MAXIMAL ENTROPY MEASURE FOR RATIONAL MAPS AND A RANDOM ITERATION ALGORITHM FOR JULIA SETS

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Abstract. We prove that for any rational map of degree d, the unique invariant measure of entropy $\log d$ can be obtained as the weak limit of atomic probability measures obtained by taking a random backward orbit along an arbitrary nonexceptional point. This proves that the computer algorithm suggested by Barnsley in some specific cases works for all rational maps and is the algorithm used to produce Julia sets in the paper.

1. Introduction

If f is an analytic map of the sphere, then f can be written as a rational map of degree d. We will denote the Riemann sphere, or the extended complex plane by $\mathbb{C}_\infty$. We are interested in understanding the dynamics of rational maps $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ of degree ≥ 2, and graphical computer approximations of the Julia set of f often provide valuable insight into the theory of the subject. This study arose in answer to a question directed by John Milnor to the first author about a computer algorithm she was using to view nonhyperbolic Julia sets and the support of a particular invariant measure. The program was written by both the present authors together with Lorelei Koss, using the algorithm which will be described below. In fact, such an algorithm is introduced with justification for certain hyperbolic rational maps in a book by Barnsley [Barnsley, 1988]. A related theorem which is used to prove Barnsley's algorithm works for certain hyperbolic maps is proved by Elton [Elton, 1987]. Also, such an algorithm is mentioned in [Devaney, 1992], [Peitgen, Jürgens, and Saupe,1992], and [Peitgen and Richter, 1986].

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It was proved by Friere, Lopes, and Mañé, ([Freire, Lopes, and Mañé, 1983] and [Mañé,1983]), and independently by Lyubich [Lyubich, 1983], that there exists a unique invariant measure μ for f with measure theoretic entropy $\log d$, (which is the topological entropy); we write $h_\mu(f) = \log d$. We call a point w on the sphere *exceptional* for the rational map f if $\cup_{j=-\infty}^{+\infty} f^j(w)$ is a finite set of points. It is well-known that there can be at most two exceptional points for f. There are exactly two exceptional points if and only if f is conformally conjugate to $z \mapsto z^d$, for some integer d, $|d| \geq 2$. If there is one exceptional point for f, then f is conformally conjugate to a polynomial. Otherwise a rational map has no exceptional points. Let us denote by $\mathcal{E}$ the set of exceptional points, and set $\mathbb{C}_\infty^{ne} = \mathbb{C}_\infty \setminus \mathcal{E}$. As is well known, if $z \in \mathbb{C}_\infty^{ne}$, then its backward orbit is infinite.

We briefly describe the Mañé-Lyubich measure as the weak* limit of some atomic measures. We endow the sphere with the σ-algebra of Borel sets, and let z denote any nonexceptional point on the sphere. If δ_z denotes the Dirac measure

$$\delta_z(B) = \begin{cases} 1 & : \quad z \in B \\ 0 & : \quad z \notin B \end{cases},$$

then for each $n \in \mathbb{N}$, we define the measure

$$\mu_n^z := \frac{1}{d^n} \sum_{y \in f^{-n} z} \delta_y.$$

We count multiple roots with multiplicity.

Theorem 1.1 (Freire, Lopes, and Mañé, 1983)**.** *[Lyubich, 1983] If z is not an exceptional point, then the sequence of measures μ_n^z converges in the weak* topology to a measure μ independent of z. The measure μ is invariant, has support the Julia set of f, and $h_\mu(f) = \log d$.*

Furthermore, μ is characterized by the property that for any measurable set A on which f is injective, $\mu(f\,A) = d \cdot \mu(A)$. As mentioned above, this measure is called the Mañé-Lyubich measure.

We now turn to the setting of this section. We begin with a description of the computer algorithm. This algorithm was used to generate Julia sets in the main part of this paper by the first author; in each case treated in the paper it was known that the rational map was not hyperbolic.

A Computer Algorithm for generating the Julia Set of f

(1) Randomly choose a point z in the complex plane.
(2) Using a computation for the inverses of f, randomly choose one of the inverses of $f(z)$.

(3) Plot it.
(4) Repeat steps 2 and 3 several thousand times. (It is useful to ignore the first few hundred points and start plotting later on.)

We now formalize what the algorithm is doing. Let

$$\Sigma_d^+ = \prod_{i=0}^{\infty} \{0, \ldots, d-1\}_i$$

denote the space of one-sided sequences on d symbols. We consider the usual topology on Σ_d^+, endow it with the σ-algebra of Borel sets, and we let ν denote the $\{\frac{1}{d}, \ldots, \frac{1}{d}\}$ Bernoulli measure on Σ_d^+. It is well-known that this measure is the measure of maximal entropy with respect to the shift map σ on Σ_d^+. We use this space to define paths of inverses for a rational map. Given a rational map f of degree d, except for the critical values of f, each point z has d inverse images, hence d well-defined local inverses which we will denote $g_0, g_1, \ldots, g_{d-1}$. (One can choose a consistent convention for labelling them but we will instead use methods of stochastic processes to prove the result to avoid this technical complication.) Therefore a typical single-valued path in $f^{-k}z$ would be denoted $g_{i_0,i_1,\ldots,i_{k-1}}(z) = g_{i_{k-1}} \circ g_{i_{k-2}} \circ \ldots \circ g_{i_0}(z)$. Similarly, given a point in $i = (i_0, i_1, \ldots, i_k, \ldots) \in \Sigma_d^+$ and a point $z \in \mathbb{C}_\infty$, a path of length k in the inverse of f is determined by $g_{i_0,i_1,\ldots,i_{k-1}}(z)$. An equivalent characterization can be given in terms of a stochastic process associated to f which we will describe later. We first turn to a statement of our main results.

2. The Main Results

This is the theorem that supports the computer algorithm described by Barnsley in a restricted setting and used for arbitrary rational maps by the authors. Here we obtain a result showing that the algorithm is valid much more generally.

Theorem 2.2. *Let f denote a rational map of degree ≥ 2. Let $\{z_j\}_{j=0}^{\infty}$ denote a backward orbit of a point under f, starting at an arbitrary $z \in \mathbb{C}_\infty^{ne}$. That is, $z_j = g_{i_0,i_1,\ldots,i_j}(z)$. Then for ν a.e. backward path in Σ_d^+, for every continuous function ϕ,*

$$\frac{1}{n}\sum_{j=0}^{n-1} \phi(z_j) \to \int \phi \, d\mu.$$

Equivalently, if we define the sequence of measures

$$\mu^z_{i_0,\ldots,i_{n-1}} = \frac{1}{n}\sum_{j=0}^{n-1} \delta_{z_j},$$

then for ν *a.e. backward path in* Σ_d^+, $\mu^z_{i_0,\ldots,i_{n-1}}$ *converges weak** *to* μ.

As a corollary we obtain the following topological result. We denote by $J(f)$ the Julia set of f. Recall that it is well-known that for any point $z \in \mathbb{C}^{ne}_\infty$,

$$J(f) \subset \overline{\bigcup_{j=0}^{\infty} f^{-j} z}$$

and if $z \in J(f)$,

$$J(f) = \overline{\bigcup_{j=0}^{\infty} f^{-j} z}$$

(cf.[Beardon,1991]). We prove a stronger version of this.

Corollary 2.3. *Let* f *denote a rational map of degree* ≥ 2. *Let* $\{z_j\}_{j=0}^{\infty}$ *denote a backward orbit of a point under* f, *starting at an arbitrary* $z \in \mathbb{C}^{ne}_\infty$. *Then for* ν *a.e. backward path in* Σ_d^+,

$$J(f) \subset \overline{\bigcup_{j=0}^{\infty} z_j}.$$

If, in addition, $z \in J(f)$, *then*

$$J(f) = \overline{\bigcup_{j=0}^{\infty} z_j}.$$

Proof. We recall first that Lyubich gives a detailed proof of the fact that the support of μ is exactly equal to $J(f)$ ([Lyubich, 1983] pp. 361–362). In particular, any open set that intersects $J(f)$ has positive Mañé-Lyubich measure.

We now consider any $z \in \mathbb{C}^{ne}_\infty$ and any nonempty open set W that intersects $J(f)$. Since $\mu(W) > 0$, we can find a nonnegative continuous function ϕ, whose support is contained completely in W, and such that on some open set $V \subset W$, with $\frac{2}{3}\mu(W) < \mu(V) < \mu(W)$, ϕ is identically 1. Applying Theorem 2.2 yields for a.e. path $i_0, \ldots, i_{n-1}, \ldots$, for some large enough n,

$$0 < \frac{1}{2}\mu(W) < \int \phi \, d\mu^z_{i_0,\ldots,i_{n-1}} = \frac{1}{n}\sum_{j=0}^{n-1} \phi(z_j),$$

which means that some of the points z_j must enter the support of ϕ, hence W, for almost every path as claimed.

□

Let $\{z_j\}$ be any backward orbit such that the conclusions of Theorem 2.1 hold. Since the Mañé-Lyubich measure is supported on $J(f)$, a clear implication is that *most* of the elements of this sequence approach $J(f)$ as $j \to \infty$. The following is a desirable strengthening of this observation, which nicely complements Corollary 2.3.

Proposition 2.4. Let $\{z_j\}$ be a backward orbit satisfying the conclusions of Theorem 2.1. Then $z_j \to J(f)$ as $j \to \infty$, i.e., $\text{dist}(z_j, J(f)) \to 0$ as $j \to \infty$.

There are two ways in which Theorem 2.2 has proven useful to the first author in the preceeding paper. The first way is in terms of simplicity of writing a program to get an accurate picture of a Julia set. In addition, in studying Lebesgue ergodic rational maps whose Julia set is necessarily the whole sphere, it is known that except in some rare cases (parabolic orbifolds) the Mañé-Lyubich measure is completely singular with repsect to the 2-dimensional Lebesgue measure (Riemannian volume form) on $\mathbb{C}_\infty$ [Zdunik, 1990]. It is illustrative to view the density of Mañé-Lyubich measure in these cases.

The simplicity of this random iteration method, as opposed to computing all the d^j elements of $f^{-j}(z_0)$ for a certain range of j, has been noted before, e.g., in [Peitgen and Richter, 1986] and [Peitgen, Jürgens, and Saupe, 1992]. Our contribution is to supply a proof that the method works for all rational maps of degree $d \geq 2$. These references also note that some parts of the Julia set may be reached rather slowly by this method; they describe an alternative, the "modified inverse iteration method," or MIIM, which covers the Julia set in fewer iterations. These phenomena illustrate that the Mañé-Lyubich measure can be relatively sparse in some regions of the Julia set. While the MIIM has advantages over the random iteration method analyzed here, in rendering the Julia set, the MIIM does not capture the behavior of the Mañé-Lyubich measure.

The ingredients in our proof of Theorem 2.2 include both some of the analysis behind the proof of Lyubich's theorem and a theorem of Furstenberg & Kifer [1983] about stochastic processes. To prove Proposition 2.4, we will also make use of Sullivan's nonwandering theorem and the classification of the periodic components of the Fatou set. We proceed to introduce the various needed concepts and provide the proofs of Theorem 2.2 and Proposition 2.4.

Rational Maps as Stochastic Processes

By $\mathcal{P}(\mathbb{C}_\infty)$ we denote the space of probability Borel measures on $\mathbb{C}_\infty$, and by $\mathcal{C}(\mathbb{C}_\infty)$ the space of continuous functions on it. $\mathcal{P}(\mathbb{C}_\infty)$ is a compact metric space in the topology of weak* convergence.

We note that, for each fixed rational map f of degree $d \geq 2$, the map

$$z \mapsto \frac{1}{d} \sum_{y \in f^{-1}z} \delta_y = \mu_1^z$$

is continuous from $\mathbb{C}_\infty$ to $\mathcal{P}(\mathbb{C}_\infty)$. (We count multiple roots of $z = f(y)$ by multiplicity.) This collection of measures defines the following operator $Q : \mathcal{C}(\mathbb{C}_\infty) \to \mathcal{C}(\mathbb{C}_\infty)$: for each function $\phi \in \mathcal{C}(\mathbb{C}_\infty)$,

$$Q\phi(z) = \int \phi(y)\, d\mu_1^z(y) = \frac{1}{d} \sum_{y \in f^{-1}z} \phi(y).$$

Clearly the operator Q averages the value of $\phi(z)$ equally over the d preimages of z under f. Furthermore the adjoint operator is defined in the usual way: for each $\phi \in \mathcal{C}(\mathbb{C}_\infty)$ and probability measure ρ,

$$\langle Q\phi, \rho \rangle = \langle \phi, Q^* \rho \rangle,$$

where as usual we pair functions and measures via $\langle \quad , \rho \rangle = \int \quad (z)\, d\rho(z)$.

An important result proved by Lyubich is the following.

Lemma 2.5 (Lyubich, 1983)**.** *The Mañé-Lyubich measure μ is invariant under Q^*.*

In particular, it was shown that for any compact set $K \subset \mathbb{C}_\infty$ containing no exceptional points of f, and for any continuous function ϕ on $\mathbb{C}_\infty$, there is a Q^* invariant measure (which is μ, the unique measure of maximal entropy for f) such that

$$\|Q^m \phi - \int \phi\, d\mu\|_K \to 0 \quad \text{as} \quad n \to \infty,$$

where $\|\phi\|_K := \sup_{z \in K} |\phi(z)|$.

From the proof given by Lyubich, we obtain the important additional result.

Lemma 2.6. *The Mañé-Lyubich measure is the unique probability measure that has support disjoint from the set of exceptional points and that is invariant under Q^*.*

Proof. Suppose that the probability measure ν is supported on a compact set K containing no exceptional points and $Q^*\nu = \nu$. Then we claim that for every continuous function ϕ, $\int \phi d\nu = \int \phi d\mu$, so $\nu = \mu$. To prove the claim we note that by the previous lemma we have for every $n \geq 1$,

$$\langle Q^n \phi, \nu \rangle = \langle \phi, (Q^*)^n \nu \rangle = \langle \phi, \nu \rangle = \int \phi(z)\, d\nu(z),$$

but since $Q^n \phi \to \int \phi d\mu$ as $n \to \infty$, uniformly on K, we have that $\int \phi(z) d\mu(z) = \int \phi(z) d\nu(z)$ as claimed.

□

We now turn to the construction of the stochastic process determined by f and the collection of measures $\{\mu_1^z\}_{z \in \mathbb{C}_\infty}$.

The stochastic process associated to f is the collection of inverse paths under f on the sphere, together with a family P_z of probability measures on this set of paths, indexed by z in the set $\mathbb{C}_\infty^{ne}$ of nonexceptional points. We denote it by $(\Omega, \mathcal{F}, P_z)$, where a point $\omega \in \Omega$ is of the form $\omega = (\omega_0, \omega_1, \ldots, \omega_n, \ldots)$ with each $\omega_i \in \mathbb{C}_\infty$, and such that $f(\omega_{i+1}) = \omega_i$ for every $i > 0$. Therefore we have that $\Omega \subset \prod_{j=0}^{\infty} (\mathbb{C}_\infty)_j$. We note that for each fixed $z \in \mathbb{C}_\infty$ which is not an exceptional point for f, we obtain a collection of backward random paths under f, each determining a point in Ω. From this we see that $\Omega \subset \prod_{j=0}^{\infty} (\mathbb{C}_\infty^{ne})_j$. The measurable structure on Ω is just the Borel structure. It remains to define the measures P_z.

We make one more observation about the notation and structure of the stochastic process Ω; we just noted that if we fix a point on the sphere it determines various sequences of backward paths starting at that point (i.e., points $\omega \in \Omega$ such that $\omega_0 = z$). Similarly, if we fix an integer n, then we can view ω_{n+1} as a point in $\mathbb{C}_\infty$ whose value depends only on ω_n. In this way we define the stochastic process $\Omega = \{\omega_n : n \geq 0\}$ as a Markov process with transition probabilites $\{\mu_z\}$ by defining the measure P_{ω_0} using:

$$P_{\omega_0}\{\omega_{n+1} \in A \subset \mathbb{C}_\infty | \omega_1, \ldots, \omega_n\} = \mu_{\omega_n}(A).$$

In our case, $\mu_z = \mu_1^z$; other families of transition probabilities yield other Markov processes.

It is a classical result of Kolmogorov that this completely determines the measure P_z (cf. [Lamperti,1977] for example). Since the family of measures P_z, hence the stochastic process itself, are determined by the operator Q defined above, which in turn is completely determined by

f, we say that $\Omega = \{\omega_n : n \geq 0\}$ is *the Markov process corresponding to* Q, or *determined by* f.

The Furstenberg-Kifer Law of Large Numbers

We now state the result of Furstenberg & Kifer [1983], which we will use in the proof of Theorem 2.2.

Let X be a compact metric space and $\mathcal{P}(X)$ the space of Borel probability measures on X. Consider a continuous map from X to $\mathcal{P}(X)$ which assigns a measure μ_x to each $x \in X$. We give $\mathcal{P}(X)$ the weak* topology. We define the corresponding operator on continuous functions of X, $\mathcal{C}(X)$ by:

$$Q\phi(x) = \int \phi(y)\, d\mu_x(y),$$

and the Markov process $\{\omega_n, n \geq 0\}$ associated to Q using for each Borel set A,

$$P_{\omega_0}\{\omega_{n+1} \in A | \omega_1, \ldots, \omega_n\} = \mu_{\omega_n}(A).$$

Theorem 2.7 (Furstenberg & Kifer, 1983)**.** *Let* $\Omega = \{\omega_n : n \geq 0\}$ *be the Markov process determined by the operator* Q*. Assume that there exists a unique probability measure* μ *that is invariant under the adjoint operator* Q^* *on* $\mathcal{P}(X)$*. Let* $\phi \in \mathcal{C}(X)$*. Take any* $\omega_0 \in X$*. Then with* P_{ω_0} *measure* 1 *on* Ω*,*

$$\frac{1}{n}\sum_{j=0}^{n-1} \phi(\omega_j) \to \int \phi\, d\mu$$

as $n \to \infty$*.*

In [Furstenberg & Kifer, 1983] a more general result is given, but the result as stated above will suffice for our purposes.

Proof of Theorem 2.2

The task that remains is to show that Theorem 2.7 is applicable in our situation. We first show this in the case when the set $\mathcal{E}$ of exceptional points is empty; then we will indicate the modifications for the case when $\mathcal{E}$ is nonempty.

If $\mathcal{E}$ is empty, set $X = \mathbb{C}_\infty$. As z runs over X, use the transition probabilities $\mu_z = \mu_1^z$, defined in the introduction. The fact that $z \mapsto \mu_1^z$ is continuous in this case is well known. The fact that the Mañé-Lyubich measure μ satisfies the hypotheses of Theorem 2.7 is a consequence of Lemmas 2.5 and 2.6, so we have the desired result in this case.

If $\mathcal{E}$ is not empty, we make use of known results on the structure of f mentioned in the introduction. The set $\mathcal{E}$ has one point or two points. If $\mathcal{E}$ has one point, f is conformally conjugate to a polynomial, for which $\mathcal{E} = \{\infty\}$, and if $\mathcal{E}$ has two points, f is conformally conjugate to a map of the form $z \mapsto z^p$, for which $\mathcal{E} = \{0, \infty\}$. In either case, we see that, given any open neighborhood U of $\mathcal{E}$, there is a compact $K \subset \mathbb{C}_\infty$ such that $\mathbb{C}_\infty \setminus K \subset U$, and such that $f^{-1}(K) \subset K$.

Now we can apply the Markov process setup to $X = K$, using the transition probabilities $\mu_z = \mu_1^z$ for z running over K. Since $f^{-1}(K) \subset K$, we see that μ_1^z is supported on K whenever $z \in K$. Again we can appeal to Lemmas 2.5 and 2.6 to see that Theorem 2.7 is applicable. The proof of Theorem 2.2 is complete.

Proof of Proposition 2.4

It is possible that the Fatou set $F(f)$ is empty, in which case there is nothing to prove. More generally, if $z_0 \in J(f)$, then $f^{-n}(z_0) \subset J(f)$, and the proposition holds. Otherwise, given $z \in F(f)$, it could happen that $f^{-n}(z) \not\to J(f)$ as $n \to \infty$. It is well known that this convergence fails only if there exists $w_0 \in F(f)$ such that z is an accumulation point of $\{f^j(w_0) : j \in \mathbb{Z}^+\}$. (See [Beardon,1991], p. 71.) In this case we call z an *accumulator*. Therefore it suffices to show that if $\{z_j\}$ is any backward orbit satisfying the conclusions of Theorem 2.2, then there exists some element z_j which is *not* an accumulator.

To analyze which points are accumulators, we recall Sullivan's nonwandering theorem ([Sullivan,1985]), which states that each component of the Fatou set $F(f)$ is mapped by some iterate f^m into a periodic component. Hence, if $z \in F(f)$ and z is an accumulator, then z must belong to a periodic component of $F(f)$.

Now we recall the classification of periodic components of $F(f)$ (see, e.g., [Beardon,1991], [Carleson and Gamelin, 1993], [Steinmetz, 1993]). If U is such a periodic component, there are four possibilities:

(1) U contains an attracting (or superattracting) periodic point,
(2) U is a Leau domain,
(3) U is a Siegel disk,
(4) U is a Herman ring.

It is also known that the number of distinct cycles is $\leq 2(d-1)$, but we will not need this. It remains to consider the behavior of $\{z_j\}$ when z_0 belongs to a periodic component of any one of these four types.

In the first case, z_0 is an accumulator if and only if z_0 is one of the attracting periodic points. Now if $\{z_j\}$ is a backward orbit satisfying the conclusion of Theorem 2.2, not all z_j can lie in the set of attracting periodic points, for then $\{z_j\}$ would be bounded away from $J(f)$. Hence, if z_0 lies in a periodic component of $F(f)$ of type 1, then some z_j must fail to be an accumulator.

It is known that periodic components of $F(f)$ of type 2 contain no accumulation points (cf. [Beardon,1991], p. 160), so if z_0 belongs to such a component of $F(f)$ we have no problem.

Next suppose $z_0 \in U$, a periodic component of type 3. Then $U = U_0$ is part of a cycle $U_0, \dots, U_{k-1}$, with $f : U_j \to U_{j+1}$ ($j+1$ computed mod k), and $f^k : U_0 \to U_0$ is conjugate to an irrational rotation. In such a case, U_0 contains lots of points which are accumulators. Say $\{z_j\}$ is a backward orbit satisfying the conclusions of Theorem 2.2. If $\{z_j\} \subset \cup_{\nu=0}^{k-1} U_\nu$, then by the rotation property we would have $\{z_j\}$ bounded away from $J(f)$, hence cannot satisfy the conclusion of Theorem 2.2. Thus we deduce that some z_j does not belong to $\cup_{\nu=0}^{k-1} U_\nu$; this z_j cannot be an accumulator.

The case $z_0 \in U$, of type 4, is handled similarly, and Proposition 2.4 is proven.

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