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A Construction of a Non-Measure-Preserving Endomorphism Using Quotient Relations and Automorphic Factors*

Karma Dajani

Department of Mathematics, Universiteit Utrecht, The Netherlands

and

Jane Hawkins

Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, North Carolina 27599

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1. INTRODUCTION

The main purpose of this paper is to construct ergodic, nonsingular, conservative n-to-one endomorphisms which preserve no equivalent σ-finite measure. While such examples are known to exist [6, 9, 12, 19], our examples are fundamentally different from the existing ones. It was noticed independently in [6, 19] that the two-sided type III Bernoulli shift of Hamachi [9] is the natural extension of an exact two-to-one endomorphism with no equivalent σ-finite invariant measure. Also in [12] it is shown that Cartesian products of finite measure-preserving exact with type III automorphisms yield examples of n-to-one endomorphisms with no equivalent invariant measure. In Section 5 we give an example of a two-to-one endomorphism whose ergodic nonsingular measure is neither exact nor is it equivalent to a product measure of an automorphism with an exact endomorphism.

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Our construction is based on the connections between the Radon–Nikodym derivative of the original endomorphism and that of its maximal automorphic factor. Using a chain rule for factor maps, we prove a theorem giving a condition for a maximal automorphic factor which does not preserve any equivalent measure to force the endomorphism to have that same property (Theorem 4.5 of this paper). As an important step in proving that the hypotheses of Theorem 4.5 are satisfied, we characterize the maximal automorphic factor of an endomorphism as the action of a quotient relation of two natural relations present in every endomorphism. In 1989, Feldman, Sutherland, and Zimmer studied ergodic relations, subrelations, and quotient relations [8]. They are also discussed in an earlier paper by K. Schmidt [17]. In this paper, we compute the quotient relation of the orbit relation for a noninvertible ergodic countable-to-one endomorphism by a natural subrelation which is trivial precisely when the map is invertible. We show that we obtain the group of integers acting by the maximal automorphic factor of the endomorphism as the quotient. We use duality properties of the relations to characterize exactness of an endomorphism in Proposition 3.8; this extends an earlier result from [1].

Some results about measure theoretic properties of ergodic endomorphisms are proved in Section 2; the characterization of the maximal automorphic factor as an action of a quotient relation is presented in Section 3. The construction is done in Sections 4 and 5. This paper is the third in a study conducted by the authors on nonsingular endomorphisms, their factors, and their natural extensions (cf. [5, 6]).

2. PRELIMINARY MEASURE THEORETIC RESULTS ABOUT NONSINGULAR ENDOMORPHISMS

Throughout this paper we will assume that $(X, \mathscr{B}, \mu)$ is a Lebesgue probability space and $T: X \to X$ is a nonsingular conservative ergodic endomorphism which is surjective and countable-to-one μ almost everywhere. (The assumption that $\mu(X) < \infty$ results in no loss of generality when T is n-to-one; this is shown in Lemma 2.4.) By a result of Rohlin [16] (see also [20]), we can assume by replacing X by a measurable T-invariant subset of full measure if necessary that T is forward nonsingular as well, so that T satisfies that for all $A \in \mathscr{B}$, $\mu(A) = 0 \Leftrightarrow \mu(T^{-1}A) = 0 \Leftrightarrow \mu(TA) = 0$. We apply a well-known result of Rohlin [16] to obtain a measurable partition $\zeta = \{A_1, A_2, A_3, \dots\}$ of X into at most countably many pieces satisfying:

(i) $\mu(A_i) > 0$ for each i;

(ii) the restriction of T to each A_i, which we will write as T_i, is one-to-one;

(iii) each A_i is of maximal measure in $X \setminus \bigcup_{j<i} A_j$ with respect to property (ii);

(iv) T_1 is one-to-one and onto X.

When we say that the endomorphism T is n-to-one, we mean that every partition $\zeta = \{A_1, A_2, A_3, \dots\}$ satisfying (i)–(iv) contains precisely n atoms and that T_i is one-to-one and onto X for each $i = 1, \dots, n$. Equivalently, for μ-a.e. $x \in X$, the set $T^{-1}x$ contains exactly n points.

For each $x \in A_i$, let $J_{\mu T_i}(x) = (d\mu\, T_i / d\mu)(x)$, and for $x \in X$, let $J_{\mu T}(x) = \sum_i J_{\mu T_i} \chi_{A_i}(x)$. This is the Jacobian function for T, defined by W. Parry [15], and is independent of the choice of ζ. Our nonsingularity assumptions imply that $J_{\mu T} > 0$ μ a.e. In order to define the Radon–Nikodym derivative of T, we consider the following identities holding μ a.e. (see [10] or [18]):

$$\theta_{\mu T}(x) \equiv \frac{d\mu\, T^{-1}}{d\mu}(x) = \sum_{y \in T^{-1}x} \frac{1}{J_{\mu T}(y)}; \tag{1}$$

$$\omega_{\mu T}(x) \equiv \frac{d\mu}{d\mu\, T^{-1}}(Tx) = \frac{1}{\theta_{\mu T}(Tx)}. \tag{2}$$

The function $\omega_{\mu T}$ satisfies for every $f \in L^1(X, \mathscr{B}, \mu)$,

$$\int_X f(Tx) \cdot \omega_{\mu T}(x)\, d\mu(x) = \int_X f(x)\, d\mu(x). \tag{3}$$

For any function ω satisfying (3) in place of $\omega_{\mu T}$, we say that ω is Markovian for T and μ, and it was shown in [19] that $\omega_{\mu T}$ is the only $T^{-1}\mathscr{B}$ measurable function which is Markovian for T and μ. Thus, if ω is a $\mathscr{B}$ measurable function which is Markovian for T and μ then $\omega \in L^1(X, \mathscr{B}, \mu)$ and $E_\mu(\omega \mid T^{-1}\mathscr{B}) = \omega_{\mu T}$. Here $E_\mu(h \mid T^{-1}\mathscr{B})$ denotes the conditional expectation of $h \in L^1(X, \mathscr{B}, \mu)$ with respect to the sub-σ-algebra $T^{-1}\mathscr{B}$. We call the function $\omega_{\mu T}$ the *Radon–Nikodym derivative of* T. Similarly, $\omega_{\mu T^k}(x) = (d\mu / d\mu\, T^{-k})(T^k x)$ is the unique $T^{-k}\mathscr{B}$ measurable function which is Markovian for T^k and μ. It is easily seen that $\omega_\mu(k, x) \equiv \omega_{\mu T}(x) \cdot \omega_{\mu T}(Tx) \cdot \cdots \cdot \omega_{\mu T}(T^{k-1}x)$ is a $\mathscr{B}$ measurable function which is Markovian for T^k and μ so that $E_\mu(\omega_\mu(k, \cdot) \mid T^{-k}\mathscr{B}) = \omega_{\mu T^k}$.

From these observations we show that even though we do not have a chain rule we have a related identity which we call the Pseudo-Chain rule (Proposition 2.3 below). Before we can prove that, we generalize a result mentioned in [12].

LEMMA 2.1. *Assume that T is a nonsingular endomorphism which is n-to-one on $(X, \mathscr{B}, \mu)$, with μ a σ-finite measure on $\mathscr{B}$. Then for every $k \in \mathbb{N}$, μ is σ-finite on $T^{-k}\mathscr{B}$.*

Proof. We fix any $k \geq 1$. The endomorphism T^k is $m = n^k$-to-one so we can find a measurable partition $\zeta = \{A_1, A_2, A_3, \ldots, A_m\}$ such that the restriction of T^k to each atom is one-to-one. Given any set $B \in T^{-k}\mathscr{B} \subseteq \mathscr{B}$, since μ is σ-finite on $\mathscr{B}$, we consider the set $B \cap A_1 = B_1$ and we write it as $B_1 = \bigcup_{j_1=1}^{\infty} B_{1, j_1}$ with $\mu(B_{1, j_1}) < \infty$. Now the B_{1, j_1}'s determine, via symmetric points, a unique countable partition of $B \cap A_2 = B_2$ (in fact of $B \cap A_p$ for every $p = 1, \ldots, m$). We write $B_2 = \bigcup_{j_1=1}^{\infty} B_{2, j_1}$, where $B_{2, j_1} = T_2^{-k}(T^k B_{1, j_1})$; if $\mu(B_{2, j_1}) = \infty$ for any j_1, we subdivide further if necessary since μ is σ-finite. We write $B_2 = \bigcup_{j_1=1}^{\infty} \bigcup_{j_2=1}^{\infty} B_{2, j_1 j_2}$, with $B_{2, j_1} = \bigcup_{j_2=1}^{\infty} B_{2, j_1 j_2}$ and each $B_{2, j_1 j_2}$ of finite measure; we use this partition to refine the partition of B_1 obtained previously. We proceed inductively, refining all previous partitions at each step, and this process stops after a finite number of steps, when we reach A_m. Finally we write $B = \bigcup_{j_1=1}^{\infty} \bigcup_{j_2=1}^{\infty} \cdots \bigcup_{j_m=1}^{\infty} C_{j_1 j_2 \ldots j_m}$, with $C_{j_1 j_2 \ldots j_m} = \bigcup_{p=1}^{\infty} B_{p, j_1 j_2 \ldots j_m} \in T^{-k}\mathscr{B}$ and $\mu(C_{j_1 j_2 \ldots j_m}) < \infty$. ∎

Remark 2.2. Lemma 2.1 is false when T is countable-to-one. A counterexample is $X = \mathbb{R}$ with $\mu =$ Lebesgue measure, and $Tx = \tan x$. Lebesgue measure is clearly σ-finite on $\mathbb{R}$ with respect to the σ-algebra $\mathscr{B}$ of Borel sets, but it is not σ-finite with respect to $T^{-1}\mathscr{B}$.

In [10] a generalization of the conditional expectation operator onto $T^{-1}\mathscr{B}$, denoted E_μ^1, is defined as a linear operator on the space of measurable functions on X when T is finite-to-one and μ is σ-finite. It is defined by $E_\mu^1(h)(x) = \sum_{y \in T^{-1}(Tx)} (h(y)/J_{\mu T}(y)) \cdot \omega_{\mu T}$. Similarly, we define for each $k \in \mathbb{N}$, $E_\mu^k(h)(x) = \sum_{y \in T^{-k}(T^k x)} (h(y)/J_{\mu T^k}(y)) \cdot \omega_{\mu T^k}$. Clearly for each $\mathscr{B}$ measurable h, the function $E_\mu^k(h)$ is $T^{-k}\mathscr{B}$ measurable, and if $h \in L^1(X, \mathscr{B}, \mu)$, by Lemma 2.1 we have $E_\mu^1(h) = E_\mu(h \mid T^{-1}\mathscr{B})$.

PROPOSITION 2.3 (Pseudo-Chain Rule). *Assume that T is a nonsingular endomorphism which is n-to-one on $(X, \mathscr{B}, \mu)$, with μ a σ-finite measure on $\mathscr{B}$. For each $k \in \mathbb{N}$, for every $i = 1, \ldots, k-1$, and a.e. $x \in X$ we have*

$$\omega_{\mu T^k} = E_\mu^k(\omega_{\mu T^i}) \cdot \omega_{\mu T^{k-i}} \circ T^i.$$

If $\mu(X) < \infty$, and T is countable to one, then

$$\omega_{\mu T^k} = E_\mu\Big(\omega_{\mu T^i} \mid T^{-k}\mathscr{B}\Big) \cdot \omega_{\mu T^{k-i}} \circ T^i.$$

Proof. We fix any $k \in \mathbb{N}$, and choose any $i = 1, \ldots, k-1$. It is easily shown that $\omega_{\mu T^i} \cdot \omega_{\mu T^{k-i}} \circ T^i$ is a $\mathscr{B}$ measurable Markovian function for

T^k and μ. By [19, Example 1.3], $\omega_{\mu T^k}$ is the unique $T^{-k}\mathscr{B}$ measurable Markovian. Furthermore, since $\omega_{\mu T^{k-i}} \circ T^i$ is $T^{-k}\mathscr{B}$ measurable we have $E_\mu^k(\omega_{\mu T^i} \cdot \omega_{\mu T^{k-i}} \circ T^i) = E_\mu^k(\omega_{\mu T^i}) \cdot \omega_{\mu T^{k-i}} \circ T^i$. The result follows if we show that if ω is Markovian for T^k and μ, then $E_\mu^k(\omega)$ is also Markovian. Since $E_\mu^k(\omega)(x) = \sum_{y \in T^{-k}(T^k x)}(\omega(y)/J_{\mu T^k}(y)) \cdot \omega_{\mu T^k}(x)$, we have for any $f \in L^1(X, \mathscr{B}, \mu)$,

$$\int_X f(x)\, d\mu(x) = \int_X f \circ T^k(x) \cdot \omega(x)\, d\mu(x)$$

since ω is Markovian;

$$= \sum_{j=1}^{n^k} \int_X f \circ T^k\left(T_j^{-k} x\right) \cdot \omega\left(T_j^{-k} x\right) \cdot \frac{1}{J_{\mu T^k}\left(T_j^{-k} x\right)}\, d\mu(x)$$

by a change of variables and since $\{T_j^{-k} X\}_{j=1}^{n^k}$ forms a disjoint partition of X;

$$= \sum_{j=1}^{n^k} \int_X f(x) \cdot \frac{\omega\left(T_j^{-k} x\right)}{J_{\mu T^k}\left(T_j^{-k} x\right)}\, d\mu(x)$$

$$= \int_X f(x) \cdot \left(\sum_{y \in T^{-k}(x)} \frac{\omega(y)}{J_{\mu T^k}(y)} \right) d\mu(x)$$

and since $\omega_{\mu T^k}$ is Markovian,

$$\int_X f \circ T^k(x) \cdot \left(\sum_{y \in T^{-k}(T^k x)} \frac{\omega(y)}{J_{\mu T^k}(y)} \right) \cdot \omega_{\mu T^k}(x)\, d\mu(x). \qquad \blacksquare$$

The following lemma shows that the assumption that μ is a probability measure does not result in any loss of generality for n-to-one maps.

LEMMA 2.4. *If T is a nonsingular n-to-one endomorphism on $(X, \mathscr{B}, \mu')$ with μ' a σ-finite measure, then there exists a finite measure $\mu \sim \mu'$ such that $\omega_{\mu T} = (h \circ T/h) \cdot \omega_{\mu' T}$ for some $T^{-1}\mathscr{B}$ measurable function h.*

Proof. By definition [11], a measure $\mu = h\, d\mu'$ is cohomologous to μ' if $\omega_{\mu T} = (h \circ T/h) \cdot \omega_{\mu' T}$. Then μ is cohomologous to μ' if and only if h is $T^{-1}\mathscr{B}$-measurable [11], and is finite if and only if h is integrable. Therefore it suffices to find a positive $h \in L^1(X, T^{-1}\mathscr{B}, \mu')$; such an h exists since μ' is σ-finite on $T^{-1}\mathscr{B}$ by Lemma 2.1. $\blacksquare$

3. THE QUOTIENT RELATION R_T/S_T AND THE MAXIMAL AUTOMORPHIC FACTOR FOR COUNTABLE-TO-ONE MAPS T

For each countable-to-one nonsingular endomorphism T, we define two amenable equivalence relations R_T and $S_T \subseteq R_T$ (cf. [4, 10]). The relation $R_T \subseteq X \times X$ is defined as follows: $(x, y) \in R_T$ if and only if there exist $m, n \geq 1$ such that $T^n x = T^m y$. We also associate a subrelation $S_T \subseteq R_T \subseteq X \times X$ by $(x, y) \in S_T$ if and only if $T^n x = T^n y$ for some $n \geq 1$. When the endomorphism T is clearly understood, we write R and S for R_T and S_T. For $x \in X$, let $R(x) \equiv \{y \in X: (x, y) \in R\}$ and $S(x) \equiv \{y \in X: (x, y) \in S\}$. One can verify that $S(x) = \bigcup_{n \geq 0} T^{-n} T^n x$ [10]. Similarly we define for each set $A \in \mathscr{B}$, $R(A) \equiv \{y: (x, y) \in R$ for some $x \in A\}$, and we say that R is *nonsingular* if $\mu(A) = 0 \Leftrightarrow \mu(R(A)) = 0$ for all $A \in \mathscr{B}$. We say R is *ergodic* (with respect to μ) if $R(A) = A \Rightarrow \mu(A) = 0$ or $\mu(X \setminus A) = 0$. We have identical definitions for the subrelation S.

If T is one-to-one, then S is trivial (each equivalence class consists of exactly one point) and R is the usual equivalence relation associated to orbits. In [10] the following connections were proved to exist between the map T and its associated relations R and S.

(1) T is nonsingular $\Leftrightarrow R$ is nonsingular.

(2) T is ergodic $\Leftrightarrow R$ is ergodic.

(3) T is nonsingular $\Rightarrow S$ is nonsingular (and the converse is false).

(4) T is exact $\Leftrightarrow S$ is ergodic.

Also, it is easily checked that:

(5) For all $n \in \mathbb{Z}$, $S(T^n x) = T^n(Sx)$ for a.e. $x \in X$.

(6) S is a subrelation of R of infinite index if and only if T is not invertible.

We apply a technique defined by Feldman, Sutherland, and Zimmer to define the quotient relation R_T/S_T in a measurable way [8]. In general the quotient of a countable ergodic relation by a subrelation can only be described as a groupoid, but in our case we obtain a genuine group, $\mathbb{Z}$, acting naturally by T on a factor space of X. By convention $T^0 x = x$.

We first define the choice maps of a relation and subrelation, following [8]. Let $\zeta = \{A_1, A_2, A_3, \ldots\}$ denote a partition chosen to satisfy conditions (i)–(iv) of Section 1. For each $k \in \mathbb{N}$, let $\zeta^k = \{A_1^k, A_2^k, \ldots\}$ denote the partition of X defined by $\zeta^k = \bigvee_{i=0}^{k-1} T^{-i}\zeta$; it satisfies conditions (i)–(iv) for T^k, so that the restriction of T^k to A_1^k, written as T_1^k, is one-to-one and onto X. For each $n \in \mathbb{Z}$, we define $\phi_n : X \to X$ by

$$\phi_n x = \begin{cases} T^n x & \text{if } n \geq 0 \\ \left(T_1^k\right)^{-1} x & \text{if } n = -k,\, k \in \mathbb{N}. \end{cases}$$

The maps ϕ_n are called choice maps, and are defined in this way to satisfy the following easily verified properties.

LEMMA 3.1.

(1) *For every* $n \in \mathbb{Z}$, *each* ϕ_n *is measurable*.

(2) *For each* $n \in \mathbb{Z}$, *for every* $(x, y) \in S$, $(\phi_n(x), \phi_n(y)) \in S$.

(3) *For each* $x \in X$, $R(x) = \bigcup_{n=-\infty}^{\infty} S(\phi_n(x))$.

(4) $(\phi_n \circ \phi_m(x), \phi_{n+m}(x)) \in S$ *for all* $n, m \in \mathbb{Z}$.

We now consider the ergodic decomposition of μ with respect to the relation S. We obtain a Lebesgue space $(Y, \mathscr{F}, \nu)$ and a canonical system of measures $\{\mu_y\}_{y \in Y}$ on X such that for any $A \in \mathscr{B}$, $\mu(A) = \int_Y \mu_y(A)\, d\nu(y)$ [16]. Let $\alpha : (X, \mathscr{B}, \mu) \to (Y, \mathscr{F}, \nu)$ be the canonical projection; then $\alpha^{-1}\mathscr{F} \subseteq \mathscr{B}$ is the σ-algebra of S-invariant subsets. We show that the choice maps $\{\phi_n\}$ on X induce automorphisms Φ_n on Y, by defining for each $n \in \mathbb{Z}$ and $y \in Y$, $\Phi_n(\alpha x) \equiv \alpha\phi_n(x)$.

PROPOSITION 3.2. *For each* $n \in \mathbb{Z}$, *the map* Φ_n *is an automorphism of* Y.

Proof. We assume first that $n \in \mathbb{N}$ so that $\phi_n = T^n$; then by definition $\Phi_n(\alpha x) = \alpha(T^n x)$. It suffices to show that $T^{-n}\alpha^{-1}\mathscr{F} = \alpha^{-1}\mathscr{F}$ (mod 0). Let $A \in T^{-n}\alpha^{-1}\mathscr{F}$; then $A = T^{-n}B$ for some $B \in \alpha^{-1}\mathscr{F}$ and $SB = B$. We write $A = T^{-n}B = T^{-n}SB = ST^{-n}B = SA \in \alpha^{-1}\mathscr{F}$ by applying (5) above. The reverse containment is shown by taking $B \in \alpha^{-1}\mathscr{F}$; then $B = T^{-n}T^n B = T^{-n}T^n SB = T^{-n}ST^n B \in T^{-n}\alpha^{-1}\mathscr{F}$.

The proposition is trivially true for $n = 0$ since $\Phi_0 = \mathrm{Id}_X$. For $n \in \mathbb{N}$, we now prove the result for Φ_{-n}. We have shown that Φ_n is an automorphism so Φ_n^{-1} exists and is an automorphism. We claim that $\Phi_n^{-1}(\alpha x) = \alpha(\phi_{-n} x)$ a.e., hence $\Phi_{-n} = \Phi_n^{-1}$. By Lemma 3.1(4), $(x, \phi_n \circ \phi_{-n} x) \in S$ so that $\alpha(x) = \alpha(\phi_n \circ \phi_{-n} x) = \Phi_n(\alpha(\phi_{-n} x))$ for a.e. x. ∎

The following result is immediate from Lemma 3.1 and the proof of Proposition 3.2.

COROLLARY 3.3. (1) *For every* $m, n \in \mathbb{Z}$, $\Phi_n \circ \Phi_m = \Phi_{m+n}$; (2) $\Phi_0 = \mathrm{Id}$.

Therefore we will write $\Phi_1 = \Phi$ and $\Phi_n = \Phi \circ \cdots \circ \Phi$ (n times) $\equiv \Phi^n$; i.e., the family $\{\Phi^n\}_{n \in \mathbb{Z}}$ defines a $\mathbb{Z}$-action on the space $(Y, \mathscr{F}, \nu)$. The group action generated by Φ is the factor action on Y of the semigroup action generated by T on X.

We analyze more closely the partition of X given by $\eta = \{S(x)\colon x \in X\}$. In [10] it was shown that there is a one-to-one correspondence between S-invariant sets and sets in the tail field $\bigcap_{n \geq 0} T^{-n}\mathscr{B}$. From this we see that the measurable hull of η is $\eta' = \bigcap_{n \geq 0} T^{-n}\epsilon$, where ϵ is the point partition, and η' generates the tail σ-algebra $\bigcap_{n \geq 0} T^{-n}\mathscr{B}$. We denote by $X_{\eta'}$ the factor space associated with the tail σ-algebra and by $\mu_{\eta'}$ the associated factor measure. It is clear that every automorphic factor of $(X, \mathscr{B}, \mu; T)$ is contained in the tail $(X_{\eta'}, \bigcap_{n \geq 0} T^{-n}\mathscr{B}, \mu_{\eta'}; T_{\eta'})$, which is isomorphic (via the identity map) to the space Y obtained above; therefore we have just shown the following.

PROPOSITION 3.4. *The factor* $(Y, \mathscr{F}, \nu; \Phi)$ *is the maximal automorphic factor of* $(X, \mathscr{B}, \mu; T)$.

Ergodicity of the relation S is the same as exactness of the endomorphism T [10], so we have the following corollary. We remark that the measures μ_y are not necessarily nonsingular for T.

COROLLARY 3.5. *For* ν a.e. $y \in Y$, *the measure* μ_y *is tail trivial for* T *on* $(X, \mathscr{B})$; *that is, for every* $A \in \bigcap_{n \geq 0} T^{-n}\mathscr{B}$, $\mu_y(A) = 0$ *or* 1.

Remark 3.6. (1) If $T = f \times g$ and $(X, \mathscr{B}, \mu) = (X_1 \times X_2, \mathscr{B}_1 \times \mathscr{B}_2, \mu_1 \times \mu_2)$ with f an automorphism of $(X_1, \mathscr{B}_1, \mu_1)$ and g an exact endomorphism of $(X_2, \mathscr{B}_2, \mu_2)$, then Proposition 3.4 implies that $(Y, \mathscr{F}, \nu) \simeq (X_1, \mathscr{B}_1, \mu_1)$. Furthermore, the factor endomorphism induced by T on Y is f.

(2) Since for all $n \in \mathbb{Z}$, $\alpha^{-1}\Phi^n(y) = T^n\alpha^{-1}(y)$ for ν a.e. $y \in Y$, we could define $\Phi_n = \alpha \circ T^n \circ \alpha^{-1}$ instead of using the choice function of [8].

DEFINITION 3.7 [8, 17]. For a nonsingular conservative countable-to-one endomorphism T, we define the *quotient relation* R_T/S_T to be the group $\mathbb{Z}$, endowed with the action generated by Φ on $(Y, \mathscr{F})$.

The next proposition is a reflection of some duality properties of relations and subrelations discussed by Feldman, Sutherland, and Zimmer [8, Proposition 1.5]. The "only if" direction of Proposition 3.8 is also stated in [2]. By $(\mathbb{Z}, \mathscr{M}, \delta)$ we will denote the Borel space of the integers with the discrete topology and counting measure. We give a short ergodic theoretical proof instead of using [8].

PROPOSITION 3.8. *Suppose* T *is any countable-to-one ergodic nonsingular endomorphism on* $(X, \mathscr{B}, \mu)$. *Then* T *is exact if and only if the product map* $\hat{T}\colon X \times \mathbb{Z} \to X \times \mathbb{Z}$ *given by* $\hat{T}(x, m) = (Tx, m + 1)$ *is ergodic with respect to* $\hat{\mu} = \mu \times \delta$.

Proof. ($\Leftarrow$) Suppose that T is not exact. Then there exists a set A, $0 < \mu(A) < 1$, such that $A = T^{-n}(T^n A)$ for all $n \in \mathbb{N}$. We construct a nontrivial invariant set for $\hat{T}$ of the form $\hat{A} = \bigcup_{j \in \mathbb{Z}} T^j A \times \{j\}$. Clearly $\hat{\mu}(\hat{A}) > 0$ and the same is true for its complement, and it is easy to see that $\hat{T}^{-1}(\hat{A}) = A$.

($\Rightarrow$) We now suppose that T is exact, but that $\hat{T}$ is not ergodic. Then $\bigcap_{i \geq 0} T^{-i}\mathscr{B}$ is trivial, and there exists a nontrivial invariant set for $\hat{T}$, say $B \in \mathscr{B} \times \mathscr{M}$; in particular, $B \in \bigcap_{i \geq 0} \hat{T}^{-i}(\mathscr{B} \times \mathscr{M}) = \bigcap_{i \geq 0} T^{-i}\mathscr{B} \times \mathscr{M}$. Then B is of the form $X \times C$ with $C \in \mathscr{M}$, but invariance of C implies $C = \mathbb{Z}$. The contradiction implies the result. ∎

Remark 3.9. $\hat{T}$ is orbit equivalent (i.e., isomorphic as an amenable equivalence relation [4]) to a skew product over the relation R_T by a $\mathbb{Z}$-valued "index" cocycle [8, Definition 1.4]; in this setting the cocycle appears as the constant cocycle 1 for the endomorphism T. Clearly the transformation $\hat{T}$ commutes with the action of the integers on $X \times \mathbb{Z}$ given by $\lambda_k(x, m) \equiv (x, m + k)$, $k \in \mathbb{Z}$, which is generated by the automorphism $\lambda_1 \equiv \lambda$. Taking the ergodic decomposition of $X \times \mathbb{Z}$ with respect to $\hat{T}$ gives a Lebesgue space $(Y, \mathscr{F}, \nu)$ on which λ is well-defined and generates an integer action (i.e., λ is invertible). By the proof of Proposition 3.8 and the discussion above (or the duality results in [8]) we see that the automorphism λ on $(Y, \mathscr{F}, \nu)$ is isomorphic to the maximal automorphic factor of T on $(X, \mathscr{B}, \mu)$.

4. TYPE III MAXIMAL AUTOMORPHISMS AND ENDOMORPHISMS

Throughout this section we assume that the endomorphism T on $(X, \mathscr{B}, \mu)$ is n-to-one and μ is σ-finite. Using the notation above, we denote by Φ on $(Y, \mathscr{F}, \nu)$ the maximal automorphic factor of T. Letting $\nu = \mu\alpha^{-1}$ as before, the function $d\nu\,\Phi/d\nu$ denotes the Radon−Nikodym derivative of the measure $\nu\Phi$ with respect to ν. The following identity follows from [3] (and is an immediate consequence of Lemma 4.1 below): for ν a.e. $y \in Y$,

$$\frac{d\nu\,\Phi}{d\nu}(y) = E_\mu\left(\omega_{\mu T}\middle| \bigcap_{n \geq 0} T^{-n}\mathscr{B}\right)(y) \equiv E_\mu\left(\omega_{\mu T}\middle| \bigcap_{n \geq 0} T^{-n}\mathscr{B}\right)(x)$$

for any $x \in X$ such that $\alpha(x) = y$.

The next three results are known identities about factors and conditional expectation operators (cf. [3]).

LEMMA 4.1. *For every measurable function h with $h \circ T \in L^1(X, \mu)$, and* ν a.e. $y \in Y$, *we have*

$$E_\mu\left(h \circ T \,\middle|\, \bigcap_{n \geq 0} T^{-n}\mathscr{B}\right)(y) = E_\mu\left(h \cdot \theta_{\mu T} \,\middle|\, \bigcap_{n \geq 0} T^{-n}\mathscr{B}\right) \circ \Phi(y) \cdot (d\nu\, \Phi/d\nu)(y)$$

$$= E_{\mu T^{-1}}\left(h \,\middle|\, \bigcap_{n \geq 0} T^{-n}\mathscr{B}\right) \circ \Phi(y).$$

We recall that the measure μ on $(X, \mathscr{B})$ disintegrates over the factor $(Y, \mathscr{F}, \nu)$ by $\mu(B) = \int_Y \mu_y(B)\, d\nu(y)$, and for any $h \in L^1(X, \mathscr{B}, \mu)$, the function $E_\mu(h \mid \bigcap_{n \geq 0} T^{-n}\mathscr{B})(y) = \int_{\alpha^{-1}y} h(x)\, d\mu_y(x)$.

LEMMA 4.2. *The disintegration of μ satisfies $\mu_{\Phi^{-1}y}T^{-1} \sim \mu_y$ for ν* a.e. $y \in Y$.

Proof. For any set $B \in \mathscr{B}$, we apply Lemma 4.1 to the function $h = \chi_B$ and evaluate the equality at $\Phi^{-1}y$. This gives, for ν a.e. y,

$$\mu_{\Phi^{-1}y}(T^{-1}B) \cdot \frac{d\nu\, \Phi^{-1}}{d\nu}(y) = \int_{X \cap \alpha^{-1}y} \chi_B(x) \cdot \frac{d\mu\, T^{-1}}{d\mu}(x)\, d\mu_y(x).$$

Since both $(d\nu\, \Phi^{-1}/d\nu)(y)$ and $(d\mu\, T^{-1}/d\mu)(x)$ are assumed to be strictly positive functions, and $\chi_B \geq 0$ everywhere, the result follows immediately. ∎

We have the following chain rule for nonsingular endomorphisms decomposed over a factor.

PROPOSITION 4.3. *For μ a.e. $x \in X$, $(d\mu\, T^{-1}/d\mu)(x) = (d\nu\, \Phi^{-1}/d\nu)(\alpha x) \cdot (d\mu_{\Phi^{-1}(\alpha x)}T^{-1}/d\mu_{(\alpha x)})(x)$. In addition,*

$$\omega_{\mu T}(x) = \frac{d\nu\, \Phi}{d\nu}(\alpha x) \cdot \frac{d\mu_{\Phi(\alpha x)}}{d\mu_{(\alpha x)}T^{-1}} \circ T(x).$$

Using the terminology introduced by Krieger for integer actions (and originating with von Neumann factors), we define an ergodic automorphism or endomorphism T on $(X, \mathscr{B}, \mu)$ to be *type* III if it admits no invariant σ-finite measure $\nu \sim \mu$ [14]. In [10], $(\omega_{\mu T})^{-1}$ is defined to be a conditional coboundary for T if $(\omega_{\mu T})^{-1} = (h \circ T)/E^1_\mu(h)$ for some measurable function h, and it is proved there that T admits an equivalent σ-finite invariant measure if and only if $(\omega_{\mu T})^{-1}$ is a conditional coboundary. A conditional coboundary is a coboundary if and only if h is $T^{-1}\mathscr{B}$ measurable [11]. It was shown earlier that an ergodic automorphism T is type III if and only if $\omega_{\mu T}$ is not a coboundary (cf. [14]).

LEMMA 4.4. *If* $\omega_{\mu T}(x) = (d\nu\,\Phi/d\nu)(\alpha x)$ *for* μ a.e. $x \in X$, *then* $\omega_{\mu T}$ *is a* T*-coboundary on* X *if and only if* $(d\nu\,\Phi/d\nu)(\alpha x)$ *is a* Φ*-coboundary on* Y *for* Φ. *If both are coboundaries, the transfer function is* $\bigcap_{n \geq 0} T^{-n}\mathscr{B}$*-measurable.*

Proof. $(\Rightarrow)$ It suffices to show that if $\omega_{\mu T}$ is measurable with respect to $\alpha^{-1}\mathscr{F} \subseteq \mathscr{B}$, and is a coboundary, then the transfer function is $\alpha^{-1}\mathscr{F}$ measurable as well. We suppose that $\omega_{\nu T} = (h \circ T)/h$; i.e., $h(x) = ((h \circ T)/\omega_{\mu T})(x)$. We show that for any $x \in X$, and any $y \in S_T(x)$, $h(x) = h(y)$. We first suppose that $Tx = Ty$. Then $h(Tx) = h(Ty)$, and $\omega_{\mu T}(x) = \omega_{\mu T}(y)$ by assumption; therefore $h(x) = h(y)$. Suppose we have shown that if $T^k x = T^k y$ for any $k \leq n$, then $h(x) = h(y)$. We now assume that $T^{n+1}x = T^{n+1}w$. Then $T^n(Tx) = T^n(Tw)$, so $h(Tx) = h(Tw)$, and we have $h(x) = ((h \circ T)/\omega_{\mu T})(x) = ((h \circ T)/\omega_{\mu T})(w) = h(w)$. By induction, we have that h is constant on $S_T(x)$ and by ergodicity of the relation S_T, it follows that h is constant $\mu_{\alpha x}$-a.e. Therefore, the transfer function depends only on αx as claimed.

$(\Leftarrow)$ This direction is immediate since $\Phi(\alpha x) = \alpha(Tx)$. ∎

THEOREM 4.5. *Suppose that* T *is a conservative ergodic* n*-to-one nonsingular endomorphism of* $(X, \mathscr{B}, \mu)$ *with maximal automorphic factor* Φ, *and* $d\mu_{\Phi y}/d\mu_y T^{-1} \equiv 1$ *for* ν a.e. y. *Then* Φ *is of type III if and only if* T *is of type III.*

Proof. $(\Rightarrow)$ Suppose T admits a σ-finite equivalent invariant measure $\mu' \sim \mu$. We can assume without loss of generality by Lemma 1.4 that μ is recurrent (see [18] or [19] for definitions) so that $\omega_{\mu T} = (h \circ T)/h$ for h a $T^{-1}\mathscr{B}$-measurable (and $\mathscr{B}$ measurable) function. By Proposition 4.3,

$$\omega_{\mu T}(x) = \frac{d\nu\,\Phi}{d\nu}(\alpha x) \cdot \frac{d\mu_{\Phi(\alpha x)}}{d\mu_{(\alpha x)}T^{-1}} \circ T(x)$$

which implies that $\dfrac{d\nu\,\Phi}{d\nu}(\alpha x) = \dfrac{h \circ T}{h}(x)$ for a.e. x.

It follows from Lemma 4.4 that $(d\nu\,\Phi/d\nu)(\alpha x)$ is a coboundary, which is a contradiction.

$(\Leftarrow)$ If Φ admits a σ-finite equivalent invariant measure then $(d\nu\,\Phi/d\nu)(\alpha x)$ is a coboundary. Then Proposition 4.3 and Lemma 4.4 imply that T cannot be of type III. ∎

5. AN EXAMPLE OF AN ENDOMORPHISM WITH NO EQUIVALENT INVARIANT MEASURE

We construct a two-to-one endomorphism T on the product space $X = Y \times Y^+ \equiv \prod_{n=-\infty}^{\infty} \{0,1\}_n \times \prod_{n=0}^{\infty} \{0,1\}_n$ and a Borel measure μ on X with the following properties: (1) T is the product of the two-sided shift with the one-sided shift; (2) T is nonsingular, conservative, and ergodic with respect to μ; (3) T admits no σ-finite invariant measure equivalent to μ; (4) T is neither exact with respect to μ, nor is μ the product measure of an exact with an automorphic measure.

We denote by $\mathscr{B} \times \mathscr{B}^+$ the product Borel σ-algebra on $Y \times Y^+$. We define the measure ν on $(Y, \mathscr{B})$ to be the type III two-sided Bernoulli shift measure constructed by Hamachi in [9]. Then the measure on $Y \times Y^+$ will be of the form $\mu(C) = \int_Y \rho_y(C \cap \alpha^{-1} y)\, d\nu(y)$. We specify the measures ρ_y on $(Y^+, \mathscr{B}^+)$ according to the following algorithm. We fix any $\lambda \in (0,1)$. We define two measures ρ^0 and ρ^1 on the space $\{0,1\}$ by $\rho^0(0) = \rho^0(1) = 1/2$, and $\rho^1(0) = 1/(1+\lambda)$, $\rho^1(1) = \lambda/(1+\lambda)$. For each $y \in Y$, we define ρ_y to be the infinite product measure given by $\rho_y = \prod_{i=0}^{\infty} \rho^{y_i}$. That is, we consider $y = (\dots, y_{-1}, y_0, y_1, \dots, y_n, \dots)$ and the ith factor in the measure ρ_y is ρ^j if and only if $y_i = j$. Each ρ_y will be an infinite product of factors of two different measures on Y^+ and, by results of Kakutani [13], for ν a.e. $y \in Y$, ρ_y will be singular with respect to the shift σ.

We denote the invertible shift on Y by Φ; i.e., $(\Phi y)_i = y_{i+1}$. It follows that $\rho_{\Phi y} = \prod_{i=0}^{\infty} \rho^{y_{i+1}} = \prod_{i=1}^{\infty} \rho^{y_i}$. We now restrict our attention to a single fiber $\{y\} \times Y^+$ with ρ_y a measure on $\mathscr{D}$, and we compute $\rho_y \sigma^{-1}$. Suppose $C \in \mathscr{D}$ is any cylinder set; then we can write

$$C = \{z \in Y^+ : z_0 = i_0, z_1 = i_1, \dots, z_n = i_n\},$$

and

$$\sigma^{-1}(C) = \{z : z_0 = 0, z_1 = i_0, \dots, z_{n+1} = i_n\} \cup \{z : z_0 = 1, z_1 = i_0, \dots, z_{n+1} = i_n\}.$$

Therefore $\rho_y(\sigma^{-1} C)$

$$\begin{aligned} &= \left[\rho^{y_0}(0) \cdot \rho^{y_1}(i_0) \cdot \rho^{y_2}(i_1) \cdots \rho^{y_{n+1}}(i_n)\right] \\ &\quad + \left[\rho^{y_0}(1) \cdot \rho^{y_1}(i_0) \cdot \rho^{y_2}(i_1) \cdots \rho^{y_{n+1}}(i_n)\right] \\ &= \prod_{j=0}^{n} \rho^{y_{j+1}}(i_j) = \rho_{\Phi y}(C). \end{aligned}$$

Since an infinite product measure is completely determined by its values on cylinder sets, we see that the two measures are equal. Consequently, $d\rho_y\,\sigma^{-1}/d\rho_{\Phi y} = 1$. To check that the endomorphism defined in this way is conservative, it is enough to see that $(d\mu(\Phi \times \sigma)^{-n}/d\mu)(y, z) = (d\nu\,\Phi^{-n}/d\nu)(y)$ for all $n \geq 0$, so $\omega_{\mu T^n}(y, z) = \omega_T(n, (y, z)) = (d\nu\,\Phi^n/d\nu)(y)$ for all $n \geq 0$. Hamachi computes that $\sum_{n=0}^{\infty}(d\nu\,\Phi^n/d\nu)(y) = \infty$ a.e. [9, p. 279], which gives the result. By Theorem 4.5, T admits no σ-finite equivalent invariant measure.

To prove that T is ergodic with respect to μ, we apply Proposition 3.4 and use the fact that all T invariant sets must be in $(Y, \mathscr{B})$ since R_T/S_T is just Φ. The ergodicity of Φ with respect to ν gives the result. The fact that μ is not equivalent to any product measure also follows from Proposition 3.4 and the uniqueness of the maximal automorphic factor; details of this argument are given in [5].

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