FAMILIES OF ERGODIC AND EXACT ONE-DIMENSIONAL MAPS

JULIA BARNES AND JANE HAWKINS

Abstract. We give a parametrized family of rational interval maps of degree two, each ergodic, exact and preserving a measure equivalent to Lebesgue measure. The family includes the unique quadratic Chebyshev polynomial as its only polynomial map. We extend the family to other settings on the circle and real line. We also give numerical approximations to the measure theoretical entropy of the equivalent invariant measure and the Hausdorff dimension of the singular measure of maximal entropy.

1. Introduction

Parametrized families of interval maps, especially families which are smooth with respect to both the variable and the parameter, have been written about extensively as they reveal important fundamental principles in the study of bifurcations of dynamical systems (see e.g., [3],[10], [15], [22]).

Most of the work has been done on polynomials; other interesting results on polynomials involve pairs of maps with the same Julia set, connecting this property to the question of whether the maps commute [1]. In addition many authors have looked at parametrized families for the onset of chaos, linking it to the existence of absolutely continuous invariant measures [15], [16].

In this paper we combine these ideas to construct a one-parameter family of real rational maps, each having the same Julia set - the interval $[-2, 2]$ - and all exhibiting similar ergodic and exact behavior. An unusual aspect of this family is that ergodicity is typically rare in families of mappings, and we show the stronger property of Lebesgue exactness for each member. Moreover we also establish the existence an invariant measure equivalent to Lebesgue measure for every member of the family.

Nevertheless the maps are not pairwise conformally conjugate; they are however topologically conjugate with topological entropy $\log 2$. Precisely one value of the parameter gives the degree two Chebyshev polynomial, which is measure theoretically isomorphic to the $(1/2, 1/2)$ Bernoulli shift using Lebesgue measure m on the interval. For all other parameter values, the respective measures of maximal entropy (which is $\log 2$) are mutually singular with respect to Lebesgue measure m. In particular, each map comes with a pair of distinguished invariant measures; a probability measure equivalent to m and a maximal entropy measure singular with respect to m (except for the Chebyshev value).

2000 *Mathematics Subject Classification.* Primary 37F10; Secondary 37F45.
Key words and phrases. Interval maps, ergodicity, Julia sets.

Each member of the parametrized family is Lebesgue ergodic and exact, and preserves a measure $\mu_a \sim m$, where μ_a depends on the parameter a. It is natural therefore to approximate (or calculate) the Liapunov exponent λ_{μ_a} or equivalently the entropy h_{μ_a}, since the Hausdorff dimension of μ_a is 1, using $HD(\mu_a) = 1 = \frac{h_{\mu_a}}{\lambda_{\mu_a}}$. The unique measure of maximal entropy, denoted ρ_a has a Hausdorff dimension which is less than 1 except in the Chebyshev case [30], so we approximate that as well using $HD(\rho_a) = \frac{\log 2}{\lambda_{\rho_a}}$.

We consider this family of mappings in several different settings, first on an interval I and next on $(-\infty, 0]$, and then a two-point extension on the unit circle in the complex plane, and an isomorphic version on $\mathbb{R}$, unifying some examples that have been studied in the literature before. In the last section we give some graphical data from estimates of the entropy of the invariant measure equivalent to m and the Hausdorff dimension of the singular maximal entropy measure. Entropy and Hausdorff dimension appear to vary smoothly with the paramater. The authors thank the referees for comments improving this paper.

2. An ergodic family of unimodal rational interval maps

Consider the real family of maps on $\mathbb{R}$ given by:

$$R_a(x) = \frac{-8 + (2 + 8a)x^2}{4 + (-1 + 4a)x^2}.$$

This family is chosen to "push off" from the Chebyshev polynomial at the tip of the Mandelbrot set using rational maps instead of polynomials; when $a = 1/4$, $R_{1/4}(x) = x^2 - 2$. It is shown in Proposition 2.1 below (and the discussion preceeding it) that R_a is not conformally conjugate to a polynomial for any other value of a; this result also follows from Proposition 3.1 (3) in this paper, proved in [11]. We begin with the following result for the parametrized family of mappings obtained when $a \in (0, 1)$.

Theorem 2.1. If $a \in (0, 1)$, then the following hold:

(1) $R_a(-2) = R_a(2) = 2$, $R_a(0) = -2$, and R_a has one critical point at $x = 0$, is strictly decreasing on $[-2, 0)$ and strictly increasing on $(0, 2]$.
(2) $R_a[-2, 2] = R_a^{-1}[-2, 2] = [-2, 2]$.
(3) $R'_a(-2) = -1/a$ and $R'_a(2) = 1/a$, so $x = 2$ is a repelling fixed point.
(4) Each R_a is finite postcritical (has a finite forward critical orbit).
(5) There is one other fixed point in $[-2, 2]$, namely $p = \dfrac{-2}{1 + 2\sqrt{a}} \in (-2, -2/3)$, with derivative $R'(p) = -(1 + 2\sqrt{a})$. That is, p is always repelling.
(6) The Schwarzian derivative of R_a,

$$S_a = \frac{R'''_a}{R'_a} - \frac{3}{2}(\frac{R''_a}{R'_a})^2,$$

satisfies $S_a(z) = \dfrac{-3}{2z^2}$. Therefore $S_a(z) < 0$ except at the critical point.

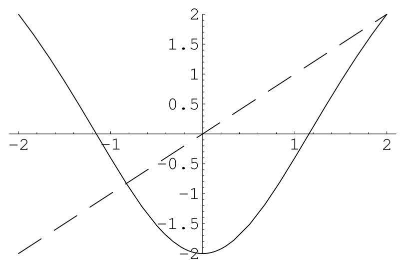

FIGURE 1. The graphs of $R_a(x)$ and x for $a = .5$

Proof. We have $R_a(2) = 2$, and by symmetry about $x = 0$, we have $R_a(-2) = 2$ as well; property (2) follows from (1). It is a straightforward calculation that $R'_a(x) = \dfrac{128ax}{(4+(4a-1)x^2)^2}$, so statement (1) holds when $a > 0$. By plugging in the values $R'_a(\pm 2) = \dfrac{\pm 256a}{(16a)^2} = \pm\dfrac{1}{a}$, (3) follows immediately, and (4) follows from (1). To show (5), we set $R_a(x) = x$. After factoring out $x - 2$ (since we know 2 is fixed), we are left with the polynomial $r_a(x) = (1-4a)x^2 + 4x + 4 = 0$, whose roots are $p = \dfrac{-2}{1 \pm 2\sqrt{a}}$. Since $0 < a < 1$, we can show that: we have $-2 < \dfrac{-2}{1+2\sqrt{a}} < \dfrac{-2}{3}$. Similarly one can show that the other fixed point is always less than $x = -2$ or greater than $x = 2$, depending on the value of a. □

A typical graph of R_a is shown in Figure 1.

2.1. **Commuting rational maps and rational maps with the same Julia set.** Studies dating back to the 1920's by Ritt [27] have shown that it is rare for two distinct polynomials to have the same Julia set. Moreover if they do, then they must commute or a multiple of one polynomial commutes with the other, where the multiplicative constant must come from a very small set relating to symmetries of the Julia set. In [1] the history of this problem is discussed and new results and examples are given to settle the question once and for all: which polynomials have an interval as their Julia sets? In our current setting, degree two rational maps, it follows from [1] that two polynomials with Julia set an interval must commute. Therefore both must be Chebyshev polynomials (see Theorems 1 and 3 in [1] for explicit statements). Since in degree 2 there is only one such polynomial corresponding to the value $a = 1/4$ in our notation, clearly our family of maps exhibits strong non-polynomial behavior. We have the following result.

Proposition 2.1. For the family $R_a(x) = \dfrac{-8+(2+8a)x^2}{4+(-1+4a)x^2}$, $a \in (0,1)$, $R_a \circ R_b = R_b \circ R_a$ if and only if $a = b$.

Proof. We apply the basic properties of Theorem 2.1. Clearly the fixed point for R_a which is in $(-2,2)$, $p_a = \dfrac{-2}{1+2\sqrt{a}}$, is different from R_b's fixed point,

$p_b = \dfrac{-2}{1+2\sqrt{b}}$, when $a \neq b$. Then since each map only has one fixed point in the open interval $(-2, 2)$, we have $R_a(R_b(p_b)) = R_a(p_b) \neq R_b(R_a(p_b))$, since $R_a(p_b)$ is neither the unique fixed point for R_b in $(-2, 2)$ nor 2. □

It is mentioned in [1] that Devaney gave simple examples of non-commuting rational and meromorphic maps with the same Julia set; here we give a family of maps with this property.

2.2. **Measure theoretic properties.** Theorem 2.1 establishes that each map R_a, $a \in (0, 1)$ is an *S-unimodal* map in the sense that it is piecewise monotone with one turning point (the critical point at 0), and has negative Schwarzian derivative throughout the domain $[-2, 2]$. Therefore we are in a setting in which the classical theory of unimodal maps applies.

Definition 2.1. On any interval $I \subset \mathbb{R}$ we put the σ-algebra of Borel sets, denoted $\mathcal{B}$. We denote m to be normalized Lebesgue measure on I, so $m(I) = 1$; a measure μ is *equivalent to* m, written $\mu \sim m$, if for any $A \in \mathcal{B}$, $\mu(A) = 0$ if and only if $m(A) = 0$. We say that a measurable map $R : I \to I$ admits an invariant probability measure equivalent to Lebesgue measure if there exists $\mu \sim m$, $\mu(I) = 1$, such that $\mu(R^{-1}A) = \mu(A)$ for every $A \in \mathcal{B}$. Throughout this paper, an *equivalent measure* always refers to a probability measure equivalent to m.

We state a result which follows from work in [25] (plus later generalizations in [20]). This result is from [20], (Theorem V.3.1); a map satisfying the hypotheses of the next theorem is usually referred to as a *Misiurewicz map.*

Theorem 2.2. Let $R : I \to I$ be a mapping satisfying the following:
(1) R is C^3 and there are only finitely many critical points;
(2) All periodic points of R are repelling;
(3) the forward orbit of each critical point of R does not accumulate onto any critical point;
(4) R is surjective.

Then R admits an equivalent measure.

Theorem 2.3. Every map $R_a(x) = \dfrac{-8 + (2 + 8a)x^2}{4 + (-1 + 4a)x^2}$, $a \in (0, 1)$, admits an equivalent measure $\mu_a \sim m$.

Proof. It is well-known (cf. [20], Singer's Theorem II.6.1) that any attracting periodic orbit of R_a either contains a critical point in its attracting basin or occurs at a boundary point of the interval. Both are impossible by Theorem 2.1 so we can apply Theorem 2.2 to obtain the result. □

For any two sets $A, B \in \mathcal{B}$ we define their *symmetric difference* by $A \triangle B = (A \setminus B) \cup (B \setminus A)$. The map R is *ergodic* if R has a trivial field of invariant sets, or equivalently, if any measurable set B with the property that $\mu(B \triangle R^{-1}B) = 0$ has either zero or full measure.

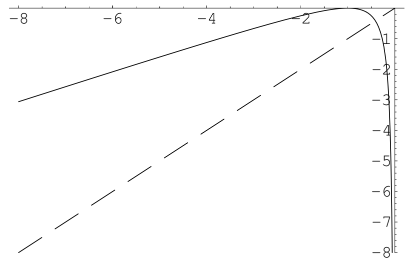

FIGURE 2. The graphs of $f_a(x)$ and x for $a = .5$

It follows from the definitions that R is measure preserving and ergodic if and only if for all sets $A, B \in \mathcal{B}$, $\mu(A) > 0, \mu(B) > 0$, there is a positive integer n such that $\mu(B \cap R^{-n}A) > 0$.

A map is *exact* if it has a trivial tail field $\cap_{n\geq 0} R^{-n}\mathcal{B} \subset \mathcal{B}$, or equivalently, if any set B with the property $\mu(R^{-n} \circ R^n(B) \triangle B) = 0$ for all n has either zero or full measure. It is clear that every exact map is also ergodic. The next result is due to Ledrappier [16] if an additional condition holds, and as stated below, can be found eg. in [4](Theorem 4.6) or in [5] (Theorem 3.2).

Theorem 2.4. Let R be C^3 S-unimodal. If R has no attractor, then R is Lebesgue exact.

Corollary 2.5. Each map $R_a(x) = \dfrac{-8 + (2+8a)x^2}{4 + (-1+4a)x^2}$, $a \in (0,1)$, is ergodic and exact with respect to an invariant probability measure $\mu_a \sim m$.

A measurable map R is *full* if for every set $B \in \mathcal{B}$, $m(B) > 0$, we have $\lim_{n\to\infty} m(R^n(B)) = 1$. Rohlin showed in [28] that in the finite measure preserving case, full is equivalent to exact, so we have the following corollary.

Corollary 2.6. For each $a \in (0,1)$, R_a is full with respect to μ_a.

3. Applications to other parametrized families of maps

In [21] Milnor discusses real quadratic maps in terms of their critical behavior on the Riemann sphere; related to this topic is a section in his book [23] about smooth Julia sets. In this section we explore the possibilities for smooth Julia sets to extend the family of mappings discussed above to rational maps, obtaining parametrized families with the same measure theoretic behavior in other classical settings.

For each real rational map f, the unique holomorphic extension of f as an analytic map of the Riemann sphere $\mathbb{C}_\infty = \mathbb{C} \cup \infty$ to itself will be denoted by $f_\#$. Each f gives rise to a map $f_\#$ with a nonempty Julia set in $\mathbb{C}_\infty$ in a natural way, cf. [3].

3.1. The quadratic rational family.

In [11], the following family of mappings was studied on the Riemann sphere:

$$f_a(z) = a(z + 1/z + 2),$$

and the following was proved.

Proposition 3.1. For each $a \in \mathbb{C} \setminus \{0\}$,

(1) The fixed points of f_a are ∞, $\frac{-\sqrt{a}}{1+\sqrt{a}}$, and $\frac{\sqrt{a}}{1-\sqrt{a}}$ if $a \neq 1$.
(2) The fixed points of f_1 are ∞ and $\frac{-1}{2}$.
(3) There is exactly one polynomial in the family f_a, occurring at $a = \frac{1}{4}$; i.e., there is a Möbius map conjugating f_a to a polynomial iff $a = \frac{1}{4}$.
(4) Each map in the family f_a for $a \in \mathbb{C} \setminus \{0\}$ is unique up to conformal equivalence so f_a is a reduced family of holomorphic maps.

This family, with $a \in \mathbb{C} \setminus 0$, was first discussed in [21] and then later by the second author in terms of parameters exhibiting ergodic behavior in [11]. For $a \in (0, 1)$, the fact that the maps f_a are all topologically conjugate follows from the stability of these maps discussed in [19], Theorem 4.2.

Theorem 3.1. The map f_a is conformally conjugate to the map R_a via the Möbius transformation $M(z) = \frac{2(z+1)}{z-1}$. Moreover, if $a \in (0, 1)$, then $J(f_a) = [-\infty, 0]$, the negative real axis.

Proof. It is easy to check by hand that $M^{-1}(z) = \frac{z+2}{z-2}$ and that $M \circ f_a \circ M^{-1} = R_a$. To prove the second statement, we note that $M([-\infty, 0]) = [-2, 2]$ and Möbius maps take one totally invariant set, $J(R_a)$, to another, $J(f_a)$. □

Corollary 3.2. For any $a \in (0, 1)$, the map $f_a(x) = a(x + 1/x + 2)$, $f_a : [-\infty, 0] \to [-\infty, 0]$ is ergodic, exact, full, and preserves a measure $\nu_a \sim m$.

The graph of $f_a(x)$ for $a = .5$ is shown in Figure 2.

3.2. **The Modified Boole family of maps.** The Boole map is defined a.e. on $\mathbb{R}$ by $f(x) = x - 1/x$. The modified Boole maps are defined by adding a parameter:

$$b_a(x) = \sqrt{a}(x - 1/x),$$

using the convention of taking the positive value.

We show that for every real number a, with $0 < a < 1$, the map b_a is ergodic, exact, and preserves a measure equivalent to m on $\mathbb{R}$. We do this by showing that each map f_a is a factor of the modified Boole map b_a, via the two-fold branched covering map $\phi(x) = -x^2$.

It is convenient to extend both maps to the Riemann sphere and restrict to the completely invariant sets (the Julia sets); in this case we see that the modified Boole family has the following properties. We denote the holomorphic extension of b_a by $(b_a)_{\#}(z) = \sqrt{a}(z - \frac{1}{z})$. In this form we have the following classical properties, most of which date back to Fatou's original work ([8] or cf. [18]).

Theorem 3.3. For each $a \in (0, 1)$, $(b_a)_{\#} : \mathbb{C}_\infty \to \mathbb{C}_\infty$ satisfies:

(1) The critical points of $(b_a)_{\#}$ are $\pm i$;
(2) The fixed points are:

$$p_\pm = \pm \frac{a^{1/4}}{\sqrt{\sqrt{a} - 1}},$$

both of which are purely imaginary;

(3) Both fixed points are attracting, with $(b_a)'_{\#}(p_\pm) = 2\sqrt{a} - 1$.
(4) $J \equiv J((b_a)_{\#}) = \mathbb{R}_\infty$
(5) $(b_a)_{\#}$ is expanding on J in the following sense: there exists a smooth Riemannian metric ρ defined in a neighborhood of J such that $||(b_a)'_{\#}(z)||_\rho > C > 1$ for all $z \in J$.
(6) $(b_a)_{\#}|_J$ is topologically conjugate to the map $z \mapsto z^2$ on the circle.

A map satisfying Property 5 in Theorem 3.3 is called *hyperbolic*.

We now regard ϕ as a map restricted to the Julia sets, or

$$\phi : (-\infty, \infty) \to (-\infty, 0],$$

(defined as above). We can extend ϕ continuously in the obvious way to a map on $\mathbb{R}_\infty \subset \mathbb{C}_\infty$, with ∞ fixed.

With respect to one-dimensional Lebesgue measure and the usual σ-algebra of Borel sets, ϕ is two-to-one except at 0 (and ∞ unless you separate $+\infty$ from $-\infty$), and a direct calculation shows that:

$$f_a \circ \phi = \phi \circ b_a = -\frac{a(z^2-1)^2}{z^2}.$$

This is true for both $\pm b_a$ (using $\sqrt{a}$ or $-\sqrt{a}$).

This shows that each map f_a studied above is a measurable factor of b_a and we have already established all the desired properties for each f_a. It remains to lift them to b_a.

We write $\mathcal{F}$ for the Borel sets in $\mathbb{R}^-$ and for each $a \in (0,1)$ we write the equivalent invariant probability measure from Theorem 3.2 as ν_a. Then $f_a : (\mathbb{R}^-, \mathcal{F}, \nu_a) \to (\mathbb{R}^-, \mathcal{F}, \nu_a)$. For any measurable set $A \in \mathbb{R} \setminus \{0\}$, we define $-A = \{-x : x \in A\}$ and $1/A = \{1/x : x \in A\}$. We have that $b_a(-A) = -b_a(A)$ and $b_a^{-1}(-A) = -[b_a^{-1}(A)]$.

For $C \in \mathcal{B}$ of the form $C = \phi^{-1} \circ \phi(C)$, we call C a *saturated set* (under ϕ). We can form the saturation of any set $B \in \mathcal{B}$ by $B_* \equiv \phi^{-1} \circ \phi(B) \supseteq B$. It is clear that $B_* = B \cup -B$, where the union may or may not be disjoint. If $B \cap -B = \emptyset$, we say B is a *a one-sheeted set*.

Theorem 3.4. For each $a \in (0,1)$, given ν_a for f_a as in Theorem 3.2, there exists an associated lifted measure m_a on $\mathbb{R}$ such that:

(1) $m_a \sim m$ is an invariant probability measure for b_a;
(2) The modified Boole map is ergodic and exact with respect to m_a.

Proof. We write $\mathbb{R}$ as the disjoint union $\mathbb{R} = P \cup N \cup \{0\}$, with $P = \{x : x > 0\}$ and $N = \{x : x < 0\}$. Given any $C \in \mathcal{B}$, we can write $C = C_P \cup C_N \cup \{0\}$, where $C_P = C \cap P$ and $C_N = C \cap N$. Clearly the union is disjoint, and any of these sets in the union could be empty. We fix a and write ν_a as ν. Given any $w \in (-\infty, 0)$, the map ϕ has two inverse branches: $\phi_P(w) = \sqrt{-w}$ and $\phi_N(w) = -\sqrt{-w}$.

Define for each $C \in \mathcal{B}$

$$\begin{aligned} m_a(C) &= \frac{1}{2}\nu(\phi(C_P)) + \frac{1}{2}\nu(\phi(C_N)) \\ &= \frac{1}{2}\nu(\phi(-C_P)) + \frac{1}{2}\nu(\phi(C_N)) \\ &= \frac{1}{2}\nu(\phi(C_P)) + \frac{1}{2}\nu(\phi(-C_N)), \end{aligned}$$

since $\phi(C_N) = \phi(-C_N)$, and $\phi(C_P) = \phi(-C_P)$. We remark that if C is a saturated set, then $m_a(C) = \nu(\phi\ C)$, and for any measurable B, $m_a(B_*) = 0$ if and only if $\nu(B) = 0$. It follows that $m_a(\mathbb{R}) = 1$.

We establish the invariance of m_a for b_a given that f_a preserves ν_a; we suppress all subscripts a, except on m_a, from now on. If C_* is a saturated set in $\mathcal{B}$, then it is easy to see that $\phi \circ b^{-1}(C_*) = f^{-1} \circ \phi(C_*)$, so it follows that

$$m_a(b^{-1}C_*) = \nu\phi(b^{-1}C_*) = \nu f^{-1}(\phi C_*) = \nu\phi(C_*) = m_a(C_*).$$

Therefore it is enough to check invariance for measurable sets $C \subset P$ (or N); we assume $C = C_P \in \mathcal{B}$. Then $C \cap (-C) = \emptyset$ and $(b^{-1}C) \cap (b^{-1}(-C)) = (b^{-1}C) \cap -(b^{-1}C) = \emptyset$. Then

$$m_a(C) = \frac{1}{2}\nu\,(\phi C) = \frac{1}{2}\nu(f^{-1}\phi C) = \frac{1}{2}\nu(\phi(b^{-1}C)), \tag{1}$$

using the invariance of ν under f. Writing $b^{-1}C = (b^{-1}C)_P \cup (b^{-1}C)_N$, we have that the right side of (1) equals

$$\frac{1}{2}[\nu(\phi(b^{-1}C)_P) + \nu(\phi(b^{-1}C)_N)] = m_a(b^{-1}C),$$

establishing the invariance of m_a.

We will now prove (2). We assume that $A \in \mathcal{B}$ is a nontrivial tail set, i.e., $m_a(A) > 0$ and up to a set of measure 0, $A = b^{-n} \circ b^n(A)$ for every $n \in \mathbb{N}$. If A is saturated, then $\phi(A)$ is a tail set for f and the full measure of A follows immediately from the exactness of f. Since $A \cap -A$ is saturated and a tail set, then the intersection has either measure 0 or 1. Since $A \cup -A$ is another saturated tail set, it has full m_a measure. Therefore $m_a(A)$ is either $\frac{1}{2}$ or 1, since $m_a(A) = m_a(-A)$; suppose that $m_a(A \cap -A) = 0$.

The uniform expansion of b on $\mathbb{R}_\infty$ with respect to a smooth metric from Theorem 3.3,(5), guarantees that for some integer n, $b^n(A) \cap b^n(-A)$ must intersect in a set of positive measure (with respect to any one of the equivalent measures under consideration, one coming from the expanding Riemannian metric). In this case, since they are tail sets, it forces $A \cap -A$ to have positive measure, which is a contradiction. Therefore, $A = -A$ up to a set of measure 0, so $m_a(A) = 1$ and the map b is exact.

□

Corollary 3.5. For any $a \in (0, 1)$, the maps $\pm b_a(x) = \pm\sqrt{a}(x - 1/x)$, on $\mathbb{R}$ are ergodic, exact, full, and preserve a measure $\mu \sim m$.

The most commonly studied modified Boole transformation is the one with $a = 1/4$ [12]; i.e., $b_{1/4}(x) = \frac{1}{2}(x - 1/x)$. We can write the conjugacy of

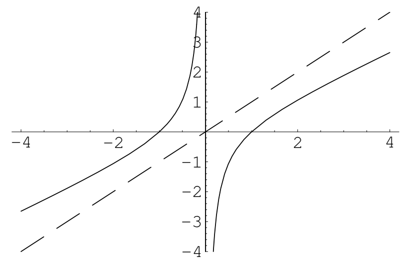

FIGURE 3. The graph of $b_a(x)$ and x for $a = .5$

Theorem 3.3,(6) explicitly and see that it is a measure theoretic isomorphism in this case.

Using the Möbius map $M(z) = \dfrac{z+i}{z-i}$, we have that

$$M \circ b_{1/4}(z) = \frac{z^2 + 2iz - 1}{z^2 - 2iz - 1} = (M(z))^2 = R \circ (M(z)),$$

letting $R(z) = z^2$. Therefore $b_{1/4}$ is conformally conjugate to $R(z) = z^2$, a one-sided Bernoulli shift with respect to m.

3.3. **Blaschke Products of degree 2.** The fourth family of maps we consider is of the form:

$$h_\alpha(z) = \frac{z(z-\alpha)}{1-\alpha z},$$

with $\alpha \in (-1, 1)$.

Using only α of the form $\alpha = 1 - 2\sqrt{a}$ with $a \in (0, 1)$, we still have that $|\alpha| < 1$, and we obtain the family:

$$B_a(z) = \frac{z(2\sqrt{a} - 1 + z)}{1 + (2\sqrt{a} - 1)z},$$

with $a \in (0, 1)$.

Each of these maps takes the unit disk to itself and $J(B_a) = S^1$; this fact will follow from the conjugacy we give below, but is also an exercise in Milnor's book ([23], Problem 7-b). Suppose now we consider the modified Boole map:

$$b_{\beta^2}(z) = \beta(z - 1/z).$$

Then using the Möbius map

$$\psi_\beta(z) = \frac{z + \sqrt{\beta/(\beta-1)}}{z - \sqrt{\beta/(\beta-1)}},$$

we have that $B_\beta(z) = \psi_\beta \circ b_{\beta^2} \circ \psi_\beta^{-1}(z)$.

Corollary 3.6. For any $a \in (0,1)$, the map $B_a(x) = \dfrac{z(2\sqrt{a}-1+z)}{1+(2\sqrt{a}-1)z}$ on S^1 is ergodic, exact, full, and preserves a measure $\mu \sim \lambda$, where λ denotes Lebesgue measure on S^1.

4. Entropy, Lyapunov exponents, and Hausdorff Dimension: Numerical approximations

The purpose of this section is to study numerically the mappings in our parametrized families, to show that they have different ergodic averages for the same continuous function, depending on whether or not the average is taken in the forward or backward direction. Moreover these averages detect information about the entropy of the maps with respect to the equivalent measure as well as the Hausdorff dimension of the maximal entropy measures (cf. [21]). We end by graphing some numerical data.

4.1. **Maximal entropy measures.** In each of the parametrized families studied above, the topological entropy of each map is $\log 2$. This follows from the result that every rational map of degree d has topological entropy $\log d$ (cf. [17]). Moreover there exists a unique invariant measure of maximal entropy which we will denote ρ called the Mañé-Lyubich measure.

We will denote it by ρ_a when viewed in a family of maps parametrized by a, and it was shown by Zdunik that $\rho_a \sim m$ if and only if the map is conformally conjugate to a Chebyshev polynomial [30]. This occurs at exactly one value: $a = 1/4$ in each family discussed above.

The second author and Taylor in [13] proved the following result about arbitrary rational maps, which had already been proved in the hyperbolic case. Let

$$\Sigma_d^+ = \prod_{i=0}^{\infty} \{0, \ldots, d-1\}_i$$

denote the space of one-sided sequences on d symbols. We consider the usual topology on Σ_d^+, and we let γ denote the $\{\frac{1}{d}, \ldots, \frac{1}{d}\}$ Bernoulli measure on Σ_d^+. It is well-known that this measure is the measure of maximal entropy with respect to the shift map σ on Σ_d^+. We use this space to define paths of inverses for a rational map. Given a rational map f of degree d, except for the critical values of f, each point z has d inverse images, hence d well-defined local inverses which we will denote $g_0, g_1, \ldots, g_{d-1}$.

Theorem 4.1. Let f denote a rational map of degree ≥ 2. Let $\{z_j\}_{j=0}^{\infty}$ denote a backward orbit of a point under f, starting at an arbitrary $z \in \mathbb{C}_\infty$ (except for at most two points that have finite backward orbits). That is, $z_j = g_{i_0, i_1, \ldots, i_j}(z)$. Then for γ a.e. backward path in Σ_d^+, for every continuous function ϕ,

$$\frac{1}{n} \sum_{j=0}^{n-1} \phi(z_j) \to \int \phi \, d\rho. \tag{2}$$

Equivalently, if we define the sequence of measures

$$\rho^z_{i_0,\ldots,i_{n-1}} = \frac{1}{n}\sum_{j=0}^{n-1} \delta_{z_j}, \tag{3}$$

then for γ a.e. backward path in Σ_d^+, $\rho^z_{i_0,\ldots,i_{n-1}}$ converges weak* to ρ.

4.2. **Bowen-Ruelle-Sinai measures and Liapunov exponents.** We have shown that all maps discussed above have equivalent invariant measures. Birkhoff's Ergodic Theorem implies the following: If $g : I \to I$ is ergodic and measure preserving with respect to a equivalent measure μ, then for every continuous function ϕ,

$$\frac{1}{n}\sum_{j=0}^{n-1} \phi(g^j) \to \int \phi \, d\mu. \tag{4}$$

Definition 4.1. If $g : I \to I$ is smooth, we say that μ is a BRS measure if the set for which (4) holds has positive Lebesgue measure, or equivalently if the sequence of measures

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} \delta_{g^j x}, \tag{5}$$

converges weak* to μ for m a.e. $x \in I$.

Note that the average in (2) is along a backward orbit while the average in (4) is taken along a forward orbit. In the next section we estimate some ergodic averages along randomly chosen forward and backward orbits; since the BRS measure is in general completely singular with respect to the measure of maximal entropy, the limits are different.

Of course the Birkhoff Ergodic theorem holds as stated in (4) for all functions $\phi \in L^1(\mu)$; in particular, if we use the function $\phi = \log |f_a'(z)| = \log |a(1-1/z^2)|$. Similarly, for the maximal entropy measure we need to use the Birkhoff-Khinchin Ergodic Theorem for stationary processes to see that the ergodic averages along backward paths converge.

A well-known result due in this setting to Ledrappier [16] says if μ is a equivalent invariant probability measure for an S-unimodal interval map f, then $h_\mu(f) = \displaystyle\int \log |f'| d\mu$. The next proposition can be found for example in [20].

Proposition 4.1. If μ is an equivalent invariant probability measure for an S-unimodal interval map f, then there exists a constant λ_μ (or sometimes written λ_f) such that for m almost every $x \in I$,

$$\lim_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)| = \lambda_\mu.$$

The constant λ_μ in the proposition is the *Liapunov exponent* of f. Therefore, since it is known that the entropy of μ is strictly less than $\log 2$ in each parametrized family except at the unique Chebyshev value $a = \frac{1}{4}$, we can use these results to approximate the entropy. Another result of Ledrappier [16] states that each of these maps is weakly Bernoulli in the sense that the natural extension is isomorphic to a Bernoulli shift. Clearly the Bernoulli shift in all but that one case is not of maximal entropy, hence it is of interest to know how that entropy changes in the parameter a.

4.3. **Hausdorff dimension of measures.** In this section we estimate the Hausdorff dimension of the unique measure of maximal entropy for each map considered. Let μ denote any of the invariant probabilty measures discussed for any of the smooth families of mappings in this paper.

Definition 4.2. The *Hausdorff dimension* of a probability measure μ on $\mathbb{R}^d$ is given by:

$$HD(\mu) := \inf\{HD(A) : A \subseteq \mathbb{R}^d \text{ Borel measurable}, \mu(A) = 1\}.$$

Let G_μ denote the set of all forward generic points of μ; i.e., the set of points for which the Birkhoff ergodic theorem holds for any continuous function on the space I under consideration. Since the families of maps b_a and B_a are hyperbolic and the maps f_a and R_a are two-point factors of them, the following theorem holds in our setting.

Theorem 4.2. ([26], Thm. 21.3) For any Borel ergodic measure μ of positive measure theoretic entropy, we have:

$$HD(\mu) = \frac{h_\mu}{\lambda_\mu} = HD(G_\mu). \tag{6}$$

In light of the above discussion, we have a numerical method for computing the measure theoretic entropy and Hausdorff dimension of the measures discussed. We describe the algorithm used to approximate the values of entropy and Hausdorff dimension in Figures 4 and 5 below.

NUMERICAL ALGORITHM FOR ENTROPY AND HAUSDORFF DIMENSION

(1) Choose one family of maps: R_a, b_a, etc. Fix a value of $a \in [0, 1]$.
(2) Pick a random value of z_0 in the Julia set of R_a.
(3) Calculate the forward average $\frac{1}{n}\sum_{j=0}^{n-1} \log |R_a'(R_a^j(z_0))|$
(4) Since the average (3) approximates $\lambda_\mu = \int \log |R_a'| d\mu$, and the measure μ is equivalent to m, $HD(\mu) = 1$ and the sum approximates h_μ.
(5) Now choose a "random" number with respect to the maximal entropy measure ρ by choosing a Lebesgue (computer-selected) random number and then taking a randomly chosen backward path for a few hundred iterations; this is the random seed we use for the next step. It is justified by the result of [13].
(6) Calculate the backward average $\frac{1}{n}\sum_{j=0}^{n-1} \log |R_a'(z_j)|$ for a randomly chosen backward path of z_0 (so $R_a^j(z_j) = z_0$).

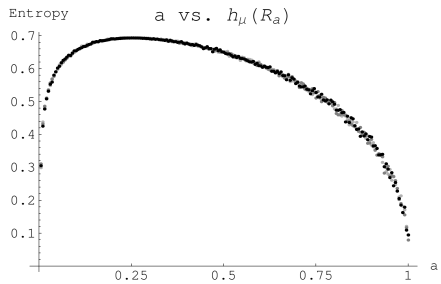

FIGURE 4. The entropy of the equivalent invariant measure

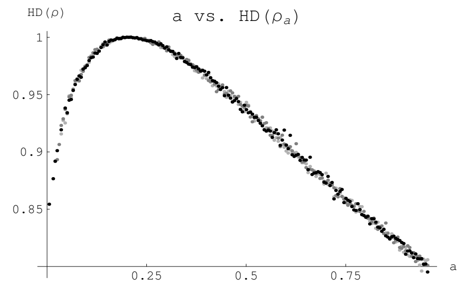

FIGURE 5. The Hausdorff dimension of the maximal entropy measure ρ_a

(7) Since the average (6) approximates $\lambda_\rho = \int \log |R_a'| d\rho$, and since $h_\rho = \log(2) \approx .69314$, we use $\log 2/\text{average}(6)$ to approximate $HD(\rho)$.
(8) In each case we choose n large enough so that the resulting output appears to be robust over different runs (which use a new random number each time). Each point on a graph is from the program run with n between 10000 and 15000.

The graph shown in Figure 4 is obtained by running the algorithm for h_μ outlined in Steps $1-4$ above on the function R_a for 200 values of a between 0 and 1. We superimpose the output of three runs each with a different random seed chosen to illustrate the robustness of the algorithm. The graph shown in Figure 5 is the output from the algorithm given in Steps $5-7$, again using 200 values of a and superimposing three runs with different random choices. One can conjecture that the values of the entropy and dimension vary smoothly with a and are unimodal with a maximum value occurring at $a = 1/4$. The same output occurs for any of the families studied here due to the conjugacies shown in this paper.

References

[1] Atela P, Hu J (1996) Commuting polynomials and polynomials with same Julia set, Intl. J. of Bif. Chaos, Vol.6, 12 A, 2427-2432.

[2] Barnes J (1999) Conservative exact rational maps of the sphere, J. Math Anal. Appl 230, 2, 350-374.

[3] Beardon A (1991) Iterations of Rational Functions, Springer-Verlag GTM 132.

[4] Blokh A, Misiurewicz M (1998) Wild attractors of polymodal negative Schwarzian maps, Commun. Math. Phys. 199, 397-416.

[5] Bruin H, Hawkins J (2001) Exactness and maximal automorphic factors of unimodal interval maps, Erg. Th. and Dyn. Sys. 21,4, 1009-1034.

[6] Carleson L, Gamelin T (1993) Complex Dynamics, Springer-Verlag.

[7] Eremenko A E, Lyubich M Y (1992) Dynamical properties of some classes of entire functions. A. Inst. Fourier (Grenoble) **42**: 989 - 1020.

[8] Fatou P (1920) Sur les equations fonctionnelles, Bull Soc. Math. France **48** 33-94.

[9] Friere A, Lopes A, Mañé R (1983) An invariant measure for rational maps, Bol.Soc.Bras Mat. 14 no, 1, 45-62.

[10] Graczyk J, Swiatek G, (1998) The Real Fatou Conjecture, Princeton University Press.

[11] Hawkins J (2003) Lebesgue Ergodic Rational Maps in parameter space, Intl Jour of Bif. and Chaos, Vol. 13, No. 6, 1423-1447.

[12] Hawkins J, Silva C (1988) Remarks on recurrence and orbit equivalence of nonsingular endomorphisms, Dynamical Systems Proceedings, Univ. of Md., Springer Lecture Notes in Math, 1342, 281–290.

[13] Hawkins J, Taylor M (2003) Maximal entropy measure for rational maps and a random iteration algorithm, Intl. J. of Bif. and Chaos, Vol. 13,6, 1442-1447.

[14] Heicklen D, Hoffman C (2002) Rational Maps are d-adic Bernoulli, Ann. of Math (2), 156, no. 1, 103-114.

[15] Jakobson M (1981) Aboslutely continuous invarian measures for one-parameter families of one-dimensional maps Comm. Math. Phys. 81, no. 1, 39-88.

[16] Ledrappier F (1981) Some properties of absolutely continuous invariant measures of an interval, Ergod. Th. & Dyn. Sys. **1**, 77-93.

[17] Lyubich M (1983) Entropy properties of rational endomorhisms of the Riemann sphere, Ergod. Th. and Dyn. Sys. **3**, 351-385.

[18] Lyubich M (1986) The dynamics of rational transforms: the topological picture, Russ. Math Surveys **41:4**, 43-117.

[19] McMullen C T, (1994) Complex Dynamics and Renormalization, Princeton University Press.

[20] de Melo W, van Strien, S (1993) One Dimensional Dynamics, Springer-Verlag.

[21] Milnor J (1993) Geometry and dynamical of quadratic rational maps, Experimental Math **2**, 37-83.

[22] Milnor J (2000) On Rational Maps with Two Critical Points, Experimental Math. **9** 480-522.

[23] Milnor J (2000) Dynamics in One Complex Variable, Friedr. Vieweg & Sohn, Verlagsgesellschaft mbH.

[24] Milnor J (2004) On Lattès Maps, Stony Brook IMS preprint #2004/01.

[25] Misiurewicz M (1981) Absolutely continuous measures for certain maps of an interval, Publ. I.H.E.S.,53, 17-51.

[26] Pesin Y (1997) Dimension Theory in Dynamical Systems Chicago Lec. in Math.

[27] Ritt J F(1923) Permutable rational functions, Trans AMS 25, 399-448.

[28] Rohlin V (1963) Exact endomorphisms of a Lebesgue space, AMS Transl. Ser. 2 **39**, 1-36.

[29] Yongcheng Y (1992) On the Julia sets of quadratic rational maps, Comp. Var. **18**, 141-147.

[30] Zdunik A (1990) Parabolic orbifolds and the dimension of the maximal measure for rational maps, Invent. Math. 99, 627–649.

Department of Mathematics and Computer Science, Western Carolina University, Cullowhee, NC 28723

E-mail address: `jbarnes@email.wcu.edu`

Department of Mathematics, University of North Carolina at Chapel Hill, CB #3250, Chapel Hill, North Carolina 27599-3250

E-mail address: `jmh@math.unc.edu`