REMARKS ON RECURRENCE AND ORBIT EQUIVALENCE OF NONSINGULAR ENDOMORPHISMS

by

Jane M. Hawkins[1]
Department of Mathematics
University of North Carolina
Chapel Hill, N.C. 27514

and

Cesar E. Silva[2]
Department of Mathematics
Williams College
Williamstown, MA. 01267

1. INTRODUCTION

Let $(X,\mathcal{B},\mu)$ denote a σ-finite Lebesgue space. An *endomorphism* is a nonsingular countable-to-1 bimeasurable transformation from (X,μ) onto itself. The Radon-Nikodym derivative of an endomorphism T is defined as the unique (mod 0) $T^{-1}\mathcal{B}$-measurable function ω_μ satisfying $\int_{T^{-1}B}\omega_\mu \, d\mu = \mu(B)$, for all $B \in \mathcal{B}$ [D]. (We assume throughout that μ is σ-finite when restricted to the sub-σ-algebra $T^{-1}\mathcal{B}$.) One can show that $\omega_\mu(x) = d\mu/d\mu T^{-1} \circ T$. The higher derivatives of T are defined by $\omega_\mu(i,x) = \omega_\mu(x)\omega_\mu(Tx)\cdots\omega_\mu(T^{i-1}x)$ $(i>0)$, and $\omega_\mu(0,x) = 1$. A measure μ is said to be *recurrent* (with respect to an endomorphism T) if for every nonnegative measurable function f, $\sum_{i\geq 0} f(T^i x)\,\omega_\mu(i,x)$ takes only the values 0 and ∞ a.e. The following two lemmas are proved in [Si].

<u>1.1 Lemma</u>. If there exists a positive function $f \in L^1(X,\mu)$ with $\sum_{i\geq 0} f(T^i x)\,\omega_\mu(i,x) = \infty$ then μ is recurrent for T. Hence, in finite measure, recurrence of μ is equivalent to $\sum_{i\geq 0}\omega_\mu(i,x) = \infty$.

<u>1.2 Lemma</u> If an endomorphism admits an equivalent recurrent measure then it is conservative.

The endomorphism T defines an action of the amenable semigroup $\mathbb{N} = \{0,1,2,\cdots\}$ on $(X,\mathcal{B},\mu)$. If H is a second countable amenable

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group, a *cocycle* for T is a Borel function $a\colon \mathbb{N}\times X \longrightarrow H$ satisfying

$$a(m+n,x) = a(m,T^n x)\cdot a(n,x), \quad \text{for all } m,n\in\mathbb{N}, \text{ and } \mu\text{-a.e. } x\in X.$$

Every cocycle a is generated by a Borel function $\varphi\colon X \longrightarrow H$ with $\varphi(x) = a(1,x)$. Then the cocycle equation gives

$$a(n,x) = \varphi(x)\varphi(Tx)\cdots\varphi(T^{n-1}x), \quad \text{for } n > 0.$$

We shall only consider free actions, i.e., $\mu\{x\colon T^n x = x \text{ for some } n \neq 0\} = 0$. We define T^0 = identity and $a(0,x) = 1 \in H$.

If the cocycle a does not depend on x, then a is a homomorphism from $\mathbb{N}$ into H. Therefore either $a(n,x) = 1 \in H$ for all n and x, or a takes values only in a sub-semigroup of H.

Two cocycles a and b are *cohomologous* if there is a Borel function $f\colon X \longrightarrow H$ such that for each $n\in\mathbb{N}$

$$a(n,x)\cdot f(T^n x) = f(x)\cdot b(n,x), \quad \text{for a.e. } x\in X.$$

The function f is called the *transfer function*. A cocycle is called a *coboundary* if it is cohomologous to 1.

The *skew product* of T by the cocycle a is the $\mathbb{N}$-action on $X\times H$ defined by $T_a(x,h) = (Tx, h\cdot a(1,x)^{-1})$.

We will be mainly interested in the cocycle ω_μ. The skew product of T by ω_μ is the Maharam skew product T_μ on $(X\times\mathbb{R}^+, \mu\times\lambda)$ defined by $T_\mu(x,y) = (Tx, y/\omega_\mu(x))$. (Throughout λ denotes Lebesgue measure.) One can easily verify that T_μ preserves $\mu\times\lambda$.

The following theorem was proved for one-to-one transformations in [M] and for endomorphisms in [Si]

<u>1.3 Theorem</u>. μ is recurrent for T if and only if T_μ is conservaive.

2. <u>RECURRENCE</u>.

In this section we explore various recurrence properties of the Radon-Nikodym derivative cocycle in the context of noninvertible transformations.

Two measures μ and ν are said to be *cohomologous* if the cocycles ω_μ and ω_ν are cohomologous. In the invertible case, μ and ν are cohomologous if and only if they are equivalent. This is not true in the case of endomorphisms as we shall see below. However we have the following lemma.

<u>2.1 Lemma</u>. If $d\nu/d\mu$ is $T^{-1}\mathcal{B}$-measurable then the cocycles ω_μ and ω_ν are cohomologous with transfer function $d\nu/d\mu$. If the cocycles ω_μ and ω_ν are cohomologous with transfer function f then f is $T^{-1}\mathcal{B}$-measurable.

Proof: Suppose $d\nu/d\mu$ is $T^{-1}\mathcal{B}$-measurable. Let $f = d\nu/d\mu$. Then $\int_{T^{-1}A} f \circ T\, \omega_\mu\, d\mu = \int_A f d\mu = \nu(A) = \int_{T^{-1}A} \omega_\nu d\nu = \int_{T^{-1}A} \omega_\nu f\, d\mu$. Since $f \circ T\, \omega_\mu$ and $\omega_\nu\, f$ are $T^{-1}\mathcal{B}$-measurable it follows that they are equal, and hence the cocycles are cohomologous. The second part is clear. □

If ω_μ and ω_ν are cohomologous by f then $\varphi(x,y) = (x, y\cdot f(x))$ gives an isomorphism of the corresponding skew products (see [Sc 1: Lemma 5.1] for the invertible case).

We say that a positive finite function f is a *μ-density* (for T) if $f(x) = f(Tx)\omega_\mu(x)$. In other words, μ admits a μ-density if and only if $\omega_\mu(x)$ is a coboundary. It follows that if ω_μ and ω_ν are cohomologous with transfer function g, and f is a μ-density then fg is a ν-density.

<u>2.2 Lemma</u>. T admits a μ-density if and only if there exists an equivalent invariant measure ν such that $d\nu/d\mu$ is $T^{-1}\mathcal{B}$-measurable.

Proof: If there exists a μ-density f then clearly f is $T^{-1}\mathcal{B}$-measurable and ν given by $d\nu = f d\mu$ is invariant. Now assume $d\nu/d\mu$ is $T^{-1}\mathcal{B}$-measurable; that $f = d\nu/d\mu$ is a μ-density follows from the fact that for all measurable sets A, $\int_{T^{-1}A} f \circ T\omega_\mu d\mu = \int_A f\, d\mu$.
$= \int_{T^{-1}A} f d\mu$ □

<u>2.3 Corollary</u>. If T admits a μ-density f and T is conservative then μ is recurrent for T.

Proof: Let $d\nu = f d\mu$. Then ω_μ and ω_ν are cohomologous and so they induce isomorphic skew products. Now $T_\nu(x,y) = (Tx,y)$ is clearly conservative, hence T_μ is conservative and Theorem 1.3 gives that μ is recurrent. □

We now compare μ recurrence with two other notions of recurrence all of which are known to be equivalent in the invertible case for finite measure. (cf. [M; Sc 1; Sc 2]).

We say that the cocycle $a: \mathbb{N} \times X \longrightarrow H$ is *recurrent* if for every measurable set A of positive measure and every $\varepsilon > 0$ we have

$$\mu(\bigcup_{n\in\mathbb{N}} (A \cap T^{-n}A \cap \{x: a(n,x) \in N_\varepsilon(1)\})) > 0,$$

where $N_\varepsilon(1)$ is an ε-neighborhood around 1 in H.

As in the invertible case, one can show that for conservative endomorphisms coboundaries are recurrent, so recurrence of cocycles is a cohomology invariant. But unlike the invertible case, recurrence of cocycles is not invariant under change to an equivalent (non cohomologous) measure.

<u>2.4 Lemma</u>. For any endomorphism T on (X,μ) there exists a cohomologous finite measure ν.

Proof: First note that since $\int_X g \circ T \, d\mu = \int_X g \, d\mu T^{-1}$ and μT^{-1} is σ-finite, there exists a positive function g with $\int_X g \circ T \, d\mu < \infty$. Let $f = g \circ T$ and $d\nu = f d\mu$. then $d\nu/d\mu$ is $T^{-1}\mathcal{B}$-measurable and Lemma 2.1 completes the proof. □

The following result appears in [Sc 1: Theorem 5.5] in the case of invertible transformations.

<u>2.5 Theorem</u>. Let T be a conservative endomorphism of a σ-finite measure space. The cocycle ω_μ is recurrent if and only if T_μ is conservative.

Proof: The statement of the theorem remains invariant under change to a cohomologous measure. By Lemma 2.4 we can pass to a cohomologous finite measure and so we can assume that μ is finite. Now assume that ω_μ is recurrent; then $\sum_{i\geq 0}\omega_\mu(i,x) = \infty$ and so Lemma 1.1 gives that μ is recurrent for T. Theorem 1.3 implies T_μ is conservative. For the converse let $\mu(A) > 0$, and for $\varepsilon > 0$ write $N_\varepsilon(1) = \{t \in \mathbb{R}^+ : e^{-\varepsilon} < t < e^{\varepsilon}\}$. Then $\mu^*(A \times N_{\varepsilon/2}(1)) > 0$, where $\mu^* = \mu \times \lambda$. The conservativity of T_μ implies that there exists $n > 0$ such that $\mu^*(A \times N_{\varepsilon/2}(1) \cap T_\mu^{-n}(A \times N_{\varepsilon/2}(1))) > 0$. Since $[A \cap T^{-n}A \cap \{x:\omega_\mu(n,x) \in N_\varepsilon(1)\}] \times N_{\varepsilon/2}(1) \supseteq A \times [N_{\varepsilon/2}(1) \cap T_\mu^{-n}(A \times N_{\varepsilon/2}(1))]$ it follows that ω_μ is recurrent. □

A cocycle a is *limit recurrent* if for a.e. x and all $\varepsilon > 0$ there exist infinitely many integers n such that $|a(n,x) - 1| < \varepsilon$. It can be shown (Theorem 2.5 and [Si: Theorem 3]) that a recurrent cocycle is limit recurrent. In finite measure, if ω_μ is limit recurrent then μ is recurrent for T (Lemma 1.1) and by Theorem 1.3 T_μ is conservative, and so ω_μ is recurrent cocycle (Theorem 2.5).

<u>2.6 Example</u>. The modified Boole tranformation on the real line $T(x) = (1/2)(x - 1/x)$ preserves the finite measure $\mu = dx/(1+x^2)$ [L] and is ergodic (see e.g. [Aa]). However one can easily compute that $\omega_\lambda = 1/2$. It follows that ω_λ is not limit recurrent and not recurrent. Therefore by Corollary 2.3 T does not admit a λ-density, and $d\mu/d\lambda$ is not $T^{-1}\mathcal{B}$-measurable. The measures μ and λ are equivalent but whereas T_μ is conservative T_λ is not, so the skew products are not isomorphic (μ and λ are not cohomologous). We note that for any n-to-1 conservative ergodic measure preserving endomorphism one can construct an equivalent measure ν with $\omega_\nu = 1/2$ [ES].

Therefore the existence of an equivalent σ-finite invariant measure for a conservative endomorphism T on (X,μ) does not imply that ω_μ is a coboundary. In Example 2.6 ω_λ has values in a semigroup not containing the identity.

In analogy with the invertible case, given a conservative endomorphism T on $(X,\mathcal{B},\mu)$ we define the *ratio set* of T *with respect to* μ [K], $r_\mu(T)$, as follows.

$$r_\mu(T) = \{\lambda \in \mathbb{R}^+ \cup \{0\}: \text{for every } A \in \mathcal{B},\ \mu(A) > 0 \text{ and every } \varepsilon > 0 \text{ there exists } n \geq 1 \text{ such that } \mu(A \cap T^{-n}A \cap \{x: \omega_\mu(n,x) \in N_\varepsilon(\lambda)\}) > 0\},$$

where $N_\varepsilon(\lambda) = \lambda N_\varepsilon(1)$ if $\lambda \neq 0$, and $N_\varepsilon(0) = \{t: t < \varepsilon\}$.

The ratio set is invariant under cohomologous measures. However Example 2.6 shows it is not invariant under equivalent measures.

<u>2.7 Proposition</u>. Let T be a conservative endomorphism of (X,μ). Then $r_\mu(T) - \{0\}$ is a sub-semigroup of $\mathbb{R}^+$ (possibly empty). Furthermore, $r_\mu(T)$ contains 1 if and only if ω_μ is recurrent.

Proof: Let $\lambda,\gamma \in r_\mu(T) - \{0\}$. Given any $A \in \mathcal{B}$, $\mu(A) > 0$ and $\varepsilon > 0$ there exists $n_1 \geq 1$ such that $\mu(A \cap T^{-n_1} A \cap \{x: \omega_\mu(n_1,x) \in N_{\varepsilon/2}(\lambda)\}) > 0$. Let $A' = A \cap T^{-n_1} A \cap \{x: \omega_\mu(n_1,x) \in N_{\varepsilon/2}(\lambda)\}$. There exists $n_2 \geq 1$ such that $\mu(A' \cap T^{-n_2} A' \cap \{x: \omega_\mu(n_2,x) \in N_{\varepsilon/2}(\lambda)\}) > 0$. This implies $\mu(A \cap T^{-(n_1+n_2)}A \cap \{x: \omega_\mu(n_1+n_2,x) \in N_\varepsilon(\lambda\gamma)\}) > 0$.

The same proof shows there exists λ such that $\lambda, \lambda^{-1} \in r_\mu(T)$ iff ω_μ is recurrent. □

We summarize the results of this section in the following theorem.

<u>2.8 Theorem</u> Let T be a conservative endomorphism of a σ-finite measure space (X,μ). The following are equivalent.

a) ω_μ is a recurrent cocycle,

b) μ is a recurrent measure,

c) T_μ is conservative,

d) $1 \in r_\mu(T)$.

Furthermore, any of a), b), c) and d) imply: e) ω_μ is limit recurrent.

If $\mu(x) < \infty$ then e) implies any of a), b), c) and d). □

3. ORBIT EQUIVALENCE.

A classification of countable ergodic amenable equivalence relations has been completed in [CFW]; this gives a notion of orbit equivalence for nonsingular ergodic endomorphisms. If we define an equivalence relation on X by $(x,y) \in R_T$ if $T^m x = T^n y$ for some nonnegative m and n it is shown in [CFW] that this relation is amenable ergodic. Then we say that T on (X,μ) and S on (X',μ') are *orbit equivalent* if there exists a nonsingular invertible (a.e.) Borel map $\psi: X \longrightarrow X'$ such that $(x,y) \in R_T$ if and only if $(\psi x, \psi y) \in R_S[Z]$.

Because many interesting questions arise concerning ω_μ (which is not the same as the Radon-Nikodym derivative cocycle of R_T), one is motivated to look at other definitions of equivalence of endomorphisms. Generalizing from group actions to semigroup actions we have the following. We write $x \le_T y$ if $T^n x = y$ for some $n \in \mathbb{N}$. Let $O^+(x)$ denote $\{x, Tx, T^2x, \cdots\}$.

3.1 Lemma. $\le_T$ defines a partial order on X. On the periodic points of X it is an equivalence relation.

Two points x and y are T-comparable, we write $x \approx_T y$, if $x \le_T y$ or $y \le_T x$. The following proposition follows from the defintions.

3.2 Proposition. Let T be an ergodic endomorphism on (X,μ). The following are equivalent.

(1) For some $k \in \mathbb{N}$, $T^k x \approx_T T^k y$.

(2) $O^+(x) \cap O^+(y) \neq \phi$.

(3) $O^+(x)$ and $O^+(y)$ differ by finitely many points.

(4) $(x,y) \in R_T$.

We say that the endomorphisms T on (X,μ) and S on (X',μ')

are *forward orbit equivalent* if there exists a nonsingular invertible (a.e.) Borel map $\psi: X \longrightarrow X'$ such that $x \leq_T y$ if and only if $\psi x \leq_S \psi y$.

3.3 Proposition. S and T are forward orbit equivalent if and only if there exist cocycles $\alpha: \mathbb{N} \times X \longrightarrow \mathbb{N}$ and $\beta: \mathbb{N} \times X' \longrightarrow \mathbb{N}$ satisfying $S^{\alpha(n,x)}\psi x = \psi(T^n x)$ and $S^m \psi x = \psi(T^{\beta(m,\psi x)}x)$ for all m,n and a.e. x.

Proof: Given $x \in X$, for each n we have $x \leq_T T^n x$. This implies $\psi x \leq_S \psi(T^n x)$, so $S^m \psi x = \psi(T^n x)$ for some m dependent on n and x, i.e., $m = \alpha(n,x)$. Since $x \leq_T T^k x$ and $T^k x \leq_T T^{k+j}x$ then

$$S^{\alpha(k+j,x)}\psi x = \psi(T^{k+j}x) = S^{\alpha(j,T^k x)} S^{\alpha(k,x)} \psi x.$$

So we obtain the cocycle equation $\alpha(k+j,x) = \alpha(j,T^k x) + \alpha(k,x)$. The existence of β is obtained in a similar way using the invertibility of ψ. The converse is straightforward. □

3.4 Theorem. S and T are forward orbit equivalent if and only if they are isomorphic.

Proof: Suppose S and T are forward orbit equivalent. By Proposition 3.3 there exist a nonsingular invertible ψ and cocycles α and β such that $S^{\alpha(n,x)}\psi x = \psi(T^n x)$ and $S^m \psi x = \psi(T^{\beta(m,\psi x)}x)$ for all m,n and a.e. $x \in X$. Then $S^n \psi x = \psi(T^{\beta(n,\psi x)}x) = S^{\alpha(\beta(n,\psi x),x)}\psi x$. Since S is aperiodic, $n = \alpha(\beta(n,\psi x),x)$ for all n and x. In particular, $1 = \alpha(\beta(1,\psi x),x) = \alpha(1,x) + \cdots + \alpha(1,T^{\beta(1,\psi x)-1}x)$. This implies that for each x either $\beta(1,\psi x)-1 = 0$ or $\alpha(1,T^i x) = 0$ for all but one i. However, since the action is free, $\alpha(1,x) = 0$ only on a set of measure 0; in particular on a set of full measure $\alpha(1,T^i x) > 0$ for all i. Therefore $\beta(1,\psi x) = 1$, so $\beta(n,x') = n$ for all n and a.e. $x' \in X'$. It immediately follows that $S^n \psi = \psi T^n$. Conversely, if S and T are isomorphic we can choose $\beta(1,x') = 1 = \alpha(1,x)$ in Proposition 3.4. □

The endomorphisms $x \longrightarrow 2x \pmod 1$ and $x \longrightarrow 3x \pmod 1$ are examples of ergodic endomorphisms which are finite measure preserving. Since they are not isomorphic they cannot be forward orbit equivalent. This is in contrast to the situation for invertible ergodic transformations where Dye's theorem [Dy] states that all finite measure preserving automorphisms are orbit equivalent.

Furthermore, it follows from [ES] that $x \longrightarrow 2x \pmod 1$ admits a measure μ equivalent to λ with $\omega_\mu = 1/2$. Since $\omega_\lambda = 1$ the ratio sets are different even though the two systems are isomorphic. In view of Theorem 3.4 we might consider alternative definitions of forward orbit equivalence. It is possible that a hierarchy analogous to the rich theory developed in [R] also exists for endomorphisms. These ideas will be further developed in a joint paper by the authors and Stanley Eigen.

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