left(\frac{\alpha - i}{\alpha + i}\right)\left(\frac{-(1+3z^2)}{3z+z^3}\right), \tag{5.11}$$

obtained by simplifying the fractions appearing.

We consider the argument of the left factor, and define for $\alpha \geq 0$, $\theta(\alpha) = \arg(\frac{\alpha-i}{\alpha+i}) = 2\arg(\alpha - i)$, where arg denotes the principal branch of the argument function, $0 \leq \arg(z) < 2\pi$. We see θ is an increasing and continuous function of α for real α; in particular, $\theta(\alpha)$ increases from 0 to 2π as α increases from 0 to ∞. Since $\left|\frac{\alpha-i}{\alpha+i}\right| = 1$ for all $\alpha \geq 0$, we have

$$e^{i\theta(\alpha)} = \frac{\alpha - i}{\alpha + i}.$$

Considering h_α restricted to $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, we have from equation (5.11), for each $t \in [0, 1)$,

$$h_\alpha(e^{2\pi it}) = e^{i(\theta(\alpha)+\pi-2\pi t)} \left(\frac{1+3e^{4\pi it}}{3+e^{4\pi it}}\right). \tag{5.12}$$

From equation (5.12) it is clear that the lifting of $h_\alpha|_{S^1}$ to $\mathbb{R}$ is increasing in α so the rotation number is continuous and increasing in α as claimed. □

As a corollary of Propositions 5.8 and 5.13 we obtain the following theorem.

Theorem 5.14. *There exist dianalytic maps of* $\mathbb{RP}^2$ *induced by maps* f_α *of the form* (3.4) *with* $J(\hat{f_\alpha}) = \mathbb{RP}^2$.

Ergodic properties with respect to (local) Lebesgue measure m of maps satisfying the conclusion of Theorem 5.14 are unknown, as each critical point has an infinite forward orbit returning near itself infinitely often. (See also examples of a similar type constructed by Herman [9].) However it is well-known that there is a unique invariant measure of entropy $\log 3$, singular with respect to m, with respect to which f_α and hence $\hat{f_\alpha}$ are one-sided Bernoulli. Discussion of one-sided Bernoulli maps and their measures can be found in [4, Section 7], and the references therein. Recall that a continuous map f of a compact space X is topologically transitive if for any nonempty open sets $U, V \subseteq X$, there exists an $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$; equivalently, f is topologically transitive if there exists a point $x \in X$ such that $\overline{\mathcal{O}_+(x)} = X$. The following corollary holds for any map with full Julia set.

Corollary 5.15. *There exist dianalytic maps of* $\mathbb{RP}^2$ *of the form* (3.4) *that are topologically transitive.*

6. Other dianalytic maps of $\mathbb{RP}^2$

We limited our study in this paper to dianalytic maps of $\mathbb{RP}^2$ with only one critical point. Even in the degree 3 case there are more complicated maps, and the analysis gets more difficult. We mention one example but do not prove most of our assertions. Consider the map

$$f(z) = \frac{z^3 + \frac{3}{2}z^2 + \frac{3}{2}i}{\frac{3}{2}iz^3 - \frac{3}{2}z + 1}. \tag{6.1}$$

By checking directly that f commutes with the antipodal map ϕ or by applying the condition given in [1], we see that f gives rise to a dianalytic map on $\mathbb{RP}^2$. Now we no longer have double critical points at 0 and ∞; instead we have two antipodal pairs of critical points. We can calculate 4 distinct critical points for f:

$$c_1 \approx -1.016 + 2.117i, \quad c_2 = -1/\overline{c_1}$$

and

$$c_3 \approx 1.409 - .676i, \quad c_4 = -1/\overline{c_3}.$$

Numerical estimates show that the four fixed points are repelling but there are 2 noncollapsing attracting period 2 cycles for f given by

$$\{p_1, p_2\} \approx \{1.65 - .308i, .035 - 1.214i\} \text{ and } \{q_1, q_2\} = \{-1/\overline{p_1}, -1/\overline{p_2}\}.$$

Moreover, $\lim_{n\to\infty} f^n(c_3) = \{p_1, p_2\}$ so $\lim_{n\to\infty} f^n(c_4) = \{q_1, q_2\}$. However, it does not seem that c_1 and c_2 are attracted to these cycles as the attraction is quite robust and they do not approach them. In Figure 8 we color points in $\mathbb{C}$ dark blue

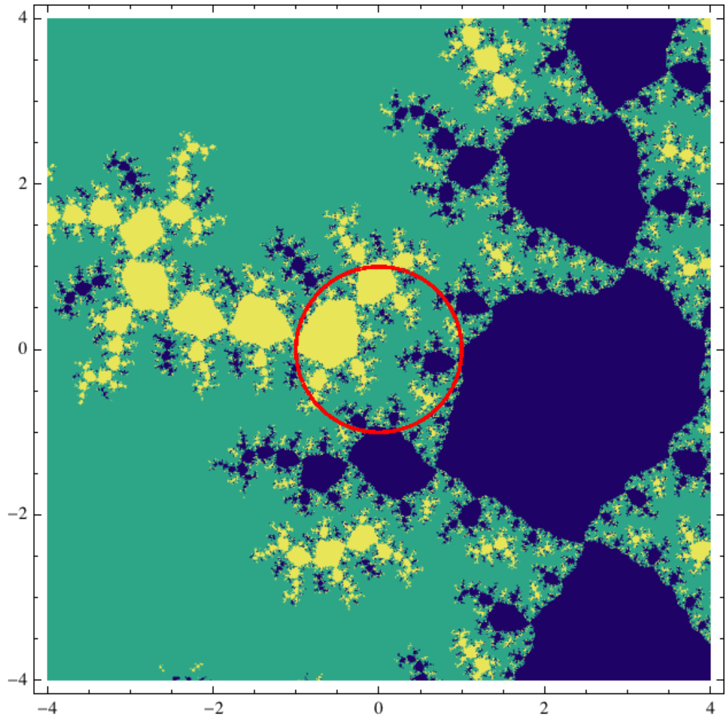

FIGURE 8. $J(f)$ is the boundary between colored regions. We include the unit circle in red so that the antipodal symmetry can be seen.The antipodal symmetry of Fatou components shows that every yellow point at z_0 has a corresponding blue point at $w_0 = -1/\overline{z_0}$, as can be seen by reflecting antipodally about the red unit circle.

if they are attracted to $\{p_1, p_2\}$, yellow if they are attracted to $\{q_1, q_2\}$, and green otherwise. It remains somewhat of a mystery as to what is going on in the green region.

While many of the results of this paper go through for general dianalytic maps of $\mathbb{RP}^2$, the connectivity of $J(\hat{f})$ for this example is not easy to determine without further analysis.

Acknowledgement

The authors are grateful to the referee for many valuable comments that improved this paper.

References

[1] Ilie Barza and Dorin Ghisa, *Dynamics of dianalytic transformations of Klein surfaces*, Math. Bohem. **129** (2004), no. 2, 129–140. MR2073510 (2005j:30058)

[2] Alan F. Beardon, *Iteration of rational functions*, Graduate Texts in Mathematics, vol. 132, Springer-Verlag, New York, 1991. Complex analytic dynamical systems. MR1128089 (92j:30026)

[3] K. Borsuk, *Drei Sätze über die n-dimensionale euklidische Sphäre*, Fund. Math. **20**, (1933), 177–190.

[4] Henk Bruin and Jane Hawkins, *Rigidity of smooth one-sided Bernoulli endomorphisms*, New York J. Math. **15** (2009), 451–483. MR2558792 (2011c:37013)

[5] Lennart Carleson and Theodore W. Gamelin, *Complex dynamics*, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993. MR1230383 (94h:30033)

[6] Sue Goodman and Jane Hawkins, *Ergodic and chaotic properties of Lipschitz maps on smooth surfaces*, New York J. Math. **18** (2012), 95–120. MR2900369

[7] Victor Guillemin and Alan Pollack, *Differential topology*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1974. MR0348781 (50 #1276)

[8] Boris Hasselblatt and Anatole Katok, *A first course in dynamics*, Cambridge University Press, New York, 2003. With a panorama of recent developments. MR1995704 (2004f:37001)

[9] Michael-R. Herman, *Exemples de fractions rationnelles ayant une orbite dense sur la sphère de Riemann* (French, with English summary), Bull. Soc. Math. France **112** (1984), no. 1, 93–142. MR771920 (86d:58055)

[10] John Milnor, *On rational maps with two critical points*, Experiment. Math. **9** (2000), no. 4, 481–522. MR1806289 (2001k:37074)

[11] John Milnor, *Dynamics in one complex variable*, 3rd ed., Annals of Mathematics Studies, vol. 160, Princeton University Press, Princeton, NJ, 2006. MR2193309 (2006g:37070)

[12] Welington de Melo and Sebastian van Strien, *One-dimensional dynamics*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 25, Springer-Verlag, Berlin, 1993. MR1239171 (95a:58035)

[13] James R. Munkres, *Topology: a first course*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1975. MR0464128 (57 #4063)

[14] J. Silverman, Presentation at JMM AMS Meeting Special Session on Dynamical Systems in Algebraic and Arithmetic Geometry, Boston (January, 2012).

[15] Jean-Christophe Yoccoz, *Il n'y a pas de contre-exemple de Denjoy analytique* (French, with English summary), C. R. Acad. Sci. Paris Sér. I Math. **298** (1984), no. 7, 141–144. MR741080 (85j:58134)

Department of Mathematics, University of North Carolina at Chapel Hill, CB #3250, Chapel Hill, North Carolina 27599-3250
E-mail address: seg@email.unc.edu

Department of Mathematics, University of North Carolina at Chapel Hill, CB #3250, Chapel Hill, North Carolina 27599-3250
Current address: National Science Foundation, 4201 Wilson Blvd., Arlington, Virginia 22230
E-mail address: jmh@math.unc.edu