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On C^2-Diffeomorphisms of the Circle which are of Type III_1

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Abstract. For every irrational number $\alpha \in [0, 1)$ which is not of constant type we construct a C^2-diffeomorphism of the circle with rotation number α which is of type III_1. This diffeomorphism can be chosen arbitrarily close to the rotation R_α. Our methods also allow us to construct, for every Liouville number $\alpha \in [0, 1)$, a C^∞-diffeomorphism of the circle with rotation number α which is of type III_1.

§ 1. Introduction

In [4], Katznelson constructs ergodic C^∞-diffeomorphisms of the circle which do not admit any σ-finite invariant measure equivalent to Lebesgue measure. The rotation numbers of these diffeomorphisms are Liouville numbers of a very special kind and are, in particular, contained in a subset of the circle of Lebesgue measure zero. Subsequently Herman [1] proved that the set of C^∞-diffeomorphisms of the circle which are of type III_1 form a dense G_δ in F^∞, the Baire space formed by taking the C^∞-closure of the set of C^∞-diffeomorphisms of the circle with irrational rotation numbers. Herman's theorem implies, however, that the rotation numbers of all these type III_1-diffeomorphisms must be contained in a set of measure zero (cf. [2, Chap. 5]). The methods of [2] also show that the situation remains basically unchanged if one looks at C^3-diffeomorphisms of the circle. C^2-diffeomorphisms, however, exhibit a different behaviour, and this is the starting point for this paper. Using a modification of Katznelson's construction we obtain, for every irrational number $\alpha \in [0, 1)$ which is not of constant type, a C^2-diffeomorphism of the circle of type III_1 with rotation number α. To formulate our result more clearly we need some terminology. We write $T^1 = R/Z = [0, 1)$ for the circle and denote, for every $r \geqq 0$, by $D^r(T^1)$ the group of orientation preserving C^r-diffeomorphisms of T^1. The rotation number of $f \in D^r(T^1)$ will be written as $\rho(f)$, and will be norma-

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lized so that $0 \leqq \rho(f) < 1$. Under this assumption the map $\rho: D^r(T^1) \to T^1$ is continuous for every $r \geqq 0$.

We remind the reader that $f \in D^2(T^1)$ is always non-singular with respect to Lebesgue measure m on T^1, and ergodic if and only if $\rho(f)$ is irrational. For every $\alpha \in T^1$, R_α will denote rotation by α. If $f \in D^r(T^1)$ is of the form $f = h^{-1} R_\alpha h$, where $h: T^1 \to T^1$ is a C^k-diffeomorphism of T^1, we say that f is C^k-conjugate to R_α (clearly $\rho(f) = \alpha$).

Turning briefly to continued fraction expansion, we consider $\alpha \in [0, 1)$ and write $\alpha = [a_1, a_2, \ldots) = 1|(a_1 + 1|(a_2 + \ldots))$ for its continued fraction expansion. The n-th convergent of α is denoted by p_n/q_n, where p_n and q_n satisfy the well known recursion formulae $p_0 = 0$, $p_1 = q_0 = 1$, $q_1 = a_1$, $p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$, for every $n \geqq 2$.

1.1. **Definition.** *Let $\alpha \in T^1 \setminus Q$ (Q is the set of rational numbers), and let $\alpha = [a_1, a_2, \ldots)$ be its continued fraction expansion. We say that α is of constant type if $\sup_n a_n < \infty$.*

Next we introduce the notion of a *ratio set* (cf. [5]). Let $f \in D^2(T^1)$ with $\rho(f) \notin Q$. A non-negative real number t is said to lie in the ratio set $r^*(f)$ of f if, for every Borel set $B \subset T^1$ with $m(B) \neq 0$ and for every $\varepsilon > 0$,

$$m\left(\bigcup_{n \in Z} \left(B \cap f^n(B) \cap \left\{x \in T^1 : \left|\frac{dm f^{-n}}{dm}(x) - t\right| < \varepsilon\right\}\right)\right) > 0.$$

Put $r(f) = r^*(f) \setminus \{0\}$. One can show that $r(f)$ is a closed subgroup of the multiplicative group of positive real numbers, and that f admits a σ-finite invariant measure m' equivalent to m if and only if $r^*(f) = \{1\}$. If f has no invariant measure equivalent to m, there are three possibilities: either $r^*(f) = \{t \in R: t \geqq 0\}$, in which case f is said to be of type III_1, or $r^*(f) = \{\lambda^n: n \in Z\} \cup \{0\}$, in which case f is said to be of type III_λ, or, finally, $r^*(f) = \{0, 1\}$, in which case f is type III_0. The main result of this paper can now be expressed as follows.

1.2. **Theorem.** *For every irrational number $\alpha \in [0, 1)$ which is not of constant type there exists a diffeomorphism $f \in D^2(T^1)$ with $\rho(f) = \alpha$, which is of type III_1.*

Theorem 1.2 will be a consequence of a slightly stronger assertion (Theorem 2.4), which shows that the diffeomorphism f can be chosen arbitrarily close to the irrational rotation R_α in $D^2(T^1)$. The proof of Theorem 2.4 can be modified to yield, for every Liouville number $\alpha \in [0, 1)$, a C^∞-diffeomorphism of type III_1 with rotation number α (cf. Remark 2.6). In an earlier, unpublished version of this paper Theorem 1.2 was stated with some additional (and, as it has since turned out, unnecessary) restrictions on the rotation number α. The authors again became interested in this problem when Herman proved the remarkable result that, for every irrational number α of constant type, there exists an $\varepsilon > 0$ such that every C^2-diffeomorphism f of T^1 with $\rho(f) = \alpha$ and $\|f - R_\alpha\|_2 < \varepsilon$ has a finite invariant measure equivalent to m, in complete contrast to the situation described by Theorem 2.4 (cf. [3]). It is not known whether there exist C^2-diffeomorphisms of the circle of type III_1 which have an irrational rotation number of constant type.

§ 2. The Proof of Theorem 1.2

Let $f \in D^2(T^1)$ with $\alpha = \rho(f) \in T^1 \backslash Q$, and let $\alpha = [a_1, a_2, \ldots)$ be the continued fraction expansion of α. For every $n \geqq 0$, $P_n(f)$ denotes the partition of T^1 given by the points $\{f^j(0): 0 \leqq j < q_n\}$. We now turn to the proof of Theorem 1.2 and start with a measure theoretic proposition closely related to ([4], II, Theorem 1.1).

2.1. **Proposition.** *Let $f \in D^2(T^1)$ with $\alpha = \rho(f) \in T^1 \backslash Q$ and let, for every n, $P_n(f)$ denote the partition of T^1 described above. Suppose that the following condition holds for infinitely many n: for every $I \in P_n(f)$ there exists a Borel set $A \subset I$ and a positive integer k with*

$$f^k(A) \subset I, \tag{2.1}$$

$$m(A) \geqq 10^{-4} m(I), \tag{2.2}$$

and such that

$$10^{-3} \leqq |\log Df^k(t)| \leqq 1 \tag{2.3}$$

for every $t \in A$. Then f admits no σ-finite invariant measure equivalent to m.

Proof. If f admits a σ-finite invariant measure m' equivalent to m, we have $dm'(t) = g(t)\, dm(t)$ for some Borel function $g > 0$ on T^1 which satisfies $g \circ f(t)\, Df(t) = g(t)$ for m a.e. $t \in T^1$. There exists a $c > 0$ such that the set $E = \{t \in T^1 : c \leqq g(t) \leqq 1.001\, c\}$ has positive measure. Let t_0 be a density point of E. By definition, we have

$$\lim_{\substack{n \to \infty \\ t_0 \in I \in P_n(f)}} \frac{m(I \cap E)}{m(I)} = 1.$$

Choose n large enough so that, for $t_0 \in I \in P_n(f)$,

$$m(I \cap E) > (1 - 10^{-6})\, m(I),$$

and such that n is one of the numbers satisfying (2.1)–(2.3). From (2.2) and (2.3) we get

$$\begin{aligned}
&m(A)/3 \leqq m(f^k(A)) \leqq 3\, m(A), \\
&m(f^k(A)) \geqq m(I)/3 \cdot 10^4, \\
&m(A \cap E) \geqq m(A) - m(I \backslash E) = m(A) - (m(I) - m(I \cap E)) \\
&\qquad \geqq m(A) - 10^{-6}\, m(I) \geqq (1 - 10^{-2})\, m(A),
\end{aligned}$$

and

$$\begin{aligned}
m(f^k(A) \cap E) &\geqq m(f^k(A)) - m(I \backslash E) > m(f^k(A)) - 10^{-6}\, m(I) \\
&> m(f^k(A)) - 3 \cdot 10^{-2}\, m(f^k(A)) = (1 - 3 \cdot 10^{-2})\, m(f^k(A)).
\end{aligned}$$

This implies

$$\begin{aligned}
m(f^{-k}(f^k(A) \cap E)) &> m(f^k(A) \cap E)/3 > (1 - 3 \cdot 10^{-2})\, m(f^k(A))/3 \\
&> (1 - 3 \cdot 10^{-2})\, m(A)/9
\end{aligned}$$

and hence

$$\begin{aligned} m(A \cap E \cap f^{-k}(E)) &\geqq m(A \cap f^{-k}(E)) - m(A \setminus E) \\ &> (1 - 3 \cdot 10^{-2})\, m(A)/9 - (m(A) - m(A \cap E)) \\ &> (1 - 3 \cdot 10^{-2})\, m(A)/9 - 10^{-2}\, m(A) \\ &> 0.09\, m(A) > 0. \end{aligned}$$

Put $B = A \cap E \cap f^{-k}(E)$. Then $m(B) > 0$, and we get $10^{-3} \leqq |\log Df^k(t)|$ for every $t \in B$, from (2.3). On the other hand,

$$Df^k(t) = \frac{g \circ f^k(t)}{g(t)}$$

and hence

$$|\log Df^k(t)| = |\log g \circ f^k(t) - \log g(t)| \leqq \log 1.001 < 10^{-3},$$

since both t and $f^k(t)$ lie in E whenever $t \in B$. This contradiction proves the non-existence of an invariant measure m' equivalent to m.

2.2. *Remark.* Any reader familiar with the notion of the ratio set $r^*(f)$ (cf. § 1) will realize that we have just proved that $r(f) \neq \{1\}$. A closer look at the proof of Proposition 2.1 shows that f must either be of type III_1 or of type III_λ for some λ with $1 > \lambda \geqq e^{-1}$.

If $f\colon T^1 \to R$ or $f\colon T^1 \to T^1$ is a C^r-map we write $\|f\|_r$ for its C^r-norm. Now let $\alpha \in [0, 1)$ be a fixed irrational number which is not of constant type, and let $[a_1, a_2, \ldots)$ be its continued fraction expansion and $(p_n/q_n, n \geqq 1)$ its sequence of convergents. For every $n \geqq 1$ we put $d_n = |q_n \alpha - p_n|$.

2.3. **Proposition.** *Let $0 < \delta, \eta \leqq 1$ and let N be a positive integer. There exists a diffeomorphism $\phi \in D^\infty(T^1)$ and integers k, M with $M > N$ and $1 \leqq k < q_{M+1}$ such that $f = \phi R_\alpha \phi^{-1}$ satisfies the following conditions.*

$$\|f - R_\alpha\|_0 < \delta\eta/10^4\, q_M^2, \tag{2.4}$$

$$\|\log Df\|_0 < 0.6\, \delta/q_{M+1} < \delta\eta/10^3\, q_M, \tag{2.5}$$

$$\sup_j \|\log Df^j\|_0 < 0.2\, \delta, \tag{2.6}$$

$$\|f - R_\alpha\|_2 < \eta, \tag{2.7}$$

and

$$0.9\, q_M^{-1} \leqq m(I) \leqq 1.1\, q_M^{-1} \qquad \text{for every } I \in P_M(f). \tag{2.8}$$

Furthermore there exists, for every $I \in P_M(f)$, an interval $A(I) \subset I$ with $f^k(A(I)) \subset I$ such that

$$m(A(I)) \geqq 0.09\, q_M^{-1} \geqq 0.08\, m(I), \tag{2.9}$$

$$|\log Df^k(t)| > 0.07\, \delta \qquad \text{for every } t \in A(I), \tag{2.10}$$

and such that the distance of $A(I) \cup f^k(A(I))$ from each of the two endpoints of I is greater than $0.1\, q_M^{-1}$. (2.11)

Proof. For every $n \geqq 1$, the partition $P_n(R_\alpha)$ consists of q_n intervals; q_{n-2} of these have length $d_{n-1}+d_n$, and $a_n q_{n-1}$ have length d_{n-1}. The partition $P_{n+1}(R_\alpha)$ divides each interval I in $P_n(R_\alpha)$ into either a_{n+1} or $a_{n+1}+1$ subintervals, all of which have length d_n, except for one of the intervals at one of the two ends of I which may have length d_n+d_{n+1}. With the exception of one of the intervals at the ends of I, f^{q_n} moves every subinterval of I to one of its neighbours. From this discussion we conclude that $a_{n+1} d_n \leqq d_{n-1} \leqq (a_{n+1}+2) d_n$ and $d_{n-1} \leqq q_n^{-1} \leqq d_{n-1}(1+a_{n+1}^{-1})$ for every $n \geqq 2$. Let $M > N$ be chosen so that $a_{M+1} \geqq 1000\eta^{-1}$, where N and η are the numbers appearing in the statement of this proposition. We define a function $\phi: T^1 \to R$ by

$$\phi(t) = t + 10^{-2}\delta q_M^{-1} \sin 2\pi q_M t.$$

The map ϕ can be considered as a C^∞-diffeomorphism of T^1, and we put $f = \phi R_\alpha \phi^{-1} \in D^\infty(T^1)$. Our choice of ϕ implies the following for f:

$$\begin{aligned}\|f - R_\alpha\|_0 &\leqq 2\pi\, 10^{-2}\delta d_M / q_M \leqq 2\pi\, 10^{-2} \delta d_{M-1}/a_{M+1} q_M \\ &< 7 \cdot 10^{-2} \delta d_{M-1}/q_{M+1}, \\ \|Df - 1\|_0 &= \|(D\phi \circ R_\alpha)(D\phi)^{-1} - 1\|_0 < 0.5\,\delta d_M \leqq 0.5\,\delta/q_{M+1} < 0.5\,\delta/a_{M+1} q_M\end{aligned}$$

and

$$\begin{aligned}\|D^2 f\|_0 &\leqq \|D^2\phi \circ R_\alpha - D^2\phi\|_0 \, \|(D\phi)^{-1}\|_0^2 \\ &\quad + \|D^2\phi\|_0 \, \|(D\phi)^{-1}\|_0^3 \, \|D\phi \circ R_\alpha - D\phi\|_0 \\ &< 3\,\delta q_M d_M + \delta q_M d_M \leqq 4\delta/a_{M+1}.\end{aligned}$$

(2.4) and (2.7) are immediate consequences of these inequalities, and (2.5) follows from $\|\log Df\|_0 \leqq 10\, \|Df - 1\|_0/9 < 0.6\,\delta/q_{M+1} < 10^{-3}\delta\eta/q_M$. For every $j \geqq 0$ we also have $\|\log Df^j\|_0 \leqq 2\, \|\log D\phi\|_0 < 0.2\,\delta$, i.e. (2.6). Next we note that $P_M(f) = \{\phi(\tilde{I}): \tilde{I} \in P_M(R_\alpha)\}$. Hence $d_{M-1}\, \|(D\phi)^{-1}\|_0^{-1} \leqq m(I) \leqq (d_{M-1}+d_M)\, \|D\phi\|_0$ for every $I \in P_M(f)$. Since $\|(D\phi)^{-1}\|_0^{-1} d_{M-1} > 0.9/q_M$ and $(d_{M-1} + d_M)\, \|D\phi\|_0 \leqq 1.064\, d_{M-1} < 1.1/q_M$, we have verified (2.8).

Now let $k = \mathrm{Int}(a_{M+1}/3)\, q_M$, where Int denotes the integral part, and define, for every ℓ with $0 \leqq \ell < q_M$, an interval J_ℓ by $J_\ell = [(\ell + 0.45)/q_M, (\ell + 0.55)/q_M]$. From our choice of a_{M+1} it is clear that, for every $\tilde{I} \in P_M(R_\alpha)$, there exists a unique integer $\ell_{\tilde{I}}$, $0 \leqq \ell_{\tilde{I}} < q_M$, with $J_{\ell_{\tilde{I}}} \subset \tilde{I}$, and we put $\tilde{A}(\tilde{I}) = J_{\ell_{\tilde{I}}}$. Clearly we have, for every $\tilde{I} \in P_M(R_\alpha)$, $R_\alpha^k(\tilde{A}(\tilde{I})) \subset \tilde{I}$, and the distance of $\tilde{A}(\tilde{I}) \cup R_\alpha^k(\tilde{A}(\tilde{I}))$ from the two endpoints of $\tilde{I}$ is greater than $0.11/q_M$. Furthermore, $|\log D\phi(t + k\alpha) - \log D\phi(t)| \geqq 0.07\,\delta$ for every $t \in \tilde{A}(\tilde{I})$. We now define, for every $I \in P_M(f)$, $A(I) = \phi(\tilde{A}(\phi^{-1}(I)))$, and observe that $|\log Df^k(t)| \geqq 0.07\,\delta$ for every $t \in A(I)$, $f^k(A(I)) \subset I$, and that the distance of $A(I) \cup f^k(A(I))$ from the two endpoints of I is still greater than $0.1/q_M$. Finally we note that $m(A(I)) \geqq m(\tilde{A}(\phi^{-1}(I))) \cdot \|(D\phi)^{-1}\|_0^{-1} \geqq 0.09/q_M \geqq 0.08\, m(I)$, and this completes the proof of (2.9)–(2.11).

2.4. **Theorem.** *Let $\alpha \in [0, 1)$ be irrational and not of constant type, and let $0 < \varepsilon < 1$ be fixed. There exists a type III_1-diffeomorphism $f^* \in D^2(T^1)$ satisfying the following conditions:*

$$\rho(f) = \alpha$$

and

$$\|f^* - R_\alpha\|_2 < \varepsilon.$$

Proof. Using an induction argument we shall construct a sequence (f_n) in $D^\infty(T^1)$ which converges in $D^2(T^1)$ to a limit f^* with the required properties. The sequence (f_n) will be obtained through repeated applications of Proposition 2.3 and of the elementary inequalities

$$\|hR_\alpha h^{-1} - hgh^{-1}\|_0 \leqq \|R_\alpha - g\|_0 \, \|h\|_1, \tag{2.12}$$

$$\begin{aligned}\|\log D(hR_\alpha h^{-1}) - \log D(hgh^{-1})\|_0 \leqq \|\log Dg\|_0 \\ + \|R_\alpha - g\|_0 (1 + \|h\|_2 + \|h^{-1}\|_2),\end{aligned} \tag{2.13}$$

and

$$\|hR_\alpha h^{-1} - hgh^{-1}\|_2 \leqq 10 \, \|R_\alpha - g\|_2 (1 + \|h\|_3 + \|h^{-1}\|_3)^2, \tag{2.14}$$

for every $g, h \in D^\infty(T^1)$. For every n, f_n will be of the form $f_n = \psi_n R_\alpha \psi_n^{-1}$ with $\psi_n \in D^\infty(T^1)$.

We now choose and fix a sequence $(\delta_n, n \geqq 1)$ of real numbers satisfying $0 < \delta_n \leqq 1$ and $\delta_{2n} = 1$ for every n, and such that $\{\delta_n \colon n \geqq 1\}$ is dense in $[0, 1]$. To start the induction process, let $\eta_1 = 2^{-1} 10^{-3} \varepsilon$, $N_1 = 1$, and apply Proposition 2.3 with $\delta = \delta_1$, $\eta = \eta_1$ and $N = N_1$ to find $f = f_1 = \phi R_\alpha \phi^{-1}$, $k = k_1$, $M = M_1$ and $\{A_1(I) = A(I) \colon I \in P_M(f)\}$ satisfying (2.4)–(2.11), and put $\psi_1 = \phi$. For the general induction step we assume to have constructed $f_\ell, \psi_\ell, \eta_\ell, N_\ell, M_\ell, k_\ell$ and $\{A_\ell(I) \colon I \in P_{M_\ell}(f_\ell)\}$ for $\ell = 1, \ldots, n$. Put

$$\eta_{n+1} = 2^{-n-1} 10^{-4} \varepsilon \big(\max_{1 \leqq \ell \leqq n} (1 + \|\psi_\ell\|_3 + \|\psi_\ell^{-1}\|_3)^2 \big)^{-1} \tag{2.15}$$

and choose $N = N_{n+1}$ such that $q_{M_n+1} q_N^{-1} < 10^{-3} 2^{-n-1}$ and

$$\begin{aligned}\max_{|s-t| \leqq 2/q_N} (|\log D\psi_n(s) - \log D\psi_n(t)| + |\log D\psi_n^{-1}(s) - \log D\psi_n^{-1}(t)|) \\ < 10^{-3}/q_{M_n}.\end{aligned} \tag{2.16}$$

We apply Proposition 2.3 with $\eta = \eta_{n+1}$, $\delta = \delta_{n+1}$, $N = N_{n+1}$ to obtain $f = \phi R_\alpha \phi^{-1}$, $k = k_{n+1}$, $M = M_{n+1}$ and $\{A(I) \colon I \in P_{M_{n+1}}(f)\}$ satisfying (2.4)–(2.11). Put $\psi_{n+1} = \psi_n \phi$, $f_{n+1} = \psi_n f \psi_n^{-1} = \psi_{n+1} R_\alpha \psi_{n+1}^{-1}$ and $A_{n+1}(I) = \psi_n(A(\psi_n^{-1}(I))$ for every $I \in P_{M_{n+1}}(f_{n+1})$. (2.8), (2.9) and (2.16) yield

$$m(A_{n+1}(I)) \geqq 0.07 \, m(I), \tag{2.17}$$

and (2.10) implies that

$$|\log Df_{n+1}^{k_{n+1}}(t)| \geqq 0.06 \, \delta_{n+1} \tag{2.18}$$

for every $t \in A_{n+1}(I)$, $I \in P_{M_{n+1}}(f_{n+1})$. From (2.6) we get

$$|\log Df_{n+1}^{k_{n+1}}(t)| \leqq 0.3 \, \delta_{n+1} \tag{2.19}$$

for every $t \in A_{n+1}(I)$, $I \in P_{M_{n+1}}(f_{n+1})$. Furthermore, the distance of $A_{n+1}(I) \cup f_{n+1}^{k_{n+1}}(A_{n+1}(I))$ from the endpoints of I is greater than

$$0.1\,\|D\psi_n^{-1}\|_0^{-1}\,q_{M_{n+1}}^{-1}. \tag{2.20}$$

From (2.4), (2.5) and (2.7), together with (2.12)–(2.14), we get

$$\|f_n - f_{n+1}\|_0 \leqq 2^{-n-1}\,10^{-8}\,\varepsilon\big(\max_{1\leqq\ell\leqq n}(1+\|\psi_\ell\|_3+\|\psi_\ell^{-1}\|_3)\big)^{-1}\,q_{M_{n+1}}^{-2}, \tag{2.21}$$

$$\|\log Df_n - \log Df_{n+1}\|_0 \leqq 0.6\,\delta_{n+1}/q_{M_{n+1}+1} \tag{2.22}$$

and

$$\|f_n - f_{n+1}\|_2 < 2^{-n-1}\,10^{-3}\,\varepsilon. \tag{2.23}$$

(f_n) is thus a Cauchy sequence in $D^2(T^1)$, and we denote its limit by f^*. Clearly $\rho(f^*)=\alpha$ and $\|f^*-R_\alpha\|_2<\varepsilon$. From (2.15), (2.21) and (2.22), and from our choice of N_ℓ it follows that

$$\|f^* - f_n\|_0 \leqq 2^{-n}\,10^{-8}\,\varepsilon\big(\max_{1\leqq\ell\leqq n}(1+\|\psi_\ell\|_3+\|\psi_\ell^{-1}\|_3)\big)^{-1}\,q_{M_{n+1}}^{-2} \tag{2.24}$$

and

$$\|\log Df^* - \log Df_n\|_0 \leqq 0.6\sum_{m=n+1}^{\infty} q_{M_m}^{-1} < 10^{-3}\,q_{M_{n+1}+1}^{-1} \tag{2.25}$$

for every $n\geqq 1$. (2.24) has the following consequence: for every $I\in P_{M_n}(f_n)$ we can find a unique interval $I^*\in P_{M_n}(f^*)$ whose endpoints differ from those of I by less than

$$\begin{aligned}\max_{0\leqq\ell<q_{M_n}}\|f^{*\ell}-f_n^\ell\|_0 &\leqq q_{M_n}\|f^*-f_n\|_0\\ &< 2^{-n}\,10^{-8}\,\varepsilon\big(\max_{1\leqq\ell\leqq n}(1+\|\psi_\ell\|_3+\|\psi_\ell^{-1}\|_3)\big)^{-1}\,q_{M_{n+1}}^{-1},\end{aligned} \tag{2.26}$$

and (2.20) now implies that $A_n(I)\cup f^{*k_n}(A_n(I))\subset I^*$. Since the correspondence $I\leftrightarrow I^*$ between $P_{M_n}(f_n)$ and $P_{M_n}(f^*)$ is a bijection, we can write $A_n(I^*)$ instead of $A_n(I)$. From (2.8) we know that $m(I)\geqq\|D\psi_n^{-1}\|_0^{-1}\cdot 0.9\,q_{M_{n+1}}^{-1}$, and (2.26) yields $m(I^*)<m(I)+2^{-n+1}\;10^{-8}\varepsilon q_{M_{n+1}}^{-1}\;\|D\psi_n^{-1}\|_0^{-1}$ and hence $m(I^*)<m(I)\cdot(1+2^{-n+2}\,10^{-8})$. We conclude that

$$m(A_n(I^*))\geqq 0.07\,m(I)>0.06\,m(I^*). \tag{2.27}$$

(2.25) and (2.18) give, for every $t\in A_n(I^*)$,

$$|\log Df^{*k_n}(t)|\geqq 0.06\,\delta_n - 10^{-3}\,a_{M_n+1}^{-1}, \tag{2.28}$$

and (2.19) shows that

$$|\log Df^{*k_n}(t)|\leqq 0.03\,\delta_n + 10^{-3}\,a_{M_n+1}^{-1} \tag{2.29}$$

for every $t\in A_n(I^*)$, $I^*\in P_{M_n}(f^*)$, $n\geqq 1$. (2.27)–(2.29) allow us to apply Proposition 2.1, bearing in mind that $\delta_{2n}=1$ for every n. Remark 2.2 shows that f is either of type III_1 or of type III_λ, and an easy argument, using (2.29) and the fact that the sequence (δ_n) is dense in the unit interval excludes the second possibility. The diffeomorphism f^* constructed above is thus of type III_1, and the theorem is proved.

2.5. *Remark.* Theorem 2.4 implies Theorem 2.1.

2.6. *Remark.* In the proofs of Proposition 2.3 and Theorem 2.4 we have used the fact that, for every $\alpha \in [0, 1)$ which is not of constant type, $\lim_n \inf |q_n \alpha - p_n| \cdot q_n = \lim_n \inf q_n d_n = 0$. The proof can easily be modified to show that, for every irrational $\alpha \in [0, 1)$ with $\lim_n \inf q_n^k d_n = 0$, $k \geqq 1$, there exists a C^{k+1}-diffeomorphism f^* of T^1 of type III_1 with $\rho(f^*) = \alpha$, and that f^* can be chosen arbitrarily close to R_α in C^{k+1}-norm. In particular, if α is a *Liouville number*, i.e. if $\lim_n \inf q_n^k d_n = 0$ for every $k \geqq 1$, we can construct a C^∞-diffeomorphism f^* of type III_1 with $\rho(f^*) = \alpha$, and we can again choose f^* arbitrarily close to R_α in the C^∞-metric.

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