Journal of Difference Equations and Applications
Vol. 00, No. 00, April 2009, 1–17

RESEARCH ARTICLE

Elliptic functions with critical orbits approaching infinity

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(*Received 00 Month 200x; in final form 00 Month 200x*)

We construct examples of elliptic functions, viewed as iterated meromorphic functions from the complex plane to the sphere, with the property that there exist one or more critical points which approach the essential singularity at ∞ under iteration but are not prepoles. We obtain many nonequivalent elliptic functions satisfying this property, including examples with Julia set the whole sphere as well as examples with nonempty Fatou set. These are the first examples of this type known to exist and provide examples to illustrate unusual chaotic measure theoretic behavior studied by the third author and others.

Keywords: complex dynamics, meromorphic functions, Julia sets

AMS Subject Classification: Primary 54H20, 37F10; Secondary 37F20

In honor of Bob Devaney's 60^{th} *birthday.*

1. Introduction

The purpose of this paper is to construct examples of elliptic maps with prescribed properties of the iterated critical orbits. It is well-known that the spectrum of types of critical orbits within a holomorphic family of meromorphic maps sheds light on the complexity of the bifurcation locus and reveals measure theoretic properties of the maps (see for example [3], [6], [7], [13], [14], and [15] and the references mentioned in these). In our setting of elliptic functions we are most interested in maps with critical points that tend to infinity under iteration, but are not prepoles. Dynamical and measure theoretic results of maps with this and other related properties are proved in [12],[13],[14], and [15], but no examples have been given up to now. The unifying property of all the maps discussed here and in [15] is that each critical point in the Julia set does not return near itself infinitely often in either the classically studied sense say of [5], [11], or [6] or in more complicated

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ISSN: 1023-6198 print/ISSN 1563-5120 online

DOI: 10.1080/1023619YYxxxxxxxx
http://www.informaworld.com

ways described in [15], unique to meromorphic maps and which will be described below.

The paper is organized as follows. In Section 2 we give the basic setting and definitions of the subject of iteration of elliptic functions. In Section 3 we discuss our specific setting; namely the elliptic functions with nonrecurrence properties of the critical orbits as defined by Kotus and Urbański in the references above. We call these maps critically tame elliptic functions and they are defined and described in Section 3 of this paper. We give the construction in Section 4 and show how it results in many nonequivalent examples, including examples with both empty and nonempty Fatou set.

2. The iteration of elliptic functions

We begin with some preliminaries about elliptic functions, the Weierstraass $\wp$ function and period lattices. Let $\lambda_1, \lambda_2 \in \mathbb{C}\backslash\{0\}$ such that $\lambda_2/\lambda_1 \notin \mathbb{R}$. A lattice $\Lambda \subset \mathbb{C}$ is defined by $\Lambda = [\lambda_1, \lambda_2]\colon = \{m\lambda_1 + n\lambda_2\colon m, n \in \mathbb{Z}\}$. Two different sets of vectors can generate the same lattice Λ; if $\Lambda = [\lambda_1, \lambda_2]$, then all other generators of Λ are obtained by multiplying the vector (λ_1, λ_2) by a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = \pm 1$. A lattice Λ forms a group that acts on $\mathbb{C}$ by translation, each $\omega \in \Lambda$ inducing the transformation of $\mathbb{C}$: $T_\omega : z \mapsto z + \omega$.

Definition 2.1. A closed, connected subset $\mathcal{Q}$ of $\mathbb{C}$ is defined to be a *fundamental region* for Λ if

(1) for each $z \in \mathbb{C}$, $\mathcal{Q}$ contains at least one point in the same Λ-orbit as z;
(2) no two points in the interior of $\mathcal{Q}$ are in the same Λ-orbit.

If $\mathcal{Q}$ is any fundamental region for Λ, then for any $s \in \mathbb{C}$, the set

$$\mathcal{Q} + s = \{z + s : z \in \mathcal{Q}\}$$

is also a fundamental region. If $\mathcal{Q}$ is a parallelogram we call $\mathcal{Q}$ a *period parallelogram* for Λ.

The ratio $\tau = \lambda_2/\lambda_1$ is an important feature of a lattice. If $\Lambda = [\lambda_1, \lambda_2]$, and $k \neq 0$ is any complex number, then $k\Lambda$ is the lattice defined by taking $k\lambda$ for each $\lambda \in \Lambda$; $k\Lambda$ is said to be *similar* to Λ. Similarity is an equivalence relation between lattices, and an equivalence class of lattices is called a *shape*. There are some distinguished shapes and types of lattices described here.

Definition 2.2.

(1) A lattice Λ is *real* if $\overline{\Lambda} = \Lambda$.
(2) $\Lambda = [\lambda_1, \lambda_2]$ is *real rectangular* if there exist generators such that $\lambda_1 \in \mathbb{R}$ and λ_2 is purely imaginary. Any lattice similar to a real rectangular lattice is *rectangular*.
(3) $\Lambda = [\lambda_1, \lambda_2]$ is *real rhombic* if there exist generators such that $\lambda_2 = \overline{\lambda_1}$. Any similar lattice is *rhombic*.
(4) A lattice Λ is *square* if $i\Lambda = \Lambda$. (Equivalently, Λ is square if it is similar to a lattice generated by $[\lambda, \lambda i]$, for some $\lambda > 0$.)
(5) A lattice is *triangular* if $\Lambda = e^{2\pi i/3}\Lambda$.

In each of cases (2) – (4) the period parallelogram with vertices $0, \lambda_1, \lambda_2$, and $\lambda_3 := \lambda_1 + \lambda_2$ can be chosen to look rectangular, rhombic, or square respectively. In case (5) a period parallelogram is comprised of two equilateral triangles.

We begin with $f\colon \mathbb{C} \to \mathbb{C}_\infty$ a meromorphic function where $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere.

An *elliptic function* is a meromorphic function in $\mathbb{C}$ which is periodic with respect to a lattice Λ. For any $z \in \mathbb{C}$ and any lattice Λ, the Weierstrass elliptic function is defined by

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right).$$

Replacing every z by $-z$ in the definition we see that $\wp_\Lambda$ is an even function. It is well-known that $\wp_\Lambda$ is meromorphic, is periodic with respect to Λ, and has order 2.

The derivative of the Weierstrass elliptic function is also an elliptic function which is periodic with respect to Λ defined by

$$\wp'_\Lambda(z) = -2 \sum_{w \in \Lambda} \frac{1}{(z-w)^3}.$$

The Weierstrass elliptic function and its derivative are related by the differential equation

$$\wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 - g_2\wp_\Lambda(z) - g_3, \tag{1}$$

where $g_2(\Lambda) = 60 \sum_{w \in \Lambda \setminus \{0\}} w^{-4}$ and $g_3(\Lambda) = 140 \sum_{w \in \Lambda \setminus \{0\}} w^{-6}$.

The numbers $g_2(\Lambda)$ and $g_3(\Lambda)$ are invariants of the lattice Λ in the following sense: if $g_2(\Lambda) = g_2(\Lambda')$ and $g_3(\Lambda) = g_3(\Lambda')$, then $\Lambda = \Lambda'$. Furthermore given any g_2 and g_3 such that $g_2^3 - 27g_3^2 \neq 0$ there exists a lattice Λ having $g_2 = g_2(\Lambda)$ and $g_3 = g_3(\Lambda)$ as its invariants [4].

For any lattice Λ, the Weierstrass elliptic function and its derivative satisfy the following properties: for $k \in \mathbb{C}\setminus\{0\}$,

$$\wp_{k\Lambda}(ku) = \frac{1}{k^2}\wp_\Lambda(u), \quad \text{(homogeneity of } \wp_\Lambda), \tag{2}$$

$$\wp'_{k\Lambda}(ku) = \frac{1}{k^3}\wp'_\Lambda(u), \quad \text{(homogeneity of } \wp'_\Lambda),$$

Verification of the homogeneity properties can be seen by substitution into the series definitions.

The following classical result characterizes all elliptic functions in terms of $\wp$ and $\wp'$ [4].

Theorem 2.3. *Every elliptic function f_Λ with period lattice Λ can be written as $f_\Lambda(z) = R(\wp_\Lambda(z)) + {\wp_\Lambda}'(z)Q(\wp_\Lambda(z))$, where R and Q are rational functions with complex coefficients. The converse is also true, namely every f of this form is elliptic.*

In Section 4 we construct elliptic functions of the form $g_\alpha = \alpha\wp$ with critical orbits having prescribed nonrecurrent properties. The following corollary is of use in the construction.

COROLLARY 2.4. *If Λ is a fixed lattice, and $\wp_\Lambda$ denotes the Weierstrass elliptic $\wp$ function with period lattice Λ, then for any $\alpha \in \mathbb{C}\setminus\{0\}$, the map $\alpha\wp_\Lambda$ is an elliptic function related to $\wp$ with a different period lattice in the following sense: for every $z \in \mathbb{C}$*

$$\alpha\wp_\Lambda(z) = \wp_{a\Lambda}(az), \text{ with } a = \frac{1}{\sqrt{\alpha}}. \tag{3}$$

Then $(\alpha\wp_\Lambda)^n = (\wp_{a\Lambda} \circ a)^n$ for each $n \in \mathbb{N}$, where $a(z) \equiv az$.

Proof. Equation (3) follows immediately from the homogeneity identity (2). □

We can determine the critical values of the Weierstrass elliptic function on an arbitrary lattice $\Lambda = [\lambda_1, \lambda_2]$. Define $\lambda_3 = \lambda_1 + \lambda_2$. For $j = 1, 2, 3$, notice that $\wp_\Lambda(\lambda_j - z) = \wp_\Lambda(z)$ for all z. Taking derivatives of both sides we obtain $-\wp'_\Lambda(\lambda_j - z) = \wp'_\Lambda(z)$. Substituting $z = \lambda_j/2$, we see that $\wp'_\Lambda(z) = 0$ at these values. We use the notation

$$c_1 = \frac{\lambda_1}{2},\ c_2 = \frac{\lambda_2}{2},\ c_3 = \frac{\lambda_3}{2},\ e_1 = \wp_\Lambda(c_1),\ e_2 = \wp_\Lambda(c_2),\ e_3 = \wp_\Lambda(c_3) \tag{4}$$

to denote the critical points and values. Since e_1, e_2, e_3 are the distinct zeros of Equation (1), we also write

$$\wp'_\Lambda(z)^2 = 4(\wp_\Lambda(z) - e_1)(\wp_\Lambda(z) - e_2)(\wp_\Lambda(z) - e_3). \tag{5}$$

Equating like terms in Equations (1) and (5), we obtain

$$e_1 + e_2 + e_3 = 0, \quad e_1e_3 + e_2e_3 + e_1e_2 = \frac{-g_2}{4}, \quad e_1e_2e_3 = \frac{g_3}{4}. \tag{6}$$

It is well-known that the singular orbits of meromorphic functions determine most of the dynamical properties. Let $\mathrm{Crit}(f)$ denote the set of critical points of f, i.e.,

$$\mathrm{Crit}(f) = \{z \colon f'(z) = 0\}.$$

If z_0 is a critical point then $f(z_0)$ is a *critical value*. The *singular set*, denoted $Sing(f)$, of an elliptic function f is the set of critical and finite asymptotic values of f and their limit points. A meromorphic function is called *Class S* if f has only finitely many critical and asymptotic values. For each lattice Λ, every elliptic function with period lattice Λ is of Class S since there are finitely many critical values and no asymptotic values.

If f is elliptic, the *postcritical set* of f is:

$$P(f) = \overline{\bigcup_{n\geq 1} f^n(\mathrm{Crit}(f))} \subset \mathbb{C}_\infty.$$

For each lattice, $\wp_\Lambda$ has three distinct critical values and no asymptotic values. Therefore the postcritical set of $\wp_\Lambda$ is:

$$P(\wp_\Lambda) = \overline{\bigcup_{n\geq 0} \wp_\Lambda^n(e_1 \cup e_2 \cup e_3)}.$$

The lattice shape relates to the properties and dynamics of the corresponding Weierstrass elliptic function to some extent, as discussed in [6], [7], and [8]; however these papers also show that within a given shape equivalence class the dynamics vary widely.

2.1 Julia and Fatou sets of elliptic functions

We review the basic dynamical definitions and properties for meromorphic functions which appear for example in [1], [2], and [3]. Let $f\colon \mathbb{C} \to \mathbb{C}_\infty$ be a meromorphic function where $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. The *Fatou set* $F(f)$ is the set of points $z \in \mathbb{C}_\infty$ such that $\{f^n \colon n \in \mathbb{N}\}$ is defined and normal in some neighborhood of z. The *Julia set* is the complement of the Fatou set on the sphere, $J(f) = \mathbb{C}_\infty \backslash F(f)$. Notice that $\mathbb{C}_\infty \backslash \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}$ is the largest open set where all iterates are defined. Since $f(\mathbb{C}_\infty \backslash \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}) \subset \mathbb{C}_\infty \backslash \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}$, Montel's theorem implies that

$$J(f) = \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}.$$

For a meromorphic function f, a point z_0 is *periodic* of period p if there exists a $p \geq 1$ such that $f^p(z_0) = z_0$. We also call the set $\{z_0, f(z_0), \ldots, f^{p-1}(z_0)\}$ a *p-cycle.* The *multiplier* of a point z_0 of period p is the derivative $(f^p)'(z_0)$. A periodic point z_0 is called *attracting, repelling,* or *neutral* if $|(f^p)'(z_0)|$ is less than, greater than, or equal to 1 respectively. If $|(f^p)'(z_0)| = 0$ then z_0 is called a *superattracting* periodic point.

Suppose U is a connected component of the Fatou set. We say that U is *preperiodic* if there exists $n > m \geq 0$ such that $f^n(U) = f^m(U)$, and the minimum of $n - m = p$ for all such n, m is the *period* of the cycle. Let $C = \{U_0, U_1, \ldots U_{p-1}\}$ be a periodic cycle of components of $F(f)$. If C is a cycle of immediate attractive basins or parabolic domains, then $U_j \cap Sing(f) \neq \emptyset$ for some $0 \leq j \leq p-1$. If C is a cycle of Siegel Disks or Herman rings, then $\partial U_j \subset \overline{\bigcup_{n\geq 0} f^n(Sing(f))}$ for all $0 \leq j \leq p - 1$.

If f is elliptic and hence of Class S, it has no wandering domains [2] or Baker domains [19]; therefore every component of $F(f)$ is preperiodic. The periodicity of elliptic functions restricts the size of Siegel disks or Herman rings. In [8], the following lemma was proved for Siegel disks, and the same proof holds for Herman rings.

Proposition 2.5. *Assume f_Λ has a cycle of Siegel disks or Herman rings $C = \{U_0, \ldots, U_{p-1}\}$. Then each U_j is completely contained in one fundamental region of Λ.*

2.1.1 Properties of special lattices Λ and $J(\wp_\Lambda)$

In Section 4 we construct examples of elliptic functions g such that there exists some $c \in \mathrm{Crit}(g)$ such that $g^n(c)$ is defined for all n and $\lim_{n\to\infty} g^n(c) = \infty$. Each construction begins with an example where some or all critical points are prepoles; we first show that this occurs for any lattice shape, and more can be said for certain other shapes. Recall that every lattice is similar to one of the form $\Lambda = [1, \lambda]$ with $\Im(\lambda) > 0$.

LEMMA 2.6. *Let $\Lambda = [1, \lambda]$ be a lattice such that $\wp_\Lambda(1/2) = e_1 \neq 0$. Suppose m is any integer and $k = \sqrt[3]{e_1/m}$. Then using the lattice $\Gamma = k\Lambda = [k, \gamma]$, $\wp_\Gamma$ has critical value $e_{1,\Gamma} = mk$, which is a pole of $\wp_\Gamma$.*

Proof. We have $e_{1,\Gamma} = e_{1,k\Lambda} = k^{-2}e_1 = mk$ using Equation (2). □

The only shape not satisfying the hypotheses of Lemma 2.6 is a square lattice in the rhombic position where $e_1 = 0$, but 0 is already a pole so we obtain the following result.

LEMMA 2.7. *If Λ is a square lattice, then the corresponding Weierstrass elliptic $\wp$ function $\wp_\Lambda$ has at least one critical point which is a prepole.*

Proof. It is well-known (cf. [4]) that square lattices are characterized by the property that $g_3 = 0$, hence by Equation (6) a critical value of $\wp_\Lambda$ is the pole at 0. □

The following family of triangular lattices $\Gamma = [\gamma_1, \gamma_2]$ where all critical points of $\wp_\Gamma$ are prepoles was given in [6].

THEOREM 2.8. *Let $\Omega = [\omega_1, \omega_2]$ be the triangular lattice associated with the invariants $g_2 = 0$ and $g_3 = 4$ so that e_1, e_2, and e_3 are the cube roots of unity. Let $k = \sqrt[3]{e^{4\pi i/3}/(m\omega_1)}$ where m is odd and negative and $\Gamma = k\Omega = [k\omega_1, k\omega_2]$.*

Then all critical points of $\wp_\Gamma$ are prepoles and $J(\wp_\Gamma) = \mathbb{C}_\infty$.

In [8] (Theorem 5.1), the following example was constructed of an elliptic $\wp$ function on a real rectangular lattice that has a superattracting fixed point and a critical point that is a prepole.

PROPOSITION 2.9. *Let $\Omega = [\omega_1, \omega_2]$, $\omega_1 > 0$ be the real rectangular lattice with invariants $g_2 = 52/9$ and $g_3 = -16/9$. Let $k = \sqrt[3]{2/(3\omega_1)}$ (taking the real root) and $\Gamma = k\Omega = [k\omega_1, k\omega_2]$. Then $\wp_\Gamma$ has the following postcritical behavior: $e_{1,\Gamma} = 3\gamma_1/2$ is a superattracting fixed point, $e_{3,\Gamma} = \gamma_1/2$ and thus $\wp_\Gamma(e_{3,\Gamma}) = e_{1,\Gamma}$, and $e_{2,\Gamma} = -2\gamma_1$ is a pole.*

3. The Kotus and Urbański conditions for critically tame elliptic functions

In this section we give a description of the critical recurrence properties of elliptic functions of interest. These properties are developed in detail in [15] and in the earlier published references [12] - [14]. We set up some notation first.

Throughout we will denote by $B_r(z_0) \subset \mathbb{C}$ the open ball of radius $r > 0$ centered at $z_0 \in \mathbb{C}$; we also define $B_R(\infty) \equiv \{z \in \mathbb{C}_\infty : |z| > R\} \subset \mathbb{C}_\infty$ for $R > 0$ (usually large).

Definition 3.1. Suppose f is an elliptic function and $f^n(z)$ is defined for all n for $z \in \mathbb{C}$. Let d denote the spherical metric on $\mathbb{C}_\infty$. A point $y \in \mathbb{C}_\infty$ is an *ω-limit point* of z for f if there exists a sequence of integers $n_k \to \infty$, such that

$$\lim_{k\to\infty} d(f^{n_k}z, y) = 0.$$

The set of all ω-limit points of z for f is called the *ω-limit set of z* and is denoted $\omega(z)$.

The ω-limit set of z is a compact subset of $\mathbb{C}$ if and only if $\infty \notin \omega(z)$. Let D denote a domain in $\mathbb{C}$.

Definition 3.2.

(1) If $g : D \to \mathbb{C}$ is an analytic map, $z \in \mathbb{C}$, and $r > 0$, then we define $U(z, g^{-1}, r)$ to be the *connected component of* $g^{-1}(B_r(g(z)))$ *that contains* z.

(2) Suppose that c is a critical point of g. Then there exists $r = r(g, c) > 0$ and $M = M(g, c) \geq 1$ such that

$$M^{-1}|z - c|^{p_c} \leq |g(z) - g(c)| \leq M|z - c|^{p_c}$$

and

$$M^{-1}|z - c|^{p_c - 1} \leq |g'(z)| \leq M|z - c|^{p_c - 1}$$

for every $z \in U(c, g^{-1}, r)$, and also such that

$$g(U(c, g^{-1}, r)) = B_r(g(c));$$

we define $p_c = p(g, c)$ to be the *order of* g *at the critical point* c. (Note that some authors call $p_c - 1$ the order of the critical point.) In particular, $p_c - 1$ is the multiplicity of the zero of g' at c.

From this point on, we assume that $f : \mathbb{C} \to \mathbb{C}_\infty$ is an elliptic function. We denote the period lattice by Λ and let $\mathcal{Q}$ denote a fundamental region as in Definition 2.1.

Definition 3.3. For each elliptic function f and for any pole b of f, let q_b denote its multiplicity. We define

$$q := \sup\{q_b : b \in f^{-1}(\infty)\} = \max\{q_b : b \in f^{-1}(\infty) \cap \mathcal{Q}\}. \tag{7}$$

We define the *prepoles of order* $n \geq 1$ of f by:

$$A_n(f) = \{z \in \mathbb{C} : f^n(z) = \infty\}.$$

It is implicit in this definition that if $z \in A_n(f)$, then f^j is defined for all $j \leq n$. We define

$$C_p(f) = \mathrm{Crit}(f) \cap \cup_{n \in \mathbb{N}} A_n; \tag{8}$$

$C_p(f)$ is the set of *prepole critical points.*

We note that $A_n(f) = f^{-1}(A_{n-1}(f))$ for $n \geq 2$, and that $A_n(f) \subset J(f)$. For each $c \in C_p(f)$, $c \in A_k(f)$ for exactly one k. A pole is an order 1 prepole, so if Λ is the period lattice of $\wp_\Lambda$, and $\alpha \in \mathbb{C} \setminus \{0\}$, then $A_1(\alpha\wp_\Lambda) = \Lambda$.

Definition 3.4.

$$\text{Let} \quad C_\infty(f) = \{c \in \mathrm{Crit}(f) \mid \lim_{n \to \infty} f^n(c) = \infty\}. \tag{9}$$

For every $c \in C_\infty(f)$ let

$$q_c = \limsup_{n \to \infty} q_{b_n}, \tag{10}$$

where $f^n(c)$ is near a pole b_n. More precisely, we see that $c \in C_\infty(f)$ if and only if:

$$\text{for all } R > 0 \text{ there exists } K \text{ such that } n \geq K \Rightarrow |f^{n+1}(c)| > R.$$

Since R is large, it then follows that

$$|f^{n+1}(c)| > R \Leftrightarrow f^n(c) \text{ is near a unique pole } b_n.$$

This defines the sequence of poles $\{b_n\}$ in Equation (10). Let

$$l_\infty = \max\{p_c q_c : c \in C_\infty(f)\} \leq \max\{p_c q : c \in C_\infty(f)\},$$

where q_c, q are as in (7) and p_c as in Definition 3.2.

By $HD(X)$ we denote the Hausdorff dimension of a set $X \subset \mathbb{C}_\infty$.

Definition 3.5. Let $f : \mathbb{C} \to \mathbb{C}_\infty$ denote an elliptic function, and let c be a critical point of f. We say f satisfies the *Kotus-Urbański conditions* or equivalently define f to be *critically tame* if the following hold:

(1) If $c \in F(f)$, then there exists either an attracting or parabolic periodic orbit of period p, $S = \{z_0, f(z_0), \ldots, f^{p-1}(z_0)\}$ such that $\omega(c) = S$.
(2) If $c \in J(f)$, then one of the following occurs:
  a) $\omega(c)$ is a compact subset of $\mathbb{C}$ such that $c \notin \omega(c)$;
  b) $c \in C_p(f)$;
  c) $c \in C_\infty(f)$, and

$$HD(J(f)) > \frac{2l_\infty}{l_\infty + 1}. \tag{11}$$

Due to the evolution of the results in [12], [13], [14], and [15] this is called *regular pseudo weakly non-recurrent* in [15].

We state one of the main theorems of [15] as it provides motivation for the construction of the examples that follow in this paper. However, since we do not use the result in the paper, we omit the definitions of the terminology and refer the reader to [15].

Theorem 3.6. *If f is a critically tame elliptic function and $h = HD(J(f))$, then there exists a nonatomic, σ-finite, ergodic, conservative and invariant measure m_h for f, equivalent to h-conformal measure, μ_h. Moreover m_h is unique up to a multiplicative constant and is supported on $J(f)$.*

The condition of regularity, inequality (11), was used in [15] to show that the h-conformal measure in Theorem 3.6 is nonatomic.

In order to establish critical tameness of the examples we construct below, the following lemma is useful and follows from [12], [13], and [14]; we give a simple proof here.

Lemma 3.7. *If every critical point c of f satisfies $c \in C_p(f)$ or $c \in C_\infty(f)$, then $J(f) = \mathbb{C}_\infty$ and f is critically tame.*

Proof. The hypotheses imply that $\mathrm{Crit}(f) \subset J(f)$, so f has no cycle of attracting or parabolic components. Suppose that $J(f) \neq \mathbb{C}_\infty$. Then there exists a periodic cycle of Fatou components $C = \{U_0, \ldots, U_{p-1}\}$. If C is a cycle of Siegel disks or Herman rings, then by Proposition 2.5, each U_j is contained within one fundamental region. Choose R large enough so that C is contained in $B_R(0)$. For any $c \in C_p(f)$, only a finite number of iterates of c are defined. Let $c_1, \ldots, c_k$ denote one representative from each equivalence class of critical points in $C_\infty(f)$. For each $c_j \in \{c_1, \ldots c_k\}$, pick n_j such that for all $n > n_j$, $|f^n(c_j)| > R$. Define

$$T = \bigcup_{j=1}^{k} \bigcup_{i=1}^{n_j} f^i(c_j).$$

T contains all forward iterates of $c \in C_\infty(f)$ that are contained in $B_R(0)$, and T is a finite set. Thus $\overline{\bigcup_{n\geq 0} f^n(P(f))} \cap B_R(0)$ is finite, which contradicts that $\partial U_j \subset \overline{\bigcup_{n\geq 0} f^n(P(f))}$ for all $0 \leq j \leq p-1$.

Since $HD(J(f)) = 2$, Definition 3.5 (2) is satisfied and f is critically tame. □

4. Examples of critically tame elliptic functions with critical point tending to infinity

We give a family of examples of elliptic functions such that a critical point escapes to infinity. We show these are critically tame elliptic functions in the sense of [15] and therefore provide the first examples known to satisfy Definition 3.5 (2) c).

4.1 The starting examples and preliminaries

We start with any lattice Λ such that $C_p(\wp_\Lambda) \neq \emptyset$; that is, at least one critical point of $\wp_\Lambda$ is a prepole. We can choose an example from Section 2.1.1, say from Theorem 2.8. While any of these examples will result, via the construction described later in this section, in an elliptic map with a critical point having an infinite orbit tending to the essential singularity at ∞, we need to determine whether the inequality (11) of Definition 3.5 is satisfied to obtain a critically tame elliptic function.

4.1.1 A critically tame starting example with a square period lattice

We set $(g_2, g_3) = (-4, 0)$; the resulting Weierstrass $\wp$ function has a real rhombic square period lattice Λ and critical values $\{e_1, e_2, e_3\} = \{-i, i, 0\}$. Writing the associated lattice generators by $[a + ai, a - ai]$, we have $a \approx 1.854$ (we do not need this approximation), and we know that $a > 0$ (cf. [6]). We refer to Λ as the *standard rhombic square lattice*. Set $k = \sqrt[3]{1/(ja)} > 0$, where j is any positive even integer, and define a new period lattice by $\Gamma = k\Lambda$.

Proposition 4.1. *The following hold for $\wp_\Gamma$:*

(1) All critical points are prepoles, $J(\wp_\Gamma) = \mathbb{C}_\infty$ and $\wp_\Gamma$ is critically tame;
(2) $\wp_\Gamma$ is ergodic with respect to Lebesgue measure on $\mathbb{C}_\infty$.

Proof. If all critical points are prepoles then $F(\wp_\Gamma) = \emptyset$ and $\wp_\Gamma$ is critically tame by Lemma 3.7 (and first proved in [12]). We show that $\wp_\Gamma$ maps the two critical points of the form $c_1 = k(a+ai)/2$ and $c_2 = k(a-ai)/2$ to poles. The third critical point, $c_3 = ka$, is still mapped to 0 by Equation (2). Note that for any square lattice, p is a pole of $\wp$ if and only if $\pm ip$ is a pole. Applying Equation (2),

$$\wp_\Gamma\left(\frac{k(a+ai)}{2}\right) = \frac{1}{k^2}\wp_\Lambda\left(\frac{a+ai}{2}\right) = \frac{-i}{k^2} = -ijka. \tag{12}$$

The point $p = -ijka$ is a pole since for every even $j \in \mathbb{N}$, $ja \in \Lambda, \pm ija \in \Lambda$, hence

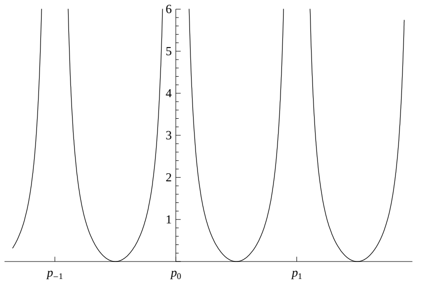

Figure 1. Graph of $\wp_\Lambda$ on a real lattice Λ with poles p_{-1}, p_0, p_1

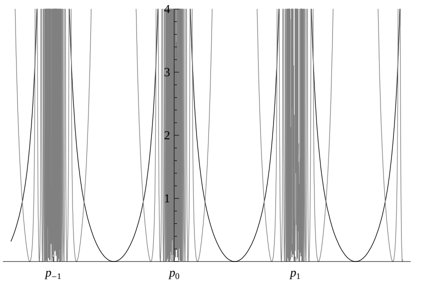

Figure 2. The poles of $\wp^2_\Lambda$ accumulate on the poles of $\wp_\Lambda$

$jka \in \Gamma, ijka \in \Gamma$. This proves (1). Statement (2) follows from (1) by applying Corollary 1.4 of [15] (cf. also [12]).

□

The main result of this section is the following.

THEOREM 4.2. *There exists a critically tame elliptic function g such that $C_\infty(g) \neq \emptyset$.*

To prove the theorem we give the construction of an example g with $C_\infty(g) \neq \emptyset$. We fix a lattice Γ satisfying the hypotheses of Proposition 4.1 using $j := j_0 = 4$; for convenience we write $\Gamma \equiv [2b + 2bi, 2b - 2bi] \equiv [\gamma_1, \gamma_2]$, with $b > 0$. We denote the Weierstrass $\wp$ function with period lattice Γ by $\wp \equiv \wp_\Gamma$ unless confusion arises. Since the three critical points and critical values are related by: $c_3 = 2b$ and $c_1 = \overline{c_2} = b + bi$, and $e_3 = 0, e_1 = -e_2$, we will fix once and for all the critical point c_1 to work with, and usually just refer to it as c. Our starting point is that all three critical values are poles, and in particular $\wp(c) = -i\,8b = e_1 \in \Gamma$.

As before $B_\varepsilon(z_0)$ is the open ball of radius $\varepsilon > 0$ centered at $z_0 \in \mathbb{C}$ and $B_R(\infty) = \{z \in \mathbb{C}_\infty : |z| > R\}$ for $R > 0$. Choose and fix any $\varepsilon < 1/4$ to start (we may make ε smaller later if necessary).

We define a parametrized family of elliptic functions as follows. For $\alpha \in B_\varepsilon(1)$, let

$$g_\alpha(z) = \alpha\wp(z). \tag{13}$$

The critical points of g_α are the same as the critical points of $\wp$, but the nonzero critical values of g_α, which we will write as $e_j(\alpha) = g_\alpha(c_j) = \alpha e_j$ clearly depend on α. However, we only have one critical orbit to control by changing α since $\alpha e_1 = -\alpha e_2$, and for any $z \in \mathbb{C} \setminus \Gamma$, $g_\alpha(-z) = \alpha\wp(-z) = \alpha\wp(z) = g_\alpha(z)$. Therefore g_α is also an even function so the critical orbit of c determines all the dynamics of g_α in the sense that $g^2_\alpha(c_1) = g^2_\alpha(c_2)$, while $g^2_\alpha(c_3) = \infty$.

The period lattice for $\wp$ and g_α can be enumerated as $\Gamma = \{\gamma_{(m,n)} : (m,n) \in \mathbb{Z} \times \mathbb{Z}\}$, where $\gamma_{(m,n)} = m\,\gamma_1 + n\,\gamma_2$. We note that each $\gamma_{(m,n)}$ is a pole for $\wp$ and g_α for every $\alpha \in B_\varepsilon(1)$. We give a definition similar to Definition 3.3.

Definition 4.3. For each $\alpha \in B_\varepsilon(1)$, we define the *prepoles of order* $n \geq 1$ of the map g_α by:

$$A_n(\alpha) = \{z \in \mathbb{C} : g^n_\alpha(z) = \infty\}.$$

A pole is an order 1 prepole, so for each $\alpha \in B_\varepsilon(1)$, $A_1(\alpha) = \Gamma$.

We note that $A_n(\alpha) = g^{-1}_\alpha(A_{n-1}(\alpha))$ for $n \geq 2$, and that $A_n(\alpha) \subset J(g_\alpha)$ for each n. Consider the (closed) period parallelogram $\mathcal{Q}$ with vertices $\{0, \gamma_1, \gamma_2, \gamma_1 + \gamma_2\}$;

$\mathcal{Q}$ is mapped by $\wp$ and g_α two-to-one onto $\mathbb{C}_\infty$ (with exceptional points on the boundary and at critical points that are still mapped in a finite-to-one manner). Therefore there are infinitely many preimages of $A_1(\alpha) = \Gamma$ in $\mathcal{Q}$, accumulating around the four poles, the vertices of $\mathcal{Q}$; but $A_2(\alpha) = g_\alpha^{-1}(A_1(\alpha))$. More generally for each $n \geq 2$ the limit points of each set $A_n(\alpha)$ are equal to the set:

$$\mathcal{L}_n(\alpha) \equiv \bigcup_{j<n} A_j(\alpha) = \bigcup_{j=1}^{n-1} g_\alpha^{-j}(\{\infty\}).$$

A simple illustration of this idea is shown in Figures 1 and 2.

This is the idea behind the construction we give here: the fact that poles of order $k \geq 2$ accumulate around poles of lower order. A similar property was shown in [10] for the meromorphic family of maps $\{\lambda \tan z\}$. We recall that we have chosen our starting example so that $c \in A_2(1)$, and that the prepoles of g_α depend holomorphically on α by Corollary 2.4 and [4].

4.2 Construction of a map g_α such that $c \in A_3(\alpha)$

To construct an example to prove Theorem 4.2, we first define a sequence of auxiliary functions G_k, $k \geq 2$, each defined on a subset of the parameter space $B_\varepsilon(1)$. We label our functions starting with $k = 2$ in order to simplify the notation, since the first function is defined using second iterates. We have:

$$G_2 : B_\varepsilon(1) \to \mathbb{C}_\infty, \quad \text{defined by} \quad G_2(\alpha) = g_\alpha^2(c). \tag{14}$$

We denote the set of poles of G_2 by $\mathcal{P}_2$. We note that G_2 has a pole at $\alpha = 1$ since $G_2(1) = g_1^2(c) = \wp^2(c) = \infty$ since $c \in A_2(1)$. Using Equation (3) from Corollary 2.4, we see that G_2 has a pole at α if and only if

$$\alpha = \frac{s}{2} \in B_\varepsilon(1), \tag{15}$$

with $s \neq 0$ an integer, or $\pm i$ times an integer, or of the form $t \pm i\,t$ with t an integer. This follows from the equations in (12).

However we have chosen ε small enough so that the point 1 is the only pole of G_2 in the domain. We then have that G_2 is a nonconstant meromorphic map and hence is open. Therefore there exists an $R_2 \geq 4$ such that $B_{R_2}(\infty) \subset G_2(B_\varepsilon(1))$; the next lemma now follows immediately from this containment.

LEMMA 4.4. *For any lattice point $\gamma_{(m,n)} \in B_{R_2}(\infty)$, there exists a parameter $\alpha_{(m,n)} \in B_\varepsilon(1) \setminus \mathcal{P}_2$ such that:*

$$G_2(\alpha_{(m,n)}) = g^2_{\alpha_{(m,n)}}(c) = \gamma_{(m,n)}.$$

Therefore $c \in A_3(\alpha_{(m,n)})$. Since $\lim_{m,n\to\infty} \gamma_{(m,n)} = \infty$, it follows that 1 is a limit point of the $\alpha_{(m,n)}$'s. That is, $\lim_{m,n\to\infty} \alpha_{(m,n)} = 1$.

This lemma provides the first step in the construction by showing that in the family of maps g_α, in any small enough neighborhood of $\alpha = 1$, there is a parameter $\beta_2 := \alpha_{(m,n)}$ with associated critical orbit:

$$c \mapsto g_{\beta_2}(c) \approx (\neq) e_1 \mapsto g^2_{\beta_2}(c) = \gamma_{(m,n)} \mapsto \infty = g^3_{\beta_2}(c). \tag{16}$$

(Recall that we start our numbering with 2, not 1 only for simplicity of notation.)

4.3 The next step: construction of g_α such that $c \in A_4(\alpha)$

We next show that each g_{β_2} defined in Section 4.2 with the properties given by (16) is itself a limit point of a sequence of elliptic functions with c a prepole of order 4; i.e., $c \in A_4(\beta_3)$ for some β_3 arbitrarily close to β_2. To this end we define the next auxiliary function:

$$G_3 : B_\varepsilon(1) \setminus \mathcal{P}_2 \to \mathbb{C}_\infty, \quad \text{by} \quad G_3(\alpha) = g^3_\alpha(c). \tag{17}$$

The map G_3 has essential singularities at $\mathcal{P}_2$ and poles at the parameter values $\alpha_{(m,n)}$ constructed above in Lemma 4.4. We denote by $\mathcal{P}_3 \subset B_\varepsilon(1)$ the set of poles of G_3. Given R_2 as in Lemma 4.4, we fix any lattice point $\gamma := \gamma_{(m,n)} \in B_{R_2}(\infty)$. Then a corresponding $\beta_2 := \alpha_{(m,n)}$ is obtained by Lemma 4.4 and labeled so that $G_2(\beta_2) = \gamma$. In this case we have that $G_3(\beta_2) = \infty$; i.e., β_2 is in $\mathcal{P}_3$ since $c \in A_3(\beta_2)$. Therefore, there exists $\varepsilon_2 := \varepsilon(\beta_2)$ so that if we define $B_3 := B_{\varepsilon_2}(\beta_2)$, then $B_3 \subset U(\beta_2, G_3^{-1}, R_2)$ and $\overline{B_3} \subset B_\varepsilon(1) \setminus \mathcal{P}_2$. Restricting G_3 to B_3, we choose $R_3 \geq 2R_2$ large enough so that for any lattice point $\gamma_j := \gamma_{(m_j,n_j)} \in B_{R_3}(\infty)$ there exists $\alpha_j := \alpha_{(m_j,n_j)} \in \overline{B_3}$ such that $G_3(\alpha_j) = \gamma_j$. Since $\lim_{(m_j,n_j)\to\infty} \gamma_{(m_j,n_j)} = \infty$, we have that the corresponding limit in parameter space is β_2; that is, $\lim_{j\to\infty} \alpha_j = \beta_2$.

This discussion gives a proof of the following lemma.

LEMMA 4.5. *For any parameter $\beta_2 \in \mathcal{P}_3 \cap B_\varepsilon(1)$, any small enough $\delta \ll \varepsilon$, and any $R > 0$, there exists a parameter $\beta_3 \in B_\delta(\beta_2) \subset B_\varepsilon(1)$ such that $G_3(\beta_3) = g^3_{\beta_3}(c) \in \Gamma \cap B_R(\infty)$ (a large lattice point). Therefore $c \in A_4(\beta_3)$.*

Since we have parameters in $\mathcal{P}_3 \cap B_\varepsilon(1)$ arbitrarily close to 1, Lemma 4.5 has the following corollary which gives a needed step in our construction.

COROLLARY 4.6. *In the family of maps $g_\alpha = \alpha\wp_\Gamma$ given in (13), in any neighborhood of $\alpha = 1$ there exist parameters β such that $c \in A_4(\beta)$.*

Remark: The construction of g_β given above shows that the following holds (compare it with (16)):

$$c \mapsto g_{\beta_3}(c) \approx e_1 \mapsto g^2_{\beta_3}(c) \approx \gamma_{(m_1,n_1)} \mapsto g^3_{\beta_3}(c) = \gamma_{(m_2,n_2)} \mapsto g^4_{\beta_3}(c) = \infty, \tag{18}$$

where none of the $\approx$ are equality, and the lattice points $\gamma_{(m_1,n_1)} \in B_{R_2}(\infty)$ and $\gamma_{(m_2,n_2)} \in B_{R_3}(\infty)$ with $R_j \geq 2^j$, for $j = 2, 3$.

4.4 The inductive step: the construction of g_α such that $c \in A_{k+1}(\alpha)$ for $k \geq 2$

Let $k \geq 2$ and assume we have defined for all $j < k$ the map $G_j(\alpha) = g^j_\alpha(c)$ with poles denoted $\mathcal{P}_j$, each defined on $B_\epsilon(1) \setminus \cup_{m<j}\mathcal{P}_m$. We now consider the map:

$$G_k : B_\varepsilon(1) \setminus \cup_{j<k}\mathcal{P}_j \to \mathbb{C}_\infty, \quad \text{by} \quad G_k : \alpha \mapsto g^k_\alpha(c). \tag{19}$$

G_k is meromorphic on the domain given above with poles which we denote by $\mathcal{P}_k$. We choose and fix a pole $\beta_{k-1} \in \mathcal{P}_k \cap B_\varepsilon(1)$, so that we have $g^k_{\beta_{k-1}}(c) = \infty$. By our previous discussion we know that there exists some $\varepsilon_{k-1} > 0$ small enough such that G_k is meromorphic from $B_k := B_{\varepsilon_{k-1}}(\beta_{k-1}) \subset B_\varepsilon(1) \setminus \cup_{j<k}\mathcal{P}_j$ to $\mathbb{C}_\infty$ with a pole at β_{k-1} and so that $B_k \subset U(\beta_{k-1}, G_k^{-1}, R_{k-1})$.

Therefore there exists a number $R_k \geq 2R_{k-1} \geq 2^k$ such that $B_{R_k}(\infty) \subset G_k(B_k)$. Again we choose and fix a lattice point $\gamma_k := \gamma_{(m,n)} \in B_{R_k}(\infty)$ and then let $\beta_k := \alpha_{(m,n)}$ be chosen and labeled so that $G_k(\beta_k) = \gamma_k \in \Gamma$.

Since the lattice point $\gamma_{(m,n)}$ can be chosen to be arbitrarily large, the resulting $\beta_k := \alpha_{(m,n)}$ mapping to it under G_k can be found arbitrarily close to the pole β_{k-1}.

We have shown that the following lemma holds.

Lemma 4.7. *For any $k \geq 3$, and for any parameter $\beta_{k-1} \in \mathcal{P}_k \cap B_\varepsilon(1)$, any small enough $\delta \ll \varepsilon$, and any $R > 0$, there exists a parameter $\beta_k \in B_\delta(\beta_{k-1}) \subset B_\varepsilon(1) \setminus \cup_{j<k}\mathcal{P}_j$ such that $G_k(\beta_k) = g^k_{\beta_k}(c) \in \Gamma \cap B_R(\infty)$ (a large lattice point). Therefore $c \in A_{k+1}(\beta_k)$.*

Since we always have parameters in $\mathcal{P}_k \cap B_\varepsilon(1)$ arbitrarily close to 1 by induction on k, Lemma 4.7 has the following corollary.

Corollary 4.8. *In the family of maps $g_\alpha = \alpha\wp_\Gamma$ given in (13), in any neighborhood of $\alpha = 1$ and for any $k \geq 3$, there exists a parameter β such that $c \in A_k(\beta)$.*

4.5 Constructing the main example

We state the main theorem that gives the elliptic function with the desired properties.

Theorem 4.9. *Suppose we are given a map $\wp_\Gamma$ on a real rhombic square lattice Γ, with c_1 (and hence every critical point) a prepole, and the parametrized family of elliptic functions $g_\alpha(z) = \alpha\wp_\Gamma(z)$, with $\alpha \in B_\varepsilon(1)$, for some small $\varepsilon > 0$. Then there exists a critically tame elliptic function g_β such that*

$$\lim_{n\to\infty} g^n_\beta(c_1) = \lim_{n\to\infty} g^n_\beta(c_2) = \infty,$$

and c_3 is a prepole.

Proof. Fix some $\varepsilon \in (0, \frac{1}{4})$ and set $\varepsilon_1 = \varepsilon$. As before, write $\wp = \wp_\Gamma$ and $c = c_1$. When $\alpha = 1$ we have that $g_1(c) = \wp(c) = \gamma_0 \in \Gamma$; so using the definitions above, $G_2(1) = \infty$ or equivalently, $1 \in \mathcal{P}_2$. By Lemma 4.4 there exists some $R_2 > 4$ such that if we fix any lattice point $\gamma_1 := \gamma_{(m,n)} \in B_{R_2}(\infty)$ there exists a parameter $\beta_2 := \alpha_{(m,n)} \in B_{\varepsilon_1}(1) \cap \mathcal{P}_3$ such that $G_2(\beta_2) = \gamma_1$. From the proof of Lemma 4.5 we know that there exists a ball $B_3 := B_{\varepsilon_2}(\beta_2) \subset B_{\varepsilon_1}(1) \setminus \mathcal{P}_2$, with $\varepsilon_2 \leq \varepsilon_1/2$, on which G_3 is well-defined and meromorphic and chosen small enough so that $B_3 \subset U(\beta_2, G_3^{-1}, R_2)$ and $\overline{B_3} \subset B_{\varepsilon_1}(1) \setminus \mathcal{P}_2$. By construction $G_3(\beta_2) = \infty$. Then by Lemma 4.5 there exists $R_3 \geq 2R_2$ such that for any lattice point $\gamma_2 := \gamma_{(m,n)} \in B_{R_2}(\infty)$ there exists a parameter $\beta_3 \in \mathcal{P}_4 \cap B_{\varepsilon_2}(\beta_2)$ such that $G_3(\beta_3) = \gamma_2$.

By induction on k, and applying Lemma 4.7, for each k we obtain β_k with $c \in A_{k+1}(\beta_k)$ and a sequence of nested balls $B_{k+1} := B_{\varepsilon_k}(\beta_k) \subset B_\varepsilon(1) \setminus \cup_{j<k}\mathcal{P}_j$, with $\varepsilon_k \leq 2^{-(k+1)}$, and a sequence $\{R_k\}$ such that each $R_k \geq 2R_{k-1}$ such that $B_{k+1} \subset U(\beta_k, G_{k+1}^{-1}, R_k)$. That is, $G_{k+1}(B_{k+1}) \subset B_{R_k}(\infty)$ or equivalently $|g^{k+1}_\alpha(c)| > 2^k$ for all $\alpha \in B_{k+1}$. Since $\varepsilon_k \to 0$ there is a parameter $\beta = \cap_{k\in\mathbb{N}}\overline{B_{\varepsilon_k}(\beta_k)}$. In fact β is an accumulation point of $\cup_{k\in\mathbb{N}}\mathcal{P}_k$. For the parameter β we see that the forward orbit of c satisfies: $g_\beta^2(c) \in B_{R_1}(\infty)$, $g_\beta^3(c) \in B_{R_2}(\infty)$, and in general $g_\beta^k(c) \in B_{R_{k-1}}(\infty)$. Since $\lim_{k\to\infty} R_k = \infty$, it follows that $\lim_{k\to\infty} g_\beta^k(c) = \infty$.

Since by our construction $\text{Crit}(g_\beta) \subset C_p(g_\beta) \cup C_\infty(g_\beta)$ it follows from Lemma 3.7 that g_β is critically tame.

□

4.6 *Extending the example to other lattices*

The example given above is by no means unique. If we begin with an elliptic function f that has a critical point c which is a prepole, then the construction given above results in an elliptic function of the form $g_\beta = \beta f$ with $\lim_{n\to\infty} g_\beta(c) = \infty$. We conclude this section by giving more examples of elliptic functions having critical points whose orbits tend to infinity.

4.6.1 Examples where all critical points escape to infinity

If we have an elliptic function of the form g_β with all critical orbits approaching infinity then g_β is critically tame by Lemma 3.7. We use this idea to provide additional examples.

THEOREM 4.10. *Let Γ be a triangular lattice such that $\wp_\Gamma$ has c_1 (and hence every critical point) a prepole, and consider the parametrized family of elliptic functions $g_\alpha(z) = \alpha\wp_\Gamma(z)$, with $\alpha \in B_\epsilon(1)$, for some small $\epsilon > 0$. Then there exists a $\beta \in B_\epsilon(1)$ giving a critically tame elliptic function $g_\beta = \beta\wp_\Gamma$ such that $\lim_{n\to\infty} g_\beta^n(c_j) = \infty$ for $j = 1, 2, 3$; i.e. all critical points tend to ∞.*

The proof of the theorem appears after the next result, which is used in the proof.

PROPOSITION 4.11.

(1) *The lattice Γ is triangular ($e^{2\pi/3}\Gamma = \Gamma$) if and only if $g_2 = 0$; in this case the critical values e_1, e_2, e_3 of $\wp_\Gamma$ all have the same modulus and are the cube roots of $g_3/4$.*
(2) *Let Γ be triangular and $g_\beta = \beta\wp_\Gamma$. If c_1, c_2, and c_3 are the critical points of $\wp_\Gamma$, then $g_\beta^n(c_2) = \rho g_\beta^n(c_1)$ and $g_\beta^n(c_3) = \rho^2 g_\beta^n(c_1)$ for all $n \in \mathbb{N}$, where $\rho = e^{2\pi i/3}$ is a cube root of unity.*

Proof. Statement (1) is a classical result that can be found in [4], including that the critical values of $\wp_\Gamma$ are the cube roots of $g_3/4$. To prove (2), we note that the critical points of g_β are the same as the critical points for $\wp_\Gamma$.

First, we use induction to prove that $g_\beta^n(c_2) = \rho g_\beta^n(c_1)$ for all $n \in \mathbb{N}$, where ρ is as above. For triangular lattices, the critical values of $\wp_\Gamma$ are related by $e_2 = \rho e_1$ and $e_3 = \rho^2 e_1$ by part (1) of the theorem. Thus

$$g_\beta(c_1) = \beta\wp_\Gamma(c_1) = \beta e_1$$

and

$$g_\beta(c_2) = \beta\wp_\Gamma(c_2) = \beta e_2 = \beta\rho e_1 = \rho g_\beta(c_1).$$

Since Γ is triangular, we have by definition that $\rho\Gamma = \Gamma$. If we assume that that $g_\beta^k(c_2) = \rho g_\beta^k(c_1)$, then

$$g_\beta^{k+1}(c_2) = g_\beta(g_\beta^k(c_2)) = g_\beta(\rho g_\beta^k(c_1)) = \beta\wp_\Gamma(\rho g_\beta^k(c_1)) = \beta\wp_{\rho\Gamma}(\rho g_\beta^k(c_1))$$

$$= \frac{\beta}{\rho^2}\wp_\Gamma(g_\beta^k(c_1)) = \rho\beta\wp_\Gamma(g_\beta^k(c_1)) = \rho g_\beta(g_\beta^k(c_1)) = \rho g_\beta^{k+1}(c_1).$$

Using an identical argument $g_\beta^n(c_3) = \rho^2 g_\beta^n(c_1)$.

□

Proof of Theorem 4.10: Since c_1 is a prepole, we apply the construction using $c = c_1$ and obtain β such that $\lim_{n\to\infty} g_\beta^n(c_1) = \infty$. By Proposition 4.11, $\lim_{n\to\infty} g_\beta^n(c_2) = \infty$ and $\lim_{n\to\infty} g_\beta^n(c_3) = \infty$, so the map g_β has all critical points tending to ∞. □

Finally, Theorem 2.8 gives infinitely many examples satisfying the hypotheses of Theorem 4.10, and from [7] they lie in infinitely many distinct conformal conjugacy classes.

4.7 An example with a nonempty Fatou set

We conclude with a construction of an elliptic function g_β that has a nonempty Fatou set, a critical point whose orbit approaches infinity, and a critical point that is a prepole. Therefore for the three critical orbits possible we have $c_1 \in F(g_\beta)$, $c_2 \in C_p(g_\beta)$, and $c_3 \in C_\infty(g_\beta)$.

The following estimate was obtained in [12] (cf. also [15]), and we use it in our construction. Suppose f is elliptic and $c \in C_p(f)$, and $c \in A_k(f)$; by a slight abuse of the notation of Definition 3.2 let $p_c := p(f^{k-1}, c)$ denote the order of f^{k-1} at the critical point c.

Theorem 4.12. *Let $f : \mathbb{C} \to \mathbb{C}_\infty$ be an elliptic function such that at least one critical value is eventually mapped onto a pole. Then*

$$HD(J(f)) > \frac{2pq}{pq+1} \geq 1,$$

where, $p = \max\{p_c : c \in C_p(f) \cap A_k\}$ and q is the maximum pole order (see Definition 3.3 (7)).

Our starting point is the standard rhombic square lattice Λ defined at the beginning of 4.1.1. We use $j = 1$ in the construction given there, and an application of Proposition 4.1 gives the following.

Lemma 4.13. *Let $\Lambda = [a + ai, a - ai]$ be the standard rhombic square lattice, set $k = \sqrt[3]{1/a} > 0$, and define $\Gamma = k\Lambda = [ka + kai, ka - kai] \equiv [d + di, d - di]$.*

Then $\wp_\Gamma((d + di)/2) = e_1(\Gamma) = -di$ and $c \in C_p(\wp_\Gamma)$ for every critical point c of $\wp_\Gamma$.

We next modify this to obtain an elliptic function with a superattracting fixed point, a critical value at 0, and another critical value at a lattice point. Given the lattice $\Gamma = [d + di, d - di]$ constructed in Lemma 4.13, define $h = \wp_\Gamma + di$. The function h is an elliptic function with poles of order two at each point of Γ. Further, h has the same critical points as $\wp_\Gamma$. We describe completely the behavior of all critical orbits of h in the next result.

Proposition 4.14. *The following hold for h:*

(1) $h((d + di)/2) = 0$ and $h((d - di)/2) = 2di$ are lattice points, and $h(d) = di$ is a superattracting fixed point.
(2) For any $\alpha \in \mathbb{C}\setminus\{0\}$, defining $g_\alpha(z) = \alpha h(z)$, we have that $HD(J(g_\alpha)) > \frac{8}{5}$.

Proof. Using Lemma 4.13, $h((d+di)/2) = \wp_\Gamma((d+di)/2) + di = 0$, and $h((d-di)/2) = \wp_\Gamma((d-di)/2)+di = di+di = 2di$. Similarly, $h(d) = \wp_\Gamma(d)+di = 0+di = di = h(di)$.

By our choice of Γ, for any $\alpha \neq 0$ we have a pole at $g_\alpha((d+di)/2) = \alpha h((d+di)/2) = 0$. We have that the poles and critical points of g_α are precisely those of h (and $\wp_\Gamma$) so the hypotheses of Theorem 4.12 are satisfied. Every pole of g_α is of order 2, and every critical point is of order 2, and thus the $HD(J(g_\alpha)) > 8/5$.

□

Finally, we use the construction from Section 4 to obtain an elliptic function with the critical behavior of interest. We label the common critical points of h, $\wp_\Gamma$, and g_α by c_1, c_2, and c_3 using the conventions established in Equation (4). In Proposition 4.14 we constructed an elliptic h such that $c_1, c_2 \in C_p(h)$ while $c_3 \in F(h)$. A perturbation of h gives the desired result.

THEOREM 4.15. *There exists a critically tame elliptic function $g_\beta = \beta h$ with a nonempty Fatou set such that* $\lim_{n\to\infty} g_\beta^n(c_2) = \infty$.

Proof. We start with h such that $c_3 = di$ is a superattracting fixed point; the family $g_\alpha = \alpha h$, $\alpha \in B_\varepsilon(1)$, moves the fixed point z_0 near di and its derivative $g'(z_0) = \alpha h'(z_0)$ holomorphically in α for α not too far from 1. Then for ε small enough, using the stability theory developed in [16] and [17], and applied to the meromorphic setting in [10] and [7], if $\alpha \in B_\varepsilon(1)$ then $g_\alpha = \alpha h$ has an attracting fixed point z_0 near $c_3 = di$ with $g_\alpha(d) = \alpha di$ contained in the immediate basin of attraction of z_0. For every $\alpha \in B_\varepsilon(1)$, Proposition 4.14 gives that $c_1 \in C_p(g_\alpha)$.

We use the construction described in Section 4 applied to the critical point c_2 to obtain β such that $\lim_{n\to\infty} g_\beta^n(c_2) = \lim_{n\to\infty} g_\beta^n((d-di)/2) = \infty$. Using Proposition 4.14 (2), the condition of Definition 3.5 (c) (11) is satisfied so g_β is critically tame.

□

Using the same technique, Proposition 2.9 leads to an example of an elliptic function of the form $g_\beta = \beta\wp_\Lambda$, with a nonsquare period lattice, that also has a nonempty Fatou set and a critical point whose orbit approaches infinity. However we are not able to determine if it is critically tame because it is unclear whether it satisfies the inequality of Definition 3.5 (11).

Acknowledgments

The authors thank the organizers of the 2008 Dynamical Systems and Topology Conference for their hospitality in Spain during which the ideas in this paper began to develop. The third author was partially supported by a grant *Chaos, fraktale i dynamika konforemna*, N N201 0222 33, and by EU FP6 Marie Curie Programmes RTN CODY and TOK SPADE2 at IMPAN. The referee is gratefully acknowledged for suggestions that improved this paper.

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