Submitted to
Topology Proceedings

SMOOTH JULIA SETS OF ELLIPTIC FUNCTIONS FOR SQUARE RHOMBIC LATTICES

JANE HAWKINS

ABSTRACT. We discuss the dynamics of iterating the Weierstrass elliptic $\wp$ function with period lattice any real rhombic square lattice in $\mathbb{C}$, that is a lattice generated by the complex numbers $a + ai$ and $a - ai$, $a > 0$. We conjecture that the Julia set of $\wp$ for each such lattice is the whole sphere and describe a holomorphic family of these maps. We prove the conjecture in a specific case and give results supporting it in general.

1. INTRODUCTION

Smooth Julia sets are somewhat of a rare occurrence in complex dynamics, and even rarer embedded in parametrized families of meromorphic maps. The only holomorphic family of rational maps with Julia set the whole sphere is due to Lattès, and a nice expository article about it can be found for example in [14].

In this study we focus on one specific lattice shape for the iterated Weierstrass elliptic $\wp$ function. While this is restrictive, it is already known there is a wide variety of dynamical behavior and topology

---

2000 *Mathematics Subject Classification.* Primary 54H20, 37F10; Secondary 37F20.

Key words and phrases. complex dynamics, meromorphic functions, Julia sets.

The author thanks Berry College and the NSF for financial support, and the organizers of the 2005 Spring Topology and Dynamics Conference, especially Eric McDowell, for their warm hospitality during which some of the ideas in this paper were developed.

among Julia sets resulting from parametrized families of elliptic functions with period lattice in the shape of a square [6, 7, 8, 9].

If the square lattice is generated by conjugate vectors of the form $\lambda_1 = a + ai$, and $\lambda_2 = a - ai$, for some real number $a > 0$, then we claim that there is much less variety among the Julia sets that can occur. In particular we conjecture that in this case $J(\wp_\Lambda) = \mathbb{C}_\infty$, the Riemann sphere, and we obtain a family of maps with this property parametrized by the real parameter $a > 0$. The starting point for this paper is the following proposition which appeared by the author and Koss in [7].

Proposition 1.1. *If Λ is rhombic square, then the Julia set is connected if and only if $J(\wp_\Lambda) = \mathbb{C}_\infty$.*

Our main theorem is that the condition is satisfied for one specific lattice and we conjecture that it is true for all square rhombic lattices. We give supporting evidence for the full conjecture in the form of traditional proofs and numerical evidence.

Conjecture 1.2. *If Λ is a real rhombic square lattice, then $J(\wp_\Lambda) = \mathbb{C}_\infty$.*

The reason that the conjecture is quite difficult to prove is that there is no closed form for the Weierstrass elliptic $\wp_\Lambda$ function, and certainly not for any fixed or periodic points of it. We proceed by studying the classical identities for these functions and by pushing them further in order to extract dynamical results from them.

2. Some preliminary definitions and notation

Let $\lambda_1, \lambda_2 \in \mathbb{C}\backslash\{0\}$ such that $\lambda_2/\lambda_1 \notin \mathbb{R}$. We define a lattice of points in the complex plane by $\Lambda = [\lambda_1, \lambda_2] \colon = \{m\lambda_1 + n\lambda_2 \colon m, n \in \mathbb{Z}\}$. Two different sets of vectors can generate the same lattice Λ; if $\Lambda = [\lambda_1, \lambda_2]$, then all other generators λ_3, λ_4 of Λ are obtained by multiplying the vector (λ_1, λ_2) by the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = \pm 1$.

We can view Λ as a group acting on $\mathbb{C}$ by translation, each $\omega \in \Lambda$ inducing the transformation of $\mathbb{C}$:

$$T_\omega : z \mapsto z + \omega.$$

Definition 2.1. A closed, connected subset Q of $\mathbb{C}$ is defined to be a *fundamental region* for Λ if

(1) for each $z \in \mathbb{C}$, Q contains at least one point in the same Λ-orbit as z;
(2) no two points in the interior of Q are in the same Λ-orbit.

If Q is any fundamental region for Λ, then for any $s \in \mathbb{C}$, the set

$$Q + s = \{z + s : z \in Q\}$$

is also a fundamental region. If we choose Q to be a parallelogram we call Q a *period parallelogram* for Λ.

The ratio $\tau = \lambda_2/\lambda_1$ is an important feature of a lattice. If $\Lambda = [\lambda_1, \lambda_2]$, and $k \neq 0$ is any complex number, then $k\Lambda$ is the lattice defined by taking $k\lambda$ for each $\lambda \in \Lambda$; $k\Lambda$ is said to be *similar* to Λ. For example, the lattice $\Lambda_\tau = [1, \tau]$ is similar to the lattice $\Lambda = \lambda_1\Lambda_\tau$. Similarity is an equivalence relation between lattices, and an equivalence class of lattices is called a *shape*.

Definition 2.2. (1) $\Lambda = [\lambda_1, \lambda_2]$ is *real rectangular* if there exist generators such that λ_1 is real and λ_2 is purely imaginary. Any lattice similar to a real rectangular lattice is *rectangular*.
(2) $\Lambda = [\lambda_1, \lambda_2]$ is *real rhombic* if there exist generators such that $\lambda_2 = \overline{\lambda_1}$. Any similar lattice is *rhombic*.
(3) A lattice Λ is *square* if $i\Lambda = \Lambda$. (Equivalently, Λ is square if it is similar to a lattice generated by $[\lambda, \lambda i]$, for some $\lambda > 0$.)

In each of cases (1) – (3) the period parallelogram with vertices $0, \lambda_1, \lambda_2$, and $\lambda_3 := \lambda_1 + \lambda_2$ can be chosen to look rectangular, rhombic, or square respectively.

2.1. Real rhombic square lattices.

Proposition 2.3. *The following are equivalent for a lattice* Λ*.*

(1) Λ *is a real rhombic square lattice.*
(2) *There exists a* $\lambda > 0$ *such that* $\Lambda = [\lambda e^{\pi i/4}, \lambda e^{-\pi i/4}]$.
(3) *There exists* $\gamma > 0$ *such that* $\Lambda = [2\gamma, \gamma + i\gamma]$.

Proof. By definition, Λ is real rhombic square if and only if: (a) $\Lambda = [\lambda_1, \lambda_2]$ with $\lambda_2 = \overline{\lambda_1}$; and (b) $i\Lambda = \Lambda$.

(2) implies (1): Obviously condition (a) is satisfied and writing $\lambda e^{\pi i/4}$ in its Cartesian form, we have $\Lambda = [\frac{\lambda}{\sqrt{2}}(1+i), \frac{\lambda}{\sqrt{2}}(1-i)]$. Then for any $\lambda \in \Lambda$, there exist $m, n \in \mathbb{Z}$ such that $\lambda = m(\frac{\lambda}{\sqrt{2}}(1+i)) + n(\frac{\lambda}{\sqrt{2}}(1-i))$. We have that

$$i\lambda = im(\frac{\lambda}{\sqrt{2}}(1+i)) + in(\frac{\lambda}{\sqrt{2}}(1-i))$$

$$= n(\frac{\lambda}{\sqrt{2}}(1+i)) - m(\frac{\lambda}{\sqrt{2}}(1-i)).$$

So $i\Lambda \subset \Lambda$; since

$$\Lambda = -\Lambda = i^2\Lambda \subset i\Lambda \subset \Lambda,$$

(b)is satisfied.

(2) and (3) are equivalent using $\gamma = \frac{\lambda}{\sqrt{2}}$ and changing the generator using the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

(1) implies (2): If (1) holds, then there exist generators satisfying $\lambda_2 = \overline{\lambda_1}$; in polar form, $\lambda_1 = re^{i\theta}$ and $\lambda_2 = re^{-i\theta}$ for some $\theta \in (0, \pi)$. Since in addition it is square, we use the remark in Definition 2.2, (3), to see that $\theta - (-\theta) = 2\theta = \pi/2$. Therefore $\theta = \pi/4$. □

We begin with $f\colon \mathbb{C} \to \mathbb{C}_\infty$ a meromorphic function where $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere.

Definition 2.4. An *elliptic function* is a meromorphic function in $\mathbb{C}$ which is periodic with respect to a lattice Λ.

For any $z \in \mathbb{C}$ and any lattice Λ, the *Weierstrass elliptic function* is defined by

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right).$$

Replacing every z by $-z$ in the definition we see that $\wp_\Lambda$ is an even function. The map $\wp_\Lambda$ is meromorphic, periodic with respect to Λ, and has order 2.

The derivative of the Weierstrass elliptic function is also an elliptic function which is periodic with respect to Λ defined by

$$\wp'_\Lambda(z) = -2\sum_{w\in\Lambda}\frac{1}{(z-w)^3}.$$

The Weierstrass elliptic function and its derivative are related by the differential equation

$$\wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 - g_2\wp_\Lambda(z) - g_3, \tag{2.1}$$

where $g_2(\Lambda) = 60\sum_{w\in\Lambda\setminus\{0\}} w^{-4}$ and $g_3(\Lambda) = 140\sum_{w\in\Lambda\setminus\{0\}} w^{-6}$.

The numbers $g_2(\Lambda)$ and $g_3(\Lambda)$ are invariants of the lattice Λ in the following sense: if $g_2(\Lambda) = g_2(\Lambda')$ and $g_3(\Lambda) = g_3(\Lambda')$, then $\Lambda = \Lambda'$. Furthermore given any g_2 and g_3 such that $g_2^3 - 27g_3^2 \neq 0$ there exists a lattice Λ having $g_2 = g_2(\Lambda)$ and $g_3 = g_3(\Lambda)$ as its invariants [5].

Theorem 2.5. [5] *For $\Lambda_\tau = [1,\tau]$, the functions $g_i(\tau) = g_i(\Lambda_\tau), i = 2, 3$, are analytic functions of τ in the open upper half plane $Im(\tau) > 0$.*

We have the following homogeneity in the invariants g_2 and g_3 [8].

Lemma 2.6. *For lattices Λ and Λ', $\Lambda' = k\Lambda \Leftrightarrow$*

$$g_2(\Lambda') = k^{-4}g_2(\Lambda) \quad \textit{and} \quad g_3(\Lambda') = k^{-6}g_3(\Lambda).$$

Theorem 2.7. [10] *The following are equivalent:*

(1) $\wp_\Lambda(\overline{z}) = \overline{\wp_\Lambda(z)}$;
(2) Λ *is a real lattice;*
(3) $g_2, g_3 \in \mathbb{R}$.

For any lattice Λ, the Weierstrass elliptic function and its derivative satisfy the following properties: for $k \in \mathbb{C}\setminus\{0\}$,

$$\wp_{k\Lambda}(ku) = \frac{1}{k^2}\wp_\Lambda(u), \quad \text{(homogeneity of } \wp_\Lambda\text{)}, \tag{2.2}$$

$$\wp'_{k\Lambda}(ku) = \frac{1}{k^3}\wp'_\Lambda(u), \quad \text{(homogeneity of } \wp'_\Lambda\text{)},$$

Verification of the homogeneity properties can be seen by substitution into the series definitions.

If $\wp'_\Lambda(z_0) = 0$ then z_0 is a *critical point* and $\wp_\Lambda(z_0)$ is a *critical value*. The critical values of the Weierstrass elliptic function on an arbitrary lattice $\Lambda = [\lambda_1, \lambda_2]$ are as follows.

For $j = 1, 2$, notice that $\wp_\Lambda(\lambda_j - z) = \wp_\Lambda(z)$ for all z. Taking derivatives of both sides we obtain $-\wp'_\Lambda(\lambda_j - z) = \wp'_\Lambda(z)$. Substituting $z = \lambda_1/2, \lambda_2/2$, or $\lambda_3/2$, we see that $\wp'_\Lambda(z) = 0$ at these values. We use the notation

$$e_1 = \wp_\Lambda(\frac{\lambda_1}{2}), \quad e_2 = \wp_\Lambda(\frac{\lambda_2}{2}), \quad e_3 = \wp_\Lambda(\frac{\lambda_3}{2})$$

to denote the critical values. Since e_1, e_2, e_3 are the distinct zeros of Equation 2.1, we also write

$$\wp'_\Lambda(z)^2 = 4(\wp_\Lambda(z) - e_1)(\wp_\Lambda(z) - e_2)(\wp_\Lambda(z) - e_3). \tag{2.3}$$

Equating like terms in Equations 2.1 and 2.3, we obtain

$$e_1 + e_2 + e_3 = 0, \quad e_1e_3 + e_2e_3 + e_1e_2 = \frac{-g_2}{4}, \quad e_1e_2e_3 = \frac{g_3}{4}. \tag{2.4}$$

Naturally, the lattice shape relates to the properties and dynamics of the corresponding Weierstrass elliptic function. Denote

$$p(x) = 4x^3 - g_2x - g_3, \tag{2.5}$$

the polynomial associated with Λ. Let $\triangle = g_2^3 - 27g_3^2 \neq 0$ denote its discriminant.

Proposition 2.8. [5]

1. *If Λ is real rhombic, then the discriminant is negative; in this case the roots of p are one real and a complex conjugate pair. If $g_3 > 0$ then the vertical diagonal of the rhombus is longer than the horizontal diagonal, and if $g_3 < 0$ then the horizontal diagonal of the rhombus is longer than the vertical diagonal.*
2. *If Λ is rhombic square, then $g_2 < 0$ and $g_3 = 0$; in this case the roots of p are $0, \pm\sqrt{g_2}/2$.*

The following corollary can be obtained using Equations 2.1 and 2.4.

Corollary 2.9.

1. *If Λ is rhombic then $e_2 = \overline{e_1}$ are the complex roots of Equation 2.5, and e_3 is real. If $g_3 > 0$ then $e_3 > 0$, and if $g_3 < 0$ then $e_3 < 0$.*
2. *If Λ is rhombic square then $e_3 = 0$, and $e_1 = \sqrt{g_2}/2 = -e_2$ are pure imaginary.*

2.2. **Fatou and Julia sets for elliptic functions.** We review the basic dynamical definitions and properties for meromorphic functions which appear in [1], [2], [3] and [4]. As above, let $f\colon \mathbb{C} \to \mathbb{C}_\infty$ be a meromorphic function where $\mathbb{C}_\infty$ denotes the Riemann sphere. The *Fatou set* $F(f)$ is the set of points $z \in \mathbb{C}_\infty$ such that $\{f^n\colon n \in \mathbb{N}\}$ is defined and normal in some neighborhood of z. The *Julia set* is the complement of the Fatou set on the sphere, $J(f) = \mathbb{C}_\infty \backslash F(f)$. Notice that $\mathbb{C}_\infty \backslash \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}$ is the largest open set where all iterates are defined. Since $f(\mathbb{C}_\infty \backslash \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}) \subset \mathbb{C}_\infty \backslash \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}$, Montel's theorem implies that

$$J(f) = \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}.$$

Let $Crit(f)$ denote the set of critical points of f, *i.e.*,

$$Crit(f) = \{z\colon f'(z) = 0\}.$$

If z_0 is a critical point then $f(z_0)$ is a *critical value.* For each lattice, $\wp_\Lambda$ has three critical values and no asymptotic values. The *singular set* $Sing(f)$ of f is the set of critical and finite asymptotic values of f and their limit points. A function is called *Class S* if f has only finitely many critical and asymptotic values; for each lattice Λ, every elliptic function with period lattice Λ is of Class S. The *postcritical set* of $\wp_\Lambda$ is:

$$P(\wp_\Lambda) = \overline{\bigcup_{n\geq 0} \wp_\Lambda^n(e_1 \cup e_2 \cup e_3)}.$$

For a meromorphic function f, a point z_0 is *periodic* of period p if there exists a $p \geq 1$ such that $f^p(z_0) = z_0$. We also call the set $\{z_0, f(z_0), \ldots, f^{p-1}(z_0)\}$ a *p-cycle.* The *multiplier* of a point z_0 of period p is the derivative $(f^p)'(z_0)$. A periodic point z_0 is called *attracting, repelling,* or *neutral* if $|(f^p)'(z_0)|$ is less than, greater than, or equal to 1 respectively. If $|(f^p)'(z_0)| = 0$ then z_0 is called a *superattracting* periodic point. As in the case of rational maps, the Julia set is the closure of the repelling periodic points [1].

Suppose U is a connected component of the Fatou set. We say that U is *preperiodic* if there exists $n > m \geq 0$ such that $f^n(U) = f^m(U)$, and the minimum of $n - m = p$ for all such n, m is the *period* of the cycle.

Proposition 2.10. *If p is an attracting fixed point or a rationally neutral fixed point for $\wp_\Lambda$, then the local coordinate chart for the point is completely contained in one fundamental period of $\wp_\Lambda$ (in fact in one half of one fundamental period).*

Proof. This is due to the periodicity of $\wp_\Lambda$; in each case the local form is invertible. If we spill into another half fundamental period or region, then injectivity fails.

□

The main conjecture of this paper is that for a real rhombic square lattice Λ, $J(\wp_\Lambda) = \mathbb{C}_\infty$. However the next result shows that there are many square lattices Γ with the property that the corresponding Weierstrass $\wp$ function with period lattice Γ has a superattracting fixed point. Hence the Julia set of $\wp_\Lambda$ does not depend on the shape of a lattice as much as on its invariants g_2 and g_3 (see Figure 1).

Proposition 2.11. *Let $\Lambda = [1, \tau]$ be a lattice such that the critical value $\wp_\Lambda(1/2) = \epsilon \neq 0$. If m is any odd integer and $k = \sqrt[3]{2\epsilon/m}$ (taking any root) then the lattice $\Gamma = k\Lambda$ has a superattracting fixed point at $mk/2$.*

Proof. Equation (2.2) for $\wp'_{k\Lambda}$ implies that $k/2$ is a critical point for $\wp_\Gamma$. Since m is odd, periodicity implies that $\wp_\Gamma(mk/2) = \wp_\Gamma(k/2)$. Further, the homogeneity property implies that

$$\wp_\Gamma(\frac{mk}{2}) = \wp_\Gamma(\frac{k}{2}) = \wp_{k\Lambda}(\frac{k}{2}) = \frac{1}{k^2}\wp_\Lambda(\frac{1}{2}) = \frac{\epsilon}{k^2} = \frac{mk}{2}.$$

□

Julia sets for square lattices exhibit additional symmetry. The following was proved in [7].

Theorem 2.12. *If Λ is rectangular square or rhombic square, then $e^{\pi i/2}J(\wp_\Lambda) = J(\wp_\Lambda)$ and $e^{\pi i/2}F(\wp_\Lambda) = F(\wp_\Lambda)$.*

Proof. Let $z \in F(\wp_\Lambda)$; then by definition, $\wp^n_\Lambda(z)$ exists and is normal for all n. By Equation 2.2, $\wp_\Lambda(iz) = -\wp_\Lambda(z)$ and since $\wp_\Lambda$ is even we know that $\wp^n_\Lambda(iz) = \wp^n_\Lambda(z)$ for all $n \geq 2$. So $\wp^n_\Lambda(iz)$ exists for all n. Let U be a neighborhood of z such that $\{\wp^n_\Lambda(U)\}$ forms a normal family. Let $V = iU$. Repeating our argument, we have that $\wp_\Lambda(V) = \wp_\Lambda(iU) = -\wp_\Lambda(U)$ and $\wp^n_\Lambda(V) = \wp^n_\Lambda(U)$ for all $n \geq 2$ and

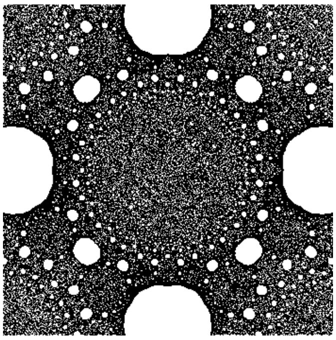

FIGURE 1. One period of the Julia set for a square lattice with a superatrracting fixed point

thus $\{\wp_\Lambda^n(V)\}$ forms a normal family. The proof of the converse is identical. So $z \in F(\wp_\Lambda)$ if and only if $iz \in F(\wp_\Lambda)$. By symmetry about the origin, $-z, -iz \in F(\wp_\Lambda)$ and the Fatou set is symmetric with respect to rotation by $\pi/2$.

□

2.3. **Summary of properties of $\wp_\Lambda$ when Λ is a real rhombic square lattice.** We collect the results for our setting and show the graph of $\wp_\Lambda$ in this case. For the rest of the paper we assume that Λ is a real rhombic square lattice. Most of these follow immediately from the results above; a few references are given for more detailed proofs.

(1) $\Lambda = [\lambda e^{\pi i/4}, \lambda e^{-\pi i/4}]$ for some $\lambda > 0$.
(2) $i\Lambda = \Lambda$, $iJ(\wp_\Lambda) = J(\wp_\Lambda)$, and $iF(\wp_\Lambda) = F(\wp_\Lambda)$.
(3) $g_3 = 0$ and $g_2 < 0$.
(4) The connection between λ and g_2 is as follows. For $k > 0$, $\Lambda' = k\Lambda$ if and only if $g_2(\Lambda') = k^{-4}g_2(\Lambda)$
(5) $e_3 = 0$, and $e_1 = -e_2$ are purely imaginary and satisfy
$$-4e_1e_2 = g_2,$$
or equivalently,
$$e_1 = -\sqrt{g_2}/2.$$
(6) We define the *standard lattice* to be the unique lattice corresponding to $g_2 = -4$, and giving $e_1 = -i$ and $e_2 = i$.

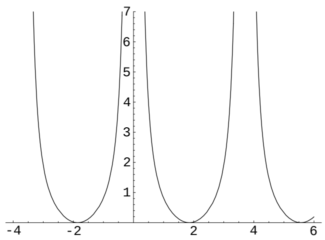

FIGURE 2. The graph of $\wp_\Lambda$ for Λ real rhombic square

(7) For the standard lattice we have $\lambda = \gamma \approx 2.62206$, and we denote the special lattice $\Gamma = [b+bi, b-bi]$, with $b \approx 1.85407$ (see eg. [12]). Note that $\gamma = \sqrt{2}b$.

3. The Main Results

As above, Γ denotes the standard square rhombic lattice with side length γ.

Theorem 3.1. *If we define a real rhombic square lattice* $\Lambda = k\Gamma$, *with* $k = (2/b)^{1/3} \approx 1.02558$, *then* $J(\wp_\Lambda) = \mathbb{C}_\infty$.

Before giving a proof of the theorem we need to recall some basic analytic properties of $\wp_\Lambda$ which follow from classical identities. There are formulas for the quarter period values of the Weierstrass elliptic function; we mention one which is of use in what follows. Let for each $i, j, k = 1, 2, 3$

$$d_i^2 = (e_i - e_j)(e_i - e_k) = 3e_i^2 - g_2/4, \tag{3.1}$$

where we choose the square root so that

$$\wp_\Lambda(\lambda_i/4) = e_i + d_i, \tag{3.2}$$

with $\Lambda = [\lambda_1, \lambda_2]$, and $\lambda_3 = \lambda_1 + \lambda_2$.

We use this to prove the following result.

Lemma 3.2. *If* $\Lambda = [a + ai, a - ai]$, $a > 0$ *is a real rhombic square lattice, then* $\wp_\Lambda(a/2) = d_3 = \sqrt{-g_2/4}$, *where* g_2 *is the invariant associated to the lattice* Λ. *In particular, for the standard lattice* Γ, $\wp_\Gamma(b/2) = 1$ *and* $\wp'_\Gamma(b/2) = -2\sqrt{2}$.

Proof. By Equation 3.2 we have that $\wp_\Gamma(b/2) = \sqrt{-g_2/4} = 1$ since $e_3 = 0$. Moreover, by Equation 2.1 we have that

$$(\wp'_\Gamma)^2 = 4 - g_2 = 8,$$

and since the function is decreasing at $b/2$, the result follows. □

Using the second equation in (2.2) we obtain the following corollary.

Corollary 3.3. *For any lattice $\Lambda = k\Gamma$, $\wp'_\Lambda(a/2) = 2\sqrt{2}k^{-3}$.*

The next result gives a simple one-point test for determining if $J(\wp_\Lambda) = \mathbb{C}_\infty$. In particular it is enough to determine if the real quarter period lattice point is in $J(\wp_\Lambda)$.

Proposition 3.4. *For any lattice $\Lambda = [a + ai, a - ai] = k\Gamma$, $\wp_\Lambda(a/2) = 1/k^2$; moreover $J(\wp_\Lambda) = \mathbb{C}_\infty$ if and only if $a/2 \in J(\wp_\Lambda)$.*

Proof. By Lemma 3.2 and Equation 2.4 we have that $\wp_\Lambda(a/2) = \sqrt{-g_2/4} = ie_1$. Since for the standard lattice $e_1(\Gamma) = -i$ and for $k\Gamma$, $e_1(k\Gamma) = -i/k^2$, the result then follows since $ie_1 = 1/k^2$.

For the second statement, we have that $J(\wp_\Lambda) \neq \mathbb{C}_\infty$ if and only if there exists a Fatou component which contains at least one critical value if and only if both e_1 and e_2 are in $F(\wp_\Lambda)$. This holds if and only if ie_1 and ie_2 are in $F(\wp_\Lambda)$, by Summary 2.3(2), which by the first statement of this proposition holds if and only if $a/2 \in F(\wp_\Lambda)$. □

The next proof uses a crude estimate on the value of γ from [12], namely that $\gamma > 1$, to produce one specific square rhombic lattice with all repelling fixed points.

Proposition 3.5. *Denoting $\gamma = \sqrt{2}b$, with b as above, and letting $k = (2/b)^{1/3}$, we have that using the lattice $\Lambda = k\Gamma$ gives a map such that $\wp_\Lambda(a/2) = a/2$ and $\wp'_\Lambda(a/2) = -\gamma = -\sqrt{2}b < -1$.*

Proof. By Lemma 3.2 we have $\wp_\Gamma(b/2) = 1$ and $\wp'_\Gamma(b/2) = -2\sqrt{2}$, so using $k = (\frac{2}{b})^{(1/3)}$, we have the following:

$$\wp_{k\Gamma}(kb/2) = \frac{1}{k^2}\wp_\Gamma(b/2) = \frac{1}{k^2} = (\frac{b}{2})^{(2/3)}, \tag{3.3}$$

and $kb/2 = (\frac{b}{2})^{(2/3)}$ so the quarter lattice point is fixed.

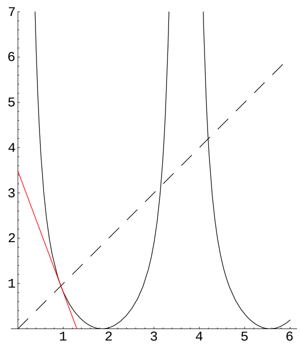

FIGURE 3. The fixed point for the standard lattice

Using Equation 2.2 to compute the derivative we have

$$\wp'_{k\Gamma}(kb/2) = 1/k^3(-2\sqrt{2}) = -b\sqrt{2} = \gamma$$

as claimed.

□

We now turn to the proof of Theorem 3.1, which follows easily from the results proved above.

Proof of Main Theorem

Proof. By Proposition 3.5 we have that $\wp_\Lambda(a/2)$ is a repelling fixed point is therefore in $J(\wp_\Lambda)$. Then by Theorem 3.4 we have that $J(\wp_\Lambda) = \mathbb{C}_\infty$.

□

We conjecture that for a real rhombic lattice, all fixed points of $\wp_\Lambda$ are repelling. The next few results establish this for large enough lattices.

Lemma 3.6. *Suppose $\Lambda = [a + ai, a - ai]$, $a > 0$. Let $D = \{z \in \mathbb{C} : |z| < \sqrt{2}a\}$, the largest open disk centered at 0 and containing no other lattice points. Then the Laurent series for $\wp_\Lambda(z)$, valid for $z \in D$, is*

$$\wp_\Lambda(z) = \frac{1}{z^2} + \frac{g_2}{30}z^2 + \frac{g_2^2}{2700}z^6 + \sum_{k=4}^{\infty} a_{2k}z^{2k}, \tag{3.4}$$

with a_n a polynomial in g_2.

In [7] it was shown that for a real square lattice, any nonrepelling fixed point is necessarily the smallest positive real one. In particular, a fixed point of $\wp_\Lambda$ in our setting lies in the interval $(0, a)$, so this is where a nonrepelling one would occur. This follows from the piecewise montonicity of $\wp_\Lambda$ and the fact that $\wp'_\Lambda$ is increasing.

Corollary 3.7. *As $a \to \infty$, $g_2 \to 0$, and the real fixed point in $z_0 \in D$ tends to 1 with $\wp_\Lambda(z_0) \to -2$. Therefore for a large enough, the smallest fixed point for $\wp_\Lambda$ in $\mathbb{R}$ is repelling.*

Proof. The function $h(z) = \frac{1}{z^2}$ has one real fixed point at $z_0 = 1$ with $h'(z_0) = -2$. Since the coefficients g_2 vary holomorphically with the lattice, by Lemma 2.6 the result follows. □

For small lattices, the evidence is even more overwhelming that all fixed points are repelling, as is shown in Figure 4. We conclude with the observation that if all fixed points are repelling for $\wp_\Lambda$, then either the Julia set is the whole sphere or it has a periodic cycle of period > 1 with derivative ± 1, because the hyperbolic case is ruled out by results from [9]. Therefore $\wp_\Lambda$ has a disconnected Julia set with very unusual properties, which are the subject of further study by the author; we conjecture this cannot happen.

References

1. Baker I N, Kotus J, Lü Y (1991) Iterates of meromorphic functions I. ETDS **11**: 241-2481
2. Bergweiler W (1993) Iteration of meromorphic functions. Bull. Amer. Math. Soc **29**, no. 2: 151 - 188
3. Devaney R, Keen L (1988) Dynamics of Tangent, Dynamical Systems, Springer Lecture Notes in Math, #1342, 105-111
4. Devaney R, Keen L (1989) Dynamics of meromorphic maps with polynomial Schwarzian derivative. Ann. Sci. Ecole Norm. Sup. (4), **22**: 55 - 81
5. DuVal P (1973) Elliptic Functions and Elliptic Curves: Cambridge University Press
6. Hawkins J, Look D (2005) Locally Sierpinski Julia sets of Weierstrass elliptic $\wp$ functions, to appear, Intl. Jour. of Bifurcations and Chaos.
7. Hawkins J, Koss L (2002) Ergodic properties and Julia sets of Weierstrass elliptic functions. Monatsh. Math., **137** no. 4: 273 - 300.
8. Hawkins J, Koss L (2004) Parametrized Dynamics of the Weierstrass Elliptic Function, Conf. Geom. Dyn. **8** 1-35.
9. Hawkins J, Koss L Connectivity Properties of Julia sets of Weierstrass Elliptic Functions, to appear, Top. and its Appl.

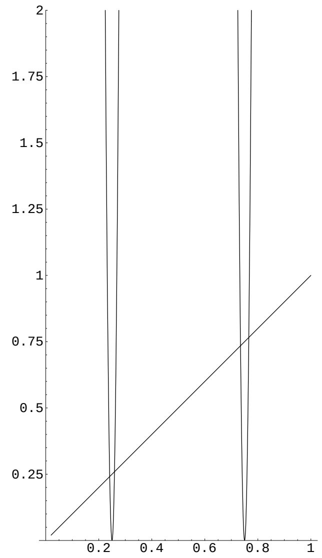

FIGURE 4. Fixed points of $\wp_\Lambda$ for a small lattice ($a = \frac{1}{4}$)

10. Jones G, Singerman D (1997) Complex Functions: An algebraic and geometric viewpoint: Cambridge Univ. Press
11. Kotus J, Urbanski M (2003) Hausdorff dimension and Hausdorff measures of Julia sets of elliptic functions. Bull. London Math. Soc. **35**, 269-275
12. Milne-Thomson L. (1950) Jacobian Elliptic Function Tables: Dover Publications, Inc
13. Milnor J (2000) Dynamics in One Complex Variable: Friedr. Vieweg & Sohn, Verlagsgesellschaft mbH.
14. Milnor J (2004) On Lattès Maps, Stony Brook IMS preprint #2004/01.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL, CB #3250, CHAPEL HILL, NORTH CAROLINA 27599-3250
E-mail address: jmh@math.unc.edu