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Families of type III_0 ergodic transformations in distinct orbit equivalent classes

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Abstract A new isomorphism invariant of certain measure preserving flows, using sequences of integers, is introduced. Using this invariant, we are able to construct large families of type III_0 systems which are not orbit equivalent. In particular we construct an uncountable family of nonsingular ergodic transformations, each having an associated flow that is approximately transitive (and therefore of zero entropy), with the property that the transformations are pairwise not orbit equivalent.

Keywords Orbit equivalence · Isomorphism of flows · Ergodic transformations

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1 Introduction

The problem of classifying invertible ergodic transformations of a measure space $(X, \mathcal{B}, \mu)$ dates back to the beginning of ergodic theory. In this paper we are interested

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in a weak form of equivalence called orbit equivalence (also known as Dye or weak equivalence); since the finite measure-preserving ergodic transformations form a single equivalence class under this equivalence relation [7], our focus is on ergodic transformations that do not preserve any measure, finite or σ-finite, equivalent to μ. These are called type III transformations and are defined and discussed below; there is also a rich literature on the subject and background material appears for example in [1,3,8–12], and the references therein. It is well-known that type III transformations include uncountably many orbit equivalence classes, and even in the subclass of type III_0 transformations which is the subject of this paper, uncountably many inequivalent examples were constructed nearly immediately after the problem was posed [1,11].

The purpose of this paper is two-fold. The first is to introduce an invariant of measure-preserving flows under the stricter notion of measure theoretic isomorphism. The new invariant is the growth rate (similar to [2]) of ergodic sums of height functions of flows associated to a type III transformation when these flows are presented as a certain canonical height function over a base transformation. The second purpose is to use this result to construct explicit product odometers whose associated flows are all non-isomorphic, hence giving an uncountable family of invertible transformations which are pairwise inequivalent under the much weaker notion of orbit equivalence.

In Sect. 2 we give the definitions and some details of the background of the problem. In Sect. 3 we define the invariant and show how it applies to our setting. We conclude in Sect. 4 by applying this invariant to produce techniques for constructing non-orbit-equivalent systems, including an explicit uncountable family. The flows we construct are all examples of flows labelled AC-flows by Osikawa in [13].

2 Background and definitions

By a *nonsingular system* $\{X, \mathcal{B}, \mu, T\}$ we mean a standard measure space $(X, \mathcal{B}, \mu)$ and a μ-measurable invertible transformation $T : X \to X$ such that μ and $\mu \circ T^{-1}$ are mutually absolutely continuous, where $(\mu \circ T^{-1})(A) = \mu(T^{-1}A)$. Two invertible nonsingular systems $\{X, \mathcal{B}, \mu, T\}$ and $\{Y, \mathcal{B}, \nu, S\}$ are called *orbit equivalent* if there is a bimeasurable invertible map $\Phi : X \to Y$ such that μ and $\nu \circ \Phi$ are mutually absolutely continuous, and the map Φ *preserves orbits*. That is, for μ-almost-every $x \in X$, we have

$$\left\{\Phi(T^n x) : n \in \mathbb{Z}\right\} = \left\{S^m(\Phi x) : m \in \mathbb{Z}\right\}.$$

We assume furthermore that T is *ergodic*; $T(A) = A$ (μ mod 0) for some $A \in \mathcal{B}$ implies either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$. The results we derive for invertible maps apply to other countable amenable group actions in light of [3].

If a nonsingular system is orbit equivalent to a transformation on a finite space, it is said to be of *type* I; the cardinality of the space classifies all such systems up to orbit equivalence. If the system is equivalent to a non-atomic system which preserves a σ-finite measure $\mu' \sim \mu$, it is said to be of *type* II_1 if $\mu'(X) < \infty$, and of *type* II_∞

if $\mu'(X) = \infty$. H. Dye's celebrated theorem states that all type II_1 systems are orbit equivalent, as are all type II_∞ systems [7]. If the system does not preserve any σ-finite measure $\sim \mu$, it is said to be of *type* III. The task of classifying nonsingular systems up to orbit equivalence reduces therefore to the case when $\{X, \mathcal{B}, \mu, T\}$ is of type III.

By the nonsingularity of T, μ and $\mu \circ T^{-n}$ are mutually absolutely continuous for all $n \in \mathbb{Z}$, so for μ a.e. $x \in X$, we define the sequence $\omega_n(x)$ of the Radon-Nikodym derivatives:

$$\omega_n(x) = \frac{d\mu \circ T^{-n}}{d\mu}(x).$$

We define the *(Krieger-Araki-Woods) ratio set* $\mathcal{R}$ [10] to be the set of all $\tau \in \mathbb{R} \cup \{\infty\}$ such that for every set $A \in \mathcal{B}$, $\mu(A) > 0$ and every $\epsilon > 0$, there is some n so that

$$\mu(A \cap T^{-n} A \cap \{x \in X : |\omega_n(x) - \tau| < \epsilon\}) > 0.$$

By the chain rule $\omega_{n+m}(x) = \omega_m(x)\omega_n(T^m x)$, we see that $\mathcal{R} \cap \mathbb{R}$ must be a closed multiplicative subgroup of $\mathbb{R}^+$, and it follows from the fact that our system is of type III that $\{0, \infty\} \subset \mathcal{R}$. A type III system is defined to be of *type* III_1 if $\mathcal{R} = [0, \infty]$, of *type* III_λ if $\mathcal{R} = \{\lambda^n : n \in \mathbb{Z}\} \cup \{0, \infty\}$ for some $0 < \lambda < 1$, and of *type* III_0 if $\mathcal{R} = \{0, 1, \infty\}$ [10].

W. Krieger showed that the ratio set is an invariant under orbit equivalence and a complete invariant in the case that the system is of type III_1 or III_λ [10]. Therefore, we turn our attention to the orbit equivalence problem of type III_0 systems, and from now on we assume that $\{X, \mathcal{B}, \mu, T\}$ is an ergodic type III_0 nonsingular system. To each such system there exists a canonical associated flow. We follow the notation and presentation of [9, §4] to build the associated flow of a type III_0 system, a notion previously introduced and developed in [4,8,12].

First, for any fixed ϵ, $0 < \epsilon < 1$, we can assume without loss of generality (by passing to a measure $\nu_\epsilon \sim \mu$ if necessary, see eg. [9, Prop. 2.3]), that the measure μ is σ-finite and has the property that there is some such that for all $n \in \mathbb{Z}$ and almost every $x \in X$ we have

$$\omega_n(x) \in (0, \epsilon] \cup \{1\} \cup [\epsilon^{-1}, \infty).$$

We set $[T] = \{S : X \to X$ such that S is a nonsingular invertible map such that for μ a.e. $x,\ S(x) = T^{j(x)}x,\ j(x) \in \mathbb{Z}\}$; we refer to $[T]$ as the *full group of* T. Next, since $1 \in \mathcal{R}$, and the ω_n do not approach the value 1 without actually achieving it, we can define for almost all x a map $T_0 \in [T]$ by setting $T_0(x) = T^{m(x)}(x)$, where

$$m(x) = \min\left\{n \in \mathbb{Z}^+ : \omega_n(x) = 1\right\}.$$

Lemma 1 ([9, Lemma 2.1]) *The automorphism T_0 is not ergodic on $(X, \mathcal{B}, \mu)$.*

Proof Assume to the contrary that T_0 is ergodic. If $\mathcal{R} \cap [a, b] = \emptyset$ and T is ergodic, then the set

$$A_{[a,b]} = \{x : \omega_n(x) \notin [a, b] \quad \forall n \in \mathbb{Z}\}$$

is of positive measure. However, the set $A_{[a,b]}$ is clearly T_0 invariant, so by our assumption that $\mathcal{R} = \{0, 1, \infty\}$, we see that for all $\epsilon > 0$, the sets $A_{[\epsilon,1-\epsilon]}$ and $A_{[1+\epsilon,1/\epsilon]}$ are all of full measure. As we almost surely have $\omega_n(x) \notin \{0, \infty\}$, it now follows that $\omega_1(x) = 1$ for μ-a.e. x, contradicting our assumption that T was of type III. □

Let $\mathcal{B}_0$ be the (nontrivial) σ-algebra of T_0-invariant sets, and define $X_0 = X/\mathcal{B}_0$, where μ_0 is the push-forward of μ onto this factor space.

Now we define the function

$$\lambda(x) = \log\{\min \omega_n(x) : n \in \mathbb{Z},\ \omega_n(x) > 1\}.$$

As λ is defined allowing n to range over all integers, $\lambda(x_0)$ is well-defined for $x_0 \in X_0$. The transformation T induces an invertible transformation $R : X_0 \to X_0$ [8,9], by defining R to send the equivalence class of x to the equivalence class of $T^j(x)$, where

$$\omega_j(x) = e^{\lambda(x)}.$$

Let $\Omega_X = \{(x, y) : x \in X_0,\ 0 \leq y < \lambda(x)\}$, and define the flow $\phi_t : \Omega_X \to \Omega_X$ by letting the point x flow up along the y-coordinate at constant speed until it hits height $y = \lambda(x)$, at which point it is sent to $(R(x), 0)$. This new system $\{\Omega_X, \mu_0 \times dy, \phi_t\}$, where the specified cross-section X_0 has first-return map R with first-return times given by $\lambda(x)$ we call the *associated Krieger flow*. Any flow with a given cross-section $\{X_0, \mathcal{C}, \mu_0\}$ with base transformation R and height function λ is called a *marked Krieger flow*, which we will write as $\{X_0, \mathcal{C}, \mu_0, R, \lambda\}$.

For a type III_0 ergodic transformation T, the isomorphism class of its Krieger flow is a complete invariant of the orbit equivalence class of T [12]. Moreover, every aperiodic nonsingular ergodic flow is measure theoretically isomorphic to a marked Krieger flow of a type III_0 system [12]. It is known that any type III transformation T is orbit equivalent to the induced transformation T_A on any set $A \in \mathcal{B}$ of positive measure, and that $\{X, \mathcal{B}, \mu, T\}$ is orbit equivalent (even isomorphic) to $\{X, \mathcal{B}, \nu, T\}$ whenever $\mu \sim \nu$.

From these properties it can be shown (cf. [9]) that if two marked flows $\{X_0, \mathcal{C}, \mu_0, R, \lambda\}$ and $\{Y_0, \mathcal{C}', \nu_0, R', \lambda'\}$ are isomorphic, then by considering them as Krieger flows of orbit equivalent systems $\{X, \mathcal{B}, \mu, T\}$ and $\{Y, \mathcal{B}', \nu, T'\}$, we can use induced transformations on subsets of positive measure in X_0 and Y_0 and also change to equivalent measures without affecting the isomorphism class of the flow. Therefore the isomorphism $\Phi : \Omega_X \to \Omega_Y$ can be assumed to take X_0 to X_0', and for μ_o a.e. $x \in X_0$, we have $\Phi \circ R(x) = R' \circ \Phi(x)$ and Φ maps one height function (i.e., the return time to the base space) to the other in the sense that $\lambda'(\Phi x) = \lambda(x)$.

An isomorphism Φ with these properties is called a *proper isomorphism of marked flows* [9] and we say the flows are *properly isomorphic*.

3 Dominating sequences as a flow invariant

A subset of the natural numbers $L = \{L_n\}$ is said to be of *lower density* δ (written $d_*(L) = \delta$) if

$$\liminf_{n\to\infty} \frac{1}{n} \# (\{1, 2, \dots, n\} \cap L) = \delta.$$

The *upper density* (written $d^*(L) = \delta$) is similarly defined using lim sup rather than lim inf. Given two increasing real-valued sequences $\{a_n\}$ and $\{b_n\}$, we say that $\{a_n\}$ *dominates* $\{b_n\}$, or $\{a_n\} \gg \{b_n\}$ if for all $L \subset \mathbb{N}$ of positive lower density,

$$\lim_{n\to\infty} \frac{a_n}{b_{L_n}} = \infty,$$

and $\{a_n\}$ *is dominated by* $\{b_n\}$, or $\{a_n\} \ll \{b_n\}$ if $\{b_n\}$ dominates $\{a_n\}$.

Lemma 2 *The following are equivalent:*

I. *The sequence $\{a_n\}$ dominates $\{b_n\}$;*
II. *there exists a collection of sequences $\{c_n^{(g)}\}_{g\in G}$ (G an arbitrary index set) such that:*
   (a) $\forall g \in G, d_*(\{c_n^{(g)}\}) > 0,$
   (b) $\inf \left\{ d^*(\{c_n^{(g)}\}) : g \in G \right\} = 0$*, and*
   (c) *for all $g \in G$*
$$\lim_{n\to\infty} \frac{a_n}{b_{c_n^{(g)}}} = \infty.$$
III. *for every collection of sequences $\{c_n^{(g)}\}_{g\in G}$ (G an arbitrary index set) such that:*
   (a) $\forall g \in G, d_*(\{c_n^{(g)}\}) > 0$*, and*
   (b) $\inf \left\{ d^*(\{c_n^{(g)}\}) : g \in G \right\} = 0,$
   we have that for all $g \in G$
$$\lim_{n\to\infty} \frac{a_n}{b_{c_n^{(g)}}} = \infty.$$

Proof I $\Rightarrow$ III is immediate; if $\{a_n\} \gg \{b_n\}$, then sequences $\{c_n^{(g)}\}$ satisfying (a) and (b) have positive lower density, so we have for all $g \in G$

$$\lim_{n\to\infty} \frac{a_n}{b_{c_n^{(g)}}} = \infty.$$

III $\Rightarrow$ II is trivial.

II $\Rightarrow$ I: Assume that there is a family of sequences $\{c_n^{(g)}\}_{g\in G}$ satisfying (a) and (b) and such that

$$\lim_{n\to\infty} \frac{a_n}{b_{c_n^{(g)}}} = \infty.$$

Let $C = \{C_n\} \subset \mathbb{N}$ have positive lower density. By choosing some $h \in G$ such that $d^*(\{c_n^{(h)}\}) < d_*(C)$, it follows that for large n, $C_n < c_n^{(h)}$. By the assumption of monotonicity of all our comparison sequences, we have $b_{C_n} < b_{c_n^{(h)}}$, so

$$\lim_{n\to\infty} \frac{a_n}{b_{C_n}} \geq \lim_{n\to\infty} \frac{a_n}{b_{c_n^{(h)}}} = \infty.$$

As our sequence $C = \{C_n\}$ was arbitrary (of positive lower density), it follows that $\{a_n\} \gg \{b_n\}$. $\square$

A sequence $\{a_n\}$ is said to dominate (be dominated by) a marked flow $\{X_0, \mathcal{C}, \mu_0, R, \lambda\}$ if it dominates (is dominated by) the sequence of ergodic sums

$$\Lambda_n(x) := \sum_{i=0}^{n-1} \lambda(R^i x)$$

for μ a.e. x. The collection of all increasing sequences which dominate a system will be denoted $D(X_0, \mu_0, R, \lambda)$. Similarly, let $d(X_0, \mu_0, R, \lambda)$ be the collection of all increasing dominated sequences.

A nonsingular transformation $\{X_0, \mathcal{C}, \mu_0, R\}$ has the *positive mean return time property* if for every $A \in \mathcal{C}$ with $\mu_0(A) > 0$ we have for almost every x the set: $S = \{n \in \mathbb{N} \colon R^n x \in A\}$ is of positive lower density. Note that R must be ergodic for this property to hold. The famous Birkhoff Ergodic Theorem guarantees that all ergodic probability measure-preserving systems have this property (the density of S is $\mu_0(A)$ in this case), but the property is not guaranteed in the nonsingular setting. For example, if R preserves a σ-finite infinite measure $\nu \sim \mu_0$, and $\nu(A) < \infty$ then S has zero density μ_0 almost everywhere.

Theorem 1 *Given two isomorphic marked flows* $\{X_0, \mathcal{C}_0, \mu_0, R, \lambda\}$ *and* $\{Y_0, \mathcal{C}_0', \nu_0, R', \lambda'\}$*, whose base transformations* R *and* R' *have the positive mean return time property, we have both*

$$D(X_0, \mu_0, R, \lambda) = D(Y_0, \nu_0, R', \lambda')$$

and

$$d(X_0, \mu_0, R, \lambda) = d(Y_0, \nu_0, R', \lambda').$$

Proof Assume first that the isomorphism is implemented via a proper isomorphism; in this case the result is obvious since $\Phi(X_0) = Y_0$ and for μ_0 a.e. x,

$$\sum_{i=0}^{n-1} \lambda(R^i x) = \sum_{i=0}^{n-1} \lambda'(R'^i(\Phi x)).$$

If the isomorphism is not proper, then using the notation introduced above we first assume that the flows are the marked Krieger flows of orbit equivalent systems $\{X, \mathcal{B}, \mu, T\}$ and $\{Y, \mathcal{B}', \nu, T'\}$ respectively. Let $\{\Omega_X, \mathcal{C}, m, \phi_t\}$ and $\{\Omega_Y, \mathcal{C}', m', \phi'_t\}$ be the associated Krieger flows given by the corresponding marked flows; here $m = \mu_0 \times dy$ and $m' = \nu_0 \times dy$. The isomorphism Φ carries X_0 onto some cross-section $B \subset \Omega_Y$, but B is not necessarily Y_0. Every set $A \in \mathcal{C}$ of positive μ_0 measure, by construction, can be identified (up to a set of measure 0) with a set in $\tilde{A} \in \mathcal{B}_0$ of positive μ measure, and $T_{\tilde{A}}$ and T are orbit equivalent. Moreover the marked Krieger flow for $T_{\tilde{A}}$ is just $\{A, \mu_A, R_A, \lambda_A\}$, where λ_A denotes the height function of the marked Krieger flow over the induced transformation R_A and μ_A is the obvious restriction measure.

For any $x \in A$, let $r_n(x)$ denote the (smallest) nth return time of R to A. Then $R_A(x) = R^{r_1(x)}(x)$, $R_A^i(x) = R^{r_i(x)}(x)$ and we have the identity

$$r_i(x) = \sum_{k=0}^{i-1} r_1(R^{r_k(x)}(x)). \tag{1}$$

If we set $A_i = \{x \in A : r_1(x) = i\}$, then $A = \cup_{i \geq 1} A_i \pmod 0$, and on A_i we have $\lambda_A(x) = \sum_{k=0}^{i-1} \lambda(R^k x)$. Then for μ_A a.e. $x \in A$:

$$\sum_{i=0}^{n-1} \lambda_A(R_A^i x) = \sum_{i=0}^{n-1} \left(\sum_{j=0}^{r_1(R_A^i x)-1} \lambda(R^{j+r_i(x)}(x)) \right) = \sum_{k=0}^{r_n(x)-1} \lambda(R^k x)$$

using Equation (1); by the positive mean return time property of R, the sequence $\{r_n(x)\} \subset \mathbb{N}$ has positive lower density except possibly on a set of measure 0 on A.

This shows that inducing on A samples the ergodic sums $\Lambda_n(x)$ of the original height function along a sequence of positive lower density, namely $\Lambda_{r_n}(x)$, and therefore any sequence $\{a_n\}$ that dominates the ergodic sums $\Lambda_n(x)$ for μ_0 a.e. x in the original flow built with R over X_0 still dominates the induced flow using R_A over $A \subset X_0$. Moreover, if $\{a_n\}$ dominates the induced flow on A, $\{a_n\}$, say $E_n(x) = \sum_{i=0}^{n-1} \lambda_A(R_A^i x)$ and

$$\lim_{n \to \infty} \frac{a_n}{E_{L_n}} = \infty,$$

for all positive lower density subsequences L_n. By "exducing" back to the original flow, we recover the original sequence $\Lambda_n(x)$ such that $E_k(x) = \Lambda_{n_k}(x)$ for all k, $\{n_k\} \subset \mathbb{N}$ has positive lower density, and

$$E_k(x) \leq \Lambda_n(x) \leq E_{k+1}(x) \quad \forall n_k \leq n \leq n_{k+1}. \tag{2}$$

It follows that for every n, and $\mu - 0$ a.e. $x \in A$

$$\Lambda_n(x) \leq E_n(x). \tag{3}$$

Hence $\frac{a_n}{E_{L_n}} \leq \frac{a_n}{\Lambda_{L_n}}$ for sequences L_n, which shows that $\{a_n\}$ dominates the exduced flow on A. We then apply the inequalities from (2) and (3) to the sets $R^n(A_i), n \leq r_1(x)$, to get the result on all of X.

A similar argument works for sequences $\{b_n\}$ that are dominated by a marked flow $\{X_0, \mathcal{C}_0, \mu_0, R, \lambda\}$. Sequences dominated by induced and exduced marked flows remain the same using subsequence arguments and inequalities (2) and (3).

We now replace μ on X by an equivalent measure $e^{\psi} d\mu$ using a $\mathcal{B}_0$ measurable function ψ with $0 < \psi < \lambda$, resulting in a new return function for the flow: $\tilde{\lambda}(x) = \lambda(x) - \psi(x) + \psi(Rx)$, and it is straightforward to show that this will not change any dominating or dominated sequences for the resulting ergodic sums

$$\tilde{\Lambda}_n(x) = \sum_{i=0}^{n-1} \tilde{\lambda}(R^i x) = \Lambda_n(x) - \psi(x) + \psi(R^n x),$$

as we have

$$\Lambda_{n-1}(Rx) \leq \tilde{\Lambda}_n(x) \leq \Lambda_{n+1}(x),$$

or equivalently,

$$\Lambda_n(x) - \lambda(x) \leq \tilde{\Lambda}_n(x) \leq \Lambda_{n+1}(x).$$

Finally, it was shown [9] that performing these two operations (inducing on sets of positive measure in the base space and a measure change) on either X, Y, or both, results in properly isomorphic marked Krieger flows; this proves the theorem. □

We remark that this isomorphism invariant can only be applied fruitfully to flows on spaces of infinite measure: all representations of ergodic finite-measure-preserving flows as height functions over base transformations with the positive mean return time property will be indistinguishable.

Proposition 1 *Let* $\{\Omega, \mathcal{B}, \nu, \phi_t\}$ *and* $\{\Omega', \mathcal{B}', \nu', \phi'_t\}$ *be two finite measure preserving ergodic flows. Then we can find representations* $\{X_0, \mathcal{C}_0, \mu_0, R, \lambda\}$ *and* $\{X'_0, \mathcal{C}'_0, \mu'_0, R', \lambda'\}$ *respectively, such that*

$$D(X_0, \mu_0, R, \lambda) = D(X'_0, \mu'_0, R', \lambda'),$$
$$d(X_0, \mu_0, R, \lambda) = d(X'_0, \mu'_0, R', \lambda').$$

Proof Any flow which preserves a finite measure can be represented as $\{X_0, \mathcal{C}_0, \mu_0, R, \lambda\}$ such that R preserves μ_0 and $\int_{X_0} \lambda(x) d\mu_0 < \infty$ [6] (in fact, we can assume that the height function takes exactly three values [14]). By the pointwise ergodic theorem, for μ_0 almost-every $x \in X_0$:

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} \lambda(R^i x)}{n} = \int\limits_{X_0} \lambda(x) d\mu_0 < \infty.$$

It now follows that a sequence dominates the marked flow if and only if it dominates the sequence $\{n\}_{n=1,2,\ldots}$ and is dominated by the marked flow if and only if it is dominated by $\{n\}_{n=1,2,\ldots}$. Since both finite-measure preserving flows admit such representations, their classes of dominated and dominating sequences coincide. □

4 Applications and examples

We will now use this flow invariant to construct type III_0 systems which are not orbit equivalent; their associated flows will necessarily preserve an infinite measure. Let $\{a_n\}$ be a sequence of positive integers such that

$$\forall n : a_n > \sum_{i=1}^{n-1} a_i, \quad \liminf_{n\to\infty} \frac{a_{n+1}}{a_n} > \delta > 1, \tag{4}$$

and let $\{X, \mathcal{B}, \mu, T\}$ be the odometer system, where

$$X = \prod_{i=1}^{\infty} \mathbb{Z}_{e^{a_i}+1},$$

T is the typical "+1 with carry" action on this space, and $\mu = \prod_{i=1}^{\infty} \nu_i$,

$$\nu_i(k) = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{1}{2e^{a_i}}, & k = 1, 2, \ldots, e^{a_i}. \end{cases}$$

We always round e^{a_n} to the nearest integer, a trivial adjustment that we do not record. By construction, T preserves μ when mapping between digits not equal to zero, and a change to/from zero in the n^{th} coordinate increases/decreases the measure by a factor of $e^{\pm a_n}$. The associated flow in this case is particularly nice; the choice of measure μ ensures that the base transformation R of the associated flow is the measure-preserving two-point odometer; as the base transformation is measure-preserving, it therefore has the positive mean return time property. If we define $A_n \subset X_0$ to be the set of binary sequences whose first 1 appears in the n^{th} coordinate, the height function $\lambda(x)$ over A_n takes the value a_n, and the μ_0-measure of each A_n is exactly 2^{-n}. As our original non-singular system is explicitly a product system, the associated flow is approximately transitive and therefore of zero entropy [5]. Entropy of associated flows, therefore, is an insufficient tool to determine whether two such nonsingular systems are orbit equivalent or not given the results that follow.

Lemma 3 *There exists an integer $M = M(\delta) > 2$ such that for μ_0 almost every binary sequence $x = x_1x_2x_3\ldots$, for sufficiently large N, and all n with $2^N \leq n < 2^{N+1}$,*

$$a_N \leq \Lambda_n(x) \leq a_{MN}.$$

Proof The lower bound is the easiest to establish, and is in fact true for all x and all N (if we set $\lambda(000\ldots) = +\infty$). Note that the lower bound follows from the claim that

$$\bigcup_{i=0}^{2^N-1} R^{-i}\left(\bigcup_{m\geq N} A_m\right) = X_0;$$

because if the orbit of every x intersects A_m where $m \geq N$ within the first $2^N - 1$ applications of the dyadic adding machine R, then the ergodic sum of the height function through time $n \geq 2^N$ is at least as large as the single value achieved when $R^i x \in A_m$.

The claim is established by induction; it is clearly true for $N = 1$. To establish the claim for $N + 1$, assume inductively that for any x, there is some $i < 2^N$ so that

$$y = R^i x \in \bigcup_{m\geq N} A_m.$$

If $y \in A_m$ with $m > N$, we are done. Assume, then, that $y \in A_N$. We therefore write $y = 0^{N-1}1^k0z$ for some $z \in X_0$ and $k \geq 1$. Performing dyadic addition on a long string of zeroes, then, we see that

$$R^{2^N} y = 0^{N-1+k}1z \in A_{N+k}.$$

We now have a point in the first $2^N + i < 2^{N+1}$ points of the orbit of x which lies in some A_m with $m \geq N + 1$. This proves the claim.

The upper bound in the inequality is more intricate; it can be thought of as a "shrinking targets" problem for the odometer. We define the sets

$$B_N = \bigcup_{i=0}^{2^N-1} R^{-i}\left(\bigcup_{m\geq 2N} A_m\right).$$

Since R preserves μ_0,

$$\mu_0(B_N) \leq 2^N \mu_0\left(\bigcup_{m\geq 2N} A_m\right) = 2^{-N+1}$$

From that inequality we see that

$$\sum_{N=1}^{\infty} \mu_0(B_N) < \infty,$$

from which it follows that

$$\mu_0 \left(\limsup_{N\to\infty} B_N \right) = 0,$$

or for μ_0 almost-every $x \in X_0$, x belongs to only finitely many B_N.

We now choose an integer $M > 2$ so that $\delta^{M-2} > 2$ using $\delta > 1$ defined in Eq. 4; then we can find N large enough so that, $\delta^{(M-2)N} > 2^{N+1}$. Using (4) again, for μ_0 almost every x and for all n such that $2^N \le n < 2^{N+1}$ the following holds:

$$\begin{aligned}
\Lambda_n(x) &\le n \cdot \max\{\lambda(R^i x) : i = 0, 1, \ldots, n-1\} \\
&\le 2^{N+1} \cdot \max\{\lambda(R^i x) : i = 0, 1, \ldots, 2^{N+1} - 1\} \\
&\le 2^{N+1} a_{2N} \\
&\le \delta^{MN-2N} a_{2N} \\
&\le a_{2N+MN-2N} \\
&\le a_{MN}.
\end{aligned}$$

□

We now use these estimates to investigate the classes $D(X_0, \mu_0, R, \lambda)$ and $d(X_0, \mu_0, R, \lambda)$ for associated flows arising from odometers where the number of states e^{a_n} grows sufficiently quickly.

Proposition 2 *Assume that a sequence* $\{a_n\}$ *satisfies*

$$\forall n : \sum_{i=1}^{n-1} a_i < a_n, \quad \liminf_{n\to\infty} \frac{a_{n+1}}{a_n} > \delta > 1$$

and $\{b_n\} \ll \{a_n\} \ll \{c_n\}$ *for some increasing sequences* $\{b_n\}$ *and* $\{c_n\}$*. Define the associated odometer* $\{X, \mathcal{B}, \mu, T\}$ *as at the beginning of this section (using the sequence* $\{a_n\}$ *in the construction), and define* $\{C_n\}$ *to be a monotone sequence such that* $C_{2^n} = c_n$*. Similarly define* $\{B_n\}$*. Then*

$$\{C_n\} \in D(X_0, \mu_0, R, \lambda), \quad \{B_n\} \in d(X_0, \mu_0, R, \lambda).$$

Proof Let $k \in \mathbb{N}$ be fixed. We apply the inequality of Lemma 3 to the sequence $\{\Lambda_n\}$, so that for sufficiently large n, with $2^N \le n < 2^{N+1}$:

$$\begin{aligned}\frac{\Lambda_n(x)}{B_{2^k n}} &\geq \frac{\Lambda_{2^N}(x)}{B_{2^{N+k+1}}}\\ &= \frac{\Lambda_{2^N}(x)}{b_{N+k+1}}\\ &\geq \frac{a_N}{b_{N+k+1}}\\ &\geq \frac{a_N}{b_{2N}}\end{aligned}$$

which diverges as $\{a_n\} \gg \{b_n\}$. Similarly, there is some constant M so that for any $k \in \mathbb{N}$:

$$\frac{C_n}{\Lambda_{2^k \cdot n}(x)} \geq \frac{c_N}{a_{M(N+k+1)}},$$

which diverges by the assumption that $\{c_n\} \gg \{a_n\}$.

As k was arbitrary and M fixed, we now apply Lemma 2 (II) to the family of sequences $\{B_{2^k n}\}_{k\in\mathbb{N}}$; the sequences $\{2^k n\}$ are of positive but arbitrarily small upper density, so for μ_0-almost every x, the sequence $\{B_n\}$ is dominated by $\{\Lambda_n(x)\}$, or $\{B_n\} \in D(X_0, \mu_0, R, \lambda)$. □

Corollary 1 *For each $\gamma, \gamma' \in (1, \infty), \gamma < \gamma'$, the odometers created by setting $a_n(\gamma) = 2^{n^\gamma}$, (resp. using γ') are not orbit equivalent.*

Proof Given any $s, t > 1$ with $s < \gamma < t$, define $A_n(\gamma)$ to be a monotone sequence such that $A_{2^n}(\gamma) = a_n(\gamma)$ as in Proposition 2. Since each $a_n(\gamma)$ satisfies the growth conditions of Proposition 2 and it is easy to verify $\{a_n(s)\} \ll \{a_n(\gamma)\} \ll \{a_n(t)\}$ (compare along the sequences nk for all $k \in \mathbb{N}$ and apply Lemma 2), we have $\{A_n(s)\} \in d(X_0, \mu_0, R, \lambda)$ and $\{A_n(t)\} \in D(X_0, \mu_0, R, \lambda)$. So for $\gamma < \gamma'$, the two associated flows will have different collections of dominated/dominating sequences. The associated Krieger flows for the odometers are therefore not isomorphic, and therefore the odometers are not orbit equivalent. □

Suppose further that we have countable collections of sequences $\{a_n^{(i)}\}$ and $\{b_n^{(i)}\}$, such that for $i > j$,

$$\{a_n^{(j)}\} \ll \{a_n^{(i)}\} \ll \{b_n^{(i)}\} \ll \{b_n^{(j)}\},$$

where each sequence obeys the conditions

$$\forall N : \sum_{n=1}^{N-1} a_n^{(j)} < a_N^{(j)}, \quad \liminf_{n\to\infty} \frac{a_{n+1}^{(j)}}{a_n^{(j)}} > 1$$

(similarly for the $\{b_n^{(j)}\}$). We also require for all i, j, n that $b_n^{(i)} > a_n^{(j)}$. For each i, construct the monotone sequences $\{A_n^{(i)}\}$ and $\{B_N^{(i)}\}$ by requiring $A_{2^n}^{(i)} = a_n^{(i)}$ (similarly for the $B_n^{(i)}$).

Corollary 2 *We can construct a type III$_0$ system $\{X, \mathcal{B}, \mu, T\}$ such that for all i, $\{A_n^{(i)}\} \in d(X_0, \mu_0, R, \lambda)$ and $\{B_n^{(i)}\} \in D(X_0, \mu_0, R, \lambda)$. We can also construct a system so that all $\{A_n^{(i)}\} \in D(X_0\mu_0, R, \lambda)$ or all $\{B_n^{(i)}\} \in d(X_0, \mu_0, R, \lambda)$.*

Proof We simply construct a sequence $\{c_n\}$ which dominates all the $\{a_n^{(i)}\}$ and is dominated by all the $\{b_n^{(i)}\}$ using diagonalization techniques (for example, $c_n = a_n^{(n)}$ suffices due to the requirement that $b_n^{(i)} > a_n^{(j)}$ for all i, j, n), or smaller than all $\{a_n^{(i)}\}$, or bigger than all $\{b_n^{(i)}\}$, respectively, and then create the associated type III$_0$ odometer as before. □

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