Houston Journal of Mathematics
© 2050 University of Houston
Volume 76, No. 1, 2050

UNBOUNDED FATOU COMPONENTS FOR ELLIPTIC FUNCTIONS OVER SQUARE LATTICES

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Communicated by George Washington

Abstract. We show the existence of elliptic functions on the complex plane C with a square period lattice L with unbounded Fatou components under iteration. For these functions, the imaginary axis lies in a single Fatou component, and projects onto a band on the torus C/L. Unlike all previous examples, the resulting Julia set is not Cantor because there are two attracting fixed points. There is an open set of parameter values within a family of functions for which these Fatou components exist. The first example of a toral band for a map with a parabolic fixed point is also given. This shows that unlike the situation for iterations of the Weierstrass elliptic P function, some Fatou components that are unbounded in only one direction can occur for square lattices.

1. Introduction

We consider the dynamics of iterated elliptic functions, doubly periodic meromorphic functions on $\mathbb{C}$; these were first studied in [**?**] though much work has been done since then. In this paper we restrict our attention to real square lattices, where a fundamental region is a square with sides parallel to the real and imaginary axes. Our main result is that Fatou components that cannot exist for the Weierstrass $\wp$ function can exist for a closely related family of elliptic functions, namely $\wp + \beta$ for some $\beta \in \mathbb{C}$. This type of Fatou component, which we call a vertical toral band, is a set which extends completely over one vertical side of the period square boundary, but not over the horizontal boundary line. All previous examples arose only in settings with special lattices (see, e.g., [**?**]). Despite the

2000 *Mathematics Subject Classification.* 37F10, 30D05.
Key words and phrases. Complex dynamics, elliptic functions, Fatou set.
The authors thank the referee for useful comments.

symmetry of the square lattice, namely that the Julia and Fatou sets of $\wp$ are invariant under rotation by $\pi/2$, π, and complex conjugation, a vertical toral band can occur. This reveals asymmetric dynamics along the vertical and horizontal sides of the square, for $\wp + \beta$ (as shown in Figures **??** - **??**), instead of the usual symmetry we see for $\wp$ shown in Figure **??**.

The dependence of the dynamics of iterated elliptic functions on the underlying lattice Λ has been well-studied ([**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**]); however up to now there was no known example of an elliptic function with a vertical toral band as a Fatou component for a square period lattice. A preperiodic Fatou component W of an elliptic function is a *toral band* if there exists $U \subset W$, such that U is open and simply connected in $\mathbb{C}$, but under the quotient map $q : \mathbb{C} \to \mathbb{C}/\Lambda$, $q(U)$ is not simply connected. In other words, U projects to a set in the torus containing a nontrivial simple closed curve. A vertical toral band is a component that contains the imaginary axis (see Definition **??**).

In this paper we show that many examples of the form $\wp_\Lambda + \beta$, with Λ a square lattice, have vertical toral band Fatou components that do not contain $\mathbb{R}$ and do not have a Cantor Julia set, including one with a parabolic fixed point. Double toral bands with non-Cantor Julia sets first appeared in [**?**].

A real square lattice Λ is generated by complex numbers of the form λ_1 and $\lambda_2 = i\lambda_1$, with $\lambda_1 > 0$, and the family of maps studied is given by:

$$F_{\Lambda,\beta} : \mathbb{C} \to \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}, \quad F_{\Lambda,\beta}(z) = \wp_\Lambda(z) + \beta. \tag{1}$$

Here, $\wp_\Lambda$ denotes the standard Weierstrass $\wp$ function with period lattice Λ.

When β is restricted to the horizontal half-period line $L = \{t + \lambda_2/2 : t \in \mathbb{R}\}$, $F_{\Lambda,\beta}$ leaves L invariant. Therefore the global longterm dynamics of $F_{\Lambda,\beta}$ are determined by dynamics along L. This allows the use of some techniques of bifurcation theory for interval maps to study the dynamics and stability of $F_{\Lambda,\beta}$ when the parameter β is restricted to L. We shift the focus of the map to the real and imaginary axes using classical identities, which are described in detail below.

In [**?**] the authors showed that for functions of the form (**??**), Cantor Julia sets occur. The first examples of elliptic functions with Λ a square lattice, with double toral bands, can be found in ([**?**], Theorem 4.5) and also later results occur in ([**?**], Theorem 4.7 and Fig. 2). The goal of this paper is to show that one-sided unbounded Fatou components are also possible in this setting. In other words, the symmetry of the square does not always carry over to the dynamics.

2. Preliminaries

By $\Lambda = [\lambda_1, \lambda_2]$ we denote the group $\Lambda = \{m\lambda_1 + n\lambda_2 \colon m, n \in \mathbb{Z}\} \subset \mathbb{C}$. Assuming $\lambda_1, \lambda_2 \in \mathbb{C}$ are non-zero and linearly independent over $\mathbb{R}$, we call Λ a *lattice*. A lattice Λ acts on $\mathbb{C}$ by translation with each $\lambda \in \Lambda$ inducing the transformation of $\mathbb{C}$ given by $z \mapsto z + \lambda$. We write $[z] = z + \Lambda$ to denote a residue class $(\mathrm{mod}\,\Lambda)$ of z.

Definition 2.1. A closed, connected set $Q \subset \mathbb{C}$ is a *fundamental region* for Λ if

(1) for each $z \in \mathbb{C}$, Q contains at least one point in the same Λ-orbit as z;
(2) no two points in the interior of Q are in the same Λ-orbit.

If Q is a fundamental region for Λ, then if $s \in \mathbb{C}$, the set

$$s + Q = \{z + s : z \in Q\}$$

is also a fundamental region. Q is a *period parallelogram* if it is a parallelogram.

If $\Lambda = [\lambda_1, \lambda_2]$, the ratio $\tau = \lambda_2/\lambda_1$ defines an equivalence relation on lattices. For $k \neq 0$ any complex number, $k\Lambda$ is the lattice defined by taking $k\lambda$ for each $\lambda \in \Lambda$ and yields the same ratio τ; $k\Lambda$ is said to be *similar* to Λ. Lattices determine double periods for elliptic functions, but it is known that equivalent lattices do not necessarily produce similar dynamics under iteration (see e.g., [**?**],[**?**]).

Definition 2.2. An *elliptic function* $f : \mathbb{C} \to \hat{\mathbb{C}}$ is a meromorphic function in $\mathbb{C}$ that is periodic with respect to a lattice Λ.

The *Weierstrass elliptic function* is defined for each $z \in \mathbb{C}$, by

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right). \tag{2}$$

The map $\wp_\Lambda$ is an even elliptic function with poles of order 2 at all lattice points. The derivative of the Weierstrass elliptic function is an odd elliptic function of order 3, also periodic with respect to Λ, and given by $\wp'_\Lambda(z) = -2\sum_{\lambda\in\Lambda} 1/(z-\lambda)^3$. The elliptic function $\wp_\Lambda$ and its derivative are related by the differential equation

$$\wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 - g_2\wp_\Lambda(z) - g_3, \tag{3}$$

where $g_2(\Lambda) = 60\sum_{\lambda\in\Lambda\setminus\{0\}} \lambda^{-4}$ and $g_3(\Lambda) = 140\sum_{\lambda\in\Lambda\setminus\{0\}} \lambda^{-6}$. The numbers $g_2(\Lambda)$ and $g_3(\Lambda)$ are invariants of the lattice Λ in the following sense: if $g_2(\Lambda) = g_2(\Lambda')$ and $g_3(\Lambda) = g_3(\Lambda')$, then $\Lambda = \Lambda'$. Furthermore given any g_2 and g_3 such that $g_2^3 - 27g_3^2 \neq 0$ there exists a lattice Λ having (g_2, g_3) as its invariants [**?**].

The following identity shows how g_2 and g_3 are related to the lattices [**?**].

Lemma 2.1. *For lattices* Λ *and* Λ', $\Lambda' = k\Lambda$ *if and only if*

$$g_2(\Lambda') = k^{-4} g_2(\Lambda) \quad \textit{and} \quad g_3(\Lambda') = k^{-6} g_3(\Lambda).$$

A lattice Λ is said to be *real* if $\Lambda = \overline{\Lambda} := \{\overline{\lambda} : \lambda \in \Lambda\}$, where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. The next result appears in [**?**].

Proposition 2.2. *The following are equivalent:*

(1) Λ *is a real lattice;*
(2) $\wp_\Lambda(\overline{z}) = \overline{\wp_\Lambda(z)}$*;*
(3) $g_2, g_3 \in \mathbb{R}$.

Since for $k \in \mathbb{C}\backslash\{0\}$, $\wp_{k\Lambda}(ku) = (1/k^2)\wp_\Lambda(u)$, the dynamics of iterated elliptic functions changes nonlinearly as k varies, even for $k \in \mathbb{R}$.

2.1. **Real square rectangular period lattices for** $\wp_\Lambda$**.** Assume that $\Lambda = [\lambda_1, i\lambda_1]$, with $\lambda_1 > 0$. Since a fundamental region Q can be chosen to be a square with two sides parallel to the real axis and two sides parallel to the imaginary axis, Λ is called a *real (rectangular) square lattice.* We gather some properties about $\wp_\Lambda$.

(1) The invariants satisfy $g_2 > 0$ and $g_3 = 0$.
(2) $\wp_\Lambda$ has infinitely many simple critical points, one located at each half lattice point, and we denote them by $[c_1], [c_2]$, and $[c_3]$, where
$$c_1 = \frac{\lambda_1}{2},\ c_2 = ic_1,\ c_3 = c_1 + c_2.$$
(3) $\wp_\Lambda$ has only three critical values $e_j = \wp_\Lambda(c_j)$ satisfying: $e_1 > 0$, $e_2 = -e_1$ and $e_3 = 0$.
(4) Since for any lattice Λ, e_1, e_2, e_3 are the distinct zeros of (**??**), we have these critical value relations:
$$\begin{aligned} \wp'_\Lambda(z)^2 =& 4(\wp_\Lambda(z) - e_1)\cdot(\wp_\Lambda(z) + e_1)\cdot\wp_\Lambda(z) \\ =& 4\left(\wp_\Lambda(z)^3 - e_1^2\wp_\Lambda(z)\right). \end{aligned} \tag{4}$$
Equating like terms in Eqn (**??**) and Eqn (**??**), we obtain
$$e_1 + e_2 + e_3 = 0, \quad e_1^2 = \frac{g_2}{4}, \quad e_1e_2e_3 = 0. \tag{5}$$
(5) The square lattice corresponding to $(g_2, g_3) = (4, 0)$, yielding $e_1 = 1, e_2 = -1$, and $e_3 = 0$ is called the *standard lattice.* We denote it by $\Lambda_s = [\gamma, i\gamma]$ where
$$\gamma = \frac{1}{4}\sqrt{\frac{2}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\right)^2 \approx 2.622,$$

and Γ denotes the classical gamma function. Setting $\kappa = \Gamma(1/4)^2/(4\sqrt{\pi})$, we have $c_1 = \kappa/(g_2)^{1/4}$.

(6) Differentiating both sides of Eqn (**??**) we have that $\wp''_\Lambda(z) = 6(\wp_\Lambda(z))^2 - g_2/2$.

A key identity is the following, from ([**?**], Theorem 2.1).

Theorem 2.3. *For a lattice Λ and $u \in \mathbb{C}$, if $i, j, k \in \{1, 2, 3\}$ are all distinct,*

$$\wp_\Lambda(u \pm c_i) = \frac{(e_i - e_j)(e_i - e_k)}{\wp_\Lambda(u) - e_i} + e_i. \tag{6}$$

Theorem **??** has the following consequence for rectangular square lattices.

Corollary 2.3. *If Λ is real square with invariant $g_2 > 0$, then*

$$\wp_\Lambda(u \pm c_3) = -\frac{e_1^2}{\wp_\Lambda(u)} = -\frac{g_2}{4\wp_\Lambda(u)} \tag{7}$$

The next result follows from (**??**).

Corollary 2.4. *If Λ is real square with $g_2 > 0$, setting $d_1 = \sqrt{3e_1^2 - g_2/4} = \sqrt{g_2/2}$, we have the following quarter lattice values for $\wp_\Lambda$: $\wp_\Lambda(c_1/2) = e_1 + d_1 = \dfrac{\sqrt{g_2}(1 + \sqrt{2})}{2}$; and for $j = 2, 3$, $\wp_\Lambda(c_1/2 + c_j) = e_1 - d_1 = \dfrac{\sqrt{g_2}(1 - \sqrt{2})}{2}$.*

The parametrized family of maps defined in Eqn (**??**) yields the following.

Corollary 2.5. *If Λ is real square, then for each $\beta \in \mathbb{C}$, setting*

$$G_\beta(z) = -\frac{g_2}{4\wp_\Lambda(z)} + \beta, \tag{8}$$

we have that $F_{\Lambda,\beta}(z - c_3) = G_\beta(z)$.

Remark 2.6. (1) In (**??**), writing $\beta = a + c_2$, $a \in \mathbb{R}$, and $z = t + c_2$ for $t \in \mathbb{R}$, in [**?**] the authors defined the real-valued map $\ell_a(t) = \wp_\Lambda(t+c_2)+a$, $t \in \mathbb{R}$. It follows that

$$\ell_a(t + c_1) = \ell_a(t - c_1) = G_a(t) = -\frac{g_2}{4\wp_\Lambda(t)} + a, \tag{9}$$

and G_a maps $\mathbb{R}$ into $\mathbb{R}$.

(2) For every $\beta \in \mathbb{C}$, we can read off the poles and critical points of G_β from (**??**). There is a double pole at $[c_3]$, the double zero of $\wp_\Lambda$ in the denominator, and the critical points are at $[0], [c_1]$, and $[c_2]$. If $\beta \in \mathbb{R}$, this gives three real critical values for G_β, namely, $\beta, \beta - e_1$, and $\beta + e_1$, respectively.

(3) In parallel with the focus in [?] we consider $\beta \in [0, \lambda_1)$. For these values of the parameter, the function G_β restricted to the real line, has a maximum occurring at 0 with maximum value β. The graph of G_β on $\mathbb{R}$ is concave down at 0 since $G_\beta''(0) = -g_2/2 < 0$. The minimum value of G_β on $\mathbb{R}$ is $\beta - e_1$, achieved at c_1, since $G_\beta''(c_1) = g_2 > 0$. Therefore $G_\beta(\mathbb{R}) = [\beta - e_1, \beta]$.

(4) Let $\mathbb{I} = \{it : t \in \mathbb{R}\}$. For parameters $\beta \in \mathbb{R}$, the map $G_\beta(it) = -\dfrac{g_2}{4\wp_\Lambda(it)} + \beta$, $t \in \mathbb{R}$, is real-valued, since for a square lattice, $\wp_\Lambda(it) \in \mathbb{R}$. In particular $G_\beta(\mathbb{I}) = [\beta, \beta + e_1]$.

To simplify the approach we therefore study the following family of functions: for $g_2 > 0, g_3 = 0$, and all $z \in \mathbb{C}$,

$$G_\beta(z) = -\frac{g_2}{4\wp_\Lambda(z)} + \beta, \text{ for } \beta \in [0, \lambda_1). \tag{10}$$

(Applying Proposition **??** shows this covers the dynamics for all $\beta \in \mathbb{R}$.) Consequently, $G_\beta(\mathbb{R}) \subset \mathbb{R}$ and the values of G_β along the boundary of a fundamental square with 0 in the lower left corner are real, and obtained by

$$\begin{aligned} G_\beta(\mathbb{R}) \cup G_\beta(\mathbb{I}) &= [-\sqrt{g_2}/2 + \beta, \beta] \cup [\beta, \beta + \sqrt{g_2}/2] \\ &= [-\sqrt{g_2}/2, \sqrt{g_2}/2] + \beta. \end{aligned} \tag{11}$$

All of the above statements follow from the next proposition.

Proposition 2.4. *Let Λ be a square lattice with invariant $g_2 > 0$. Define $L = \{t + c_2 : t \in \mathbb{R}\}$, a half lattice line parallel to the real axis. Then the following hold:*

(1) The critical points of $\wp_\Lambda$ occur at half lattice points, with residue class $[c_1] = \kappa g_2^{-1/4}$, $[c_2] = i[c_1]$, and $[c_3] = [c_1 + c_2]$.

(2) The critical values of $\wp_\Lambda$ are $e_1 = \dfrac{\sqrt{g_2}}{2}, e_2 = -e_1$ and $e_3 = 0$.

(3) Two critical points for $\wp_\Lambda'$ occur along the line L, at points $u = c_2 \pm t_0$, $t_0 \in \mathbb{R}$ where u satisfies $\wp_\Lambda(u) = -\sqrt{g_2}/(2\sqrt{3}) \in (e_2, 0)$ (the range of $\wp_\Lambda$ along L).

(4) If by V we denote the vertical half lattice line given by $V = \{z = c_1 + it : t \in \mathbb{R}\}$, then critical points for $\wp_\Lambda'$ also occur at $w = c_1 \pm it_0 \in V$, where $\wp_\Lambda(w) = \sqrt{g_2}/(2\sqrt{3}) \in \mathbb{R}$.

(5) The critical values for $\wp_\Lambda'$ are: $\wp_\Lambda'(u) = \pm\left(\dfrac{g_2}{3}\right)^{3/4}$, and $\wp_\Lambda'(w) = \pm i\left(\dfrac{g_2}{3}\right)^{3/4}$.

(6) The critical values of $\wp_\Lambda'$ on L are the maximum and minimum values of $(G_\beta)'|_{\mathbb{R}}$ (for any value of $\beta \in \mathbb{C}$).

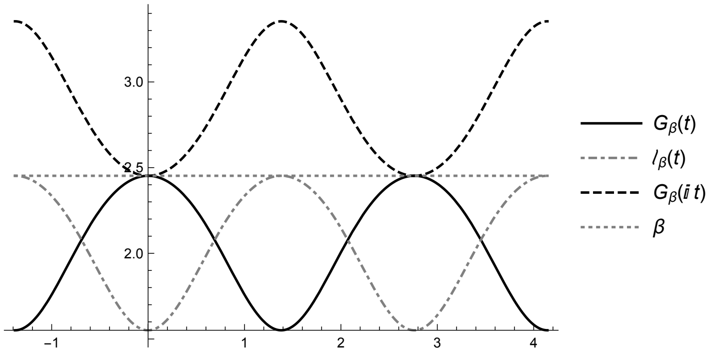

FIGURE 1. For $(g_2, g_3) = (3.25, 0)$, and $\beta \approx 2.45$, we show the graphs of $\ell_\beta(t)$ and $G_\beta(t)$, as well as the graph showing $G_\beta(it)$, restricted to $t \in \mathbb{R}$.

PROOF. Items (1) and (2) come from the standard identities. For (6), since $\wp_\Lambda : L \to \mathbb{R}$, and $\wp'_\Lambda : L \to \mathbb{R}$, it suffices to consider these as differentiable functions on the compact interval $c_2 + [-c_1, c_1] \subset L$, in which case $\wp'_\Lambda$ will achieve a maximum value and a minimum value, and there is a representative of $[c_2 + t_0]$ and $[c_2 - t_0]$ in $c_2 + (-c_1, c_1)$ since $e_2 = -\sqrt{g_2}/2 < -\sqrt{g_2}/(2\sqrt{3}) < 0$. □

In Figure **??** we show the graph of G_β restricted to $\mathbb{R}$, and restricted to $\mathbb{I}$, with the graph of the function ℓ_β for a typical value of $\beta > 0$.

2.2. **Fatou and Julia sets for elliptic functions.** Background definitions and properties for meromorphic functions appear in [**?**], [**?**], [**?**] and [**?**]. Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic function where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. The *Fatou set* $\mathcal{F}(f)$ is the set of points $z \in \hat{\mathbb{C}}$ such that $\{f^n : n \in \mathbb{N}\}$ is defined and normal in some neighborhood of z. The *Julia set* is the complement of the Fatou set on the sphere, $\mathcal{J}(f) = \hat{\mathbb{C}} \backslash \mathcal{F}(f)$. Montel's theorem implies that $\mathcal{J}(f) = \overline{\bigcup_{n \geqslant 0} f^{-n}(\infty)}$.

For each lattice Λ, every elliptic function with period lattice Λ is of Class S, as it is a meromorphic function f with only finitely many critical values and no asymptotic values. Therefore the basic dynamics are similar to those of rational maps, with the exception of the poles. The first result holds for all Class S functions (see [**?**], Theorem 12 and the references mentioned there, such as [**?**]).

Theorem 2.5. *For any lattice Λ, the Fatou set of an elliptic function f_Λ with period lattice Λ has no wandering domains and no Baker domains.*

In particular, Sullivan's No Wandering Domains Theorem holds in this setting so all Fatou components of f_Λ are preperiodic. Because there are only finitely many critical values, we have a bound on the number of attracting, superattracting, and parabolic periodic points that can occur.

The next result was proved in [**?**] for the Weierstrass elliptic function but since $\wp_\Lambda$ and $F_{\Lambda,\beta}$ have the same poles for every $\beta \in \mathbb{C}$, and $F_{\Lambda,\beta}$ is also even, the same proof works, and the result also holds for $G_\beta, \beta \in \mathbb{C}$ (see also [**?**], Theorem 1).

Theorem 2.6. *For a lattice Λ, and $\beta \in \mathbb{C}$, neither $F_{\Lambda,\beta}$ nor G_β has a cycle of Herman rings.*

We summarize this discussion with the following result.

Theorem 2.7. *For a lattice Λ, and every $\beta \in \mathbb{C}$, at most three different types of forward invariant Fatou cycles can occur for G_β, and each periodic Fatou component contains one of these:*

(1) a linearizing neighborhood of an attracting periodic point;
(2) a Böttcher neighborhood of a superattracting periodic point;
(3) an attracting Leau petal for a periodic parabolic point. The periodic point is in $\mathcal{J}(G_\beta)$;
(4) a periodic Siegel disk containing an irrationally neutral periodic point.

The proof of Lemma **??** is given for $\wp_\Lambda$ in [**?**] but remains the same for the elliptic function G_β, and in fact for any even elliptic function.

Lemma 2.7. *If Λ is a lattice and f_Λ is an even elliptic function with period lattice Λ, then*

(1) $\mathcal{J}(f_\Lambda) + \Lambda = \mathcal{J}(f_\Lambda)$, and $\mathcal{F}(f_\Lambda) + \Lambda = \mathcal{F}(f_\Lambda)$.
(2) If Λ is real, then $\overline{\mathcal{F}(f_\Lambda)} = \mathcal{F}(f_\Lambda)$ and $\overline{\mathcal{J}(f_\Lambda)} = \mathcal{J}(f_\Lambda)$.
(3) If c is any half lattice point of Λ, then $c + z \in \mathcal{J}(f_\Lambda)$ if and only if $c - z \in \mathcal{J}(f_\Lambda)$.

In particular, if F_0 is any component of $\mathcal{F}(f_\Lambda)$ that contains a half lattice point c, then F_0 is symmetric with respect to c.

Proof. To prove (3), if $w = -z \pmod{\Lambda}$, then since f_Λ is even we have $f_\Lambda(z) = f_\Lambda(w)$. Consider $c = \lambda/2$, where λ is one of $\lambda_1, i\lambda_1$, or $(\lambda_1 + i\lambda_1)$. Since $\lambda - c = c$, it follows that $f_\Lambda(c+z) = f_\Lambda(-c+z+\lambda) = f_\Lambda(c-z)$. If $c_{m,n} = c + m\lambda_1 + ni\lambda_1$, then $f_\Lambda(c_{m,n}+z) = f_\Lambda(c+z) = f_\Lambda(c-z) = f_\Lambda(c_{m,n}-z)$ for every $m, n \in \mathbb{Z}$ □

Definition 2.8. Given two elliptic functions $f = f_\Lambda$ and $g = g_{\Lambda'}$ over period lattices Λ and Λ' respectively, we say f is conformally conjugate to g if there exists a Mobius map $\phi(z) = (az+b)/(cz+d)$, such that $f \circ \phi = \phi \circ g$.

Note that necessarily $c = 0$, since neither f nor g is defined at ∞, so $\phi(z) = az + b$ is a holomorphic automorphism of the plane. The following result is known [**?**, **?**], but we restate it here for background.

Proposition 2.8. *If Λ is a lattice, then for every $\lambda \in \Lambda$ and each fixed $\beta \in \mathbb{C}$,*

(1) $\wp_\Lambda + \beta$ is conformally conjugate to $\wp_\Lambda + \beta + \lambda$;
(2) G_β is conformally conjugate to $G_{\beta+\lambda}$.

Definition 2.9. Assume that f_Λ is an elliptic function over the lattice $\Lambda = [\lambda_1, i\lambda_1]$. Then

(1) $\mathcal{J}(f)$ is a *Cantor Julia set* if $\mathcal{J}(f)$ is a compact, totally disconnected, perfect subset of $\hat{\mathbb{C}}$.
(2) A Fatou component A_0 of f_Λ is a *toral band* if A_0 contains an open subset U which is simply connected in $\mathbb{C}$, but U projects to a topological band around the torus $\mathbb{C}/\Lambda$ containing a homotopically nontrivial curve. A toral band A_0, is a *double toral band* if it projects to a set in the torus $\mathbb{C}/\Lambda$ that contains closed paths that generate the fundamental group $\pi_1(\mathbb{C}/\Lambda)$.
(3) We say A_0 is a *vertical toral band* if $\mathbb{I} + n\lambda_1 \subset A_0$ for some $n \in \mathbb{Z}$, but $\mathbb{R} \not\subset \mathcal{F}(f_\Lambda)$; and A_0 is a *horizontal toral band* if $\mathbb{R} + m\lambda_1 i \subset A_0$ for some $m \in \mathbb{Z}$, but $\mathbb{I} \not\subset \mathcal{F}(f_\Lambda)$.

Remark 2.10. The definition of a toral band sometimes appears in the literature as a forward invariant component satisfying Definition **??** (2). However recent examples show that a toral band satisfying Definition **??** (2) can fail to be either invariant or cyclic. Therefore we use the topological characterization.

When $\mathcal{J}(G_\beta)$ is a Cantor set we have a double toral band, with $\mathcal{F}(G_\beta)$ containing both real and imaginary axes in many examples, but in this same setting, for certain values of g_2 and β, vertical toral bands can occur. Often the condition is checkable as the next result shows.

Proposition 2.9. *For a lattice $\Lambda = [\lambda_1, i\lambda_1]$, with $\lambda_1 > 0$, an even elliptic function f periodic with respect to Λ has a vertical toral band if $V_0 = \{it : t \in [0, c_1]\} \subset \mathcal{F}(f)$.*

Proof. Suppose that $V_0 \subset \mathcal{F}(f)$; by the periodicity of f and the symmetry with respect to a critical point, $\mathbb{I} \subset \mathcal{F}(f)$. Since $\mathcal{F}(f)$ is open the result follows. □

3. Cantor sets for $g_2 < 3$.

We consider maps G_β of the form given in (**??**). We start with a proposition that follows from more general results proved by the authors in ([**?**], Section 6, Theorem 6.1); see also Proposition **??** below.

Proposition 3.1. *If* $0 < g_2 < 3$*, and* $g_3 = 0$*, then* $\mathcal{J}(G_\beta)$ *is a Cantor set for all* $\beta \in \mathbb{R}$ *and* $\mathcal{F}(G_\beta)$ *contains a double toral band.*

There are always real fixed points for the map G_β, and more can be said.

Proposition 3.2. *Let* Λ *be a square lattice with invariant* $g_2 > 0$*. For every* $\beta \in [0, \lambda_1]$*, the map* G_β *has at least one fixed point* $t_0 \in [0, \beta]$*. For every* $\beta \in (0, c_1 + e_1)$*,* G_β *has a fixed point* $t_0 \in (0, c_1)$*.*

(1) When $0 < g_2 < 3$*, there is exactly one fixed point on* $\mathbb{R}$ *and it is attracting for each* β*.*
(2) For $g_2 > 3$*, there is at least one fixed point for* G_β*, and the fixed point could be attracting, repelling, or rationally neutral.*
(3) If there is a fixed point $t_0 \in (0, c_1)$ *that is not attracting, then* G_β *does not have a horizontal toral band.*

Proof. For each value of $g_2 > 0$, if $\beta = 0$, using (**??**), $G_\beta(0) = 0$ and $G'_\beta(0) = 0$. For each value of $\beta \in (0, \lambda_1)$, using Remark **??** (2) and (3), the map $H(t) = G_\beta(t) - t$ satisfies $H(0) = \beta > 0$. We also have $H(\beta) = -g_2/(4\wp_\Lambda(\beta)) < 0$, so there is a fixed point between 0 and β.

To show (1), we note that $H'(t) = G'_\beta(t) - 1$ is negative since $G'_\beta(t) < 1$ on $\mathbb{R}$ by Proposition **??** (5) and (6). Therefore H is decreasing so H has at most one 0, hence G_β has at most one fixed point. By the first statement, there is exactly one fixed point and since $|G'_\beta(t)| < 1$ on $\mathbb{R}$, it is attracting.

Since (from Remark **??** (3)), $H(c_1) = -e_1 + \beta - c_1$, (2) follows. If t_0 is not attracting, then $t_0 \in \mathcal{J}(G_\beta)$. □

If $0 < g_2 < 3$, then the fixed point is attracting, and is the only fixed point for G_β for all β. The case $g_2 = 3$ is discussed in more detail in Section **??**, but there exists a point $t_\eta \in (0, \lambda_1)$ such that $G'_\beta(t_\eta) = 1$. There is also a point $s_\eta \in (0, \lambda_1)$ such that $G'_\beta(s_\eta) = -1$, leading to some parabolic fixed points for G_β for certain values of β.

4. $g_2 = 3 + \varepsilon$ and toral bands

When $g_2 > 3$, the maximum value of $G'_\beta(z)$ for $z \in \mathbb{R}$, $\eta = (g_2/3)^{3/4}$, is greater than 1. We apply a technique from [**?**] and Proposition **??** (3) to give a method for finding a parameter value $\beta \in [0, \lambda_1)$ and a point $t_\eta \in [0, \lambda_1]$ such that $G_\beta(t_\eta) = t_\eta$, and $G'_\beta(t_\eta) = \eta$, or a value very close to $\eta > 1$. We find a point u such that $\wp_\Lambda(u + c_2) = -\sqrt{g_2}/2\sqrt{3}$; we must choose u within the range of G_0, which by Remark **??**, is $[-\sqrt{g_2}/2, 0]$. The remark also shows that such a point exists (see also [**?**]).

If we set $\beta = u - \sqrt{g_2/12} + c_1$, then

$$t_\eta = u + c_1,\ G_\beta(t_\eta) = t_\eta,\ \text{ and } G'_\beta(t_\eta) = \eta. \tag{12}$$

When $g_2 > 3$, there is an interval of points u for which $\wp'_\Lambda(u + c_2) > 1$; for small $\varepsilon > 0$ and $g_2 = 3+\varepsilon$, the derivatives in that interval are very close to 1, this means that we can take a bit of a shortcut using the identity: $\wp_\Lambda^{-1}(-\sqrt{g_2/12}) - c_2 \approx \wp_\Lambda(\sqrt{g_2/12} + c_2) \in \mathbb{R}$. We use this below to reduce computation; it does not hold when g_2 gets too large.

Example 4.1. Set $g_2 = 3.15$ and $g_3 = 0$. Then using the method described above, $\beta \approx 2.459$ yields G_β with 3 fixed points in $[0, \lambda_1]$. First, there is a repelling fixed point $t_\eta \approx 1.904$ with derivative just above 1. Then, since $\wp''_\Lambda(t_\eta) = 0$, the concavity of G_β changes there from concave up to concave down.

Since by (**??**) $t_\eta > c_1$, we have that for all $t \in (c_1, \lambda_1)$: (i) $G_\beta > 0$, (ii) G_β is increasing, (iii) there are two other fixed points close to t_η, and these are each attracting fixed points. We label them p_1, with $p_1 < t_\eta$, and p_2 with $p_2 > t_\eta$.

Using Remark **??**, for $V_0 = \{it \ :\ t \in [0, c_1]\}$, $G_\beta(V_0) = [\beta, \beta + e_1]$. Since $G_\beta(V_0)$ lies completely in the attracting basin of p_2, we have a toral band. This is illustrated in Figure **??**.

Remark 4.2. The computer algorithm used for Figures ?? – ??. In each figure we show the complex plane with points in the attracting basins of the real parabolic or attracting points colored according to how many iterations it takes to land close to the fixed point. Therefore the points in the Julia set do not get colored at all (so they are usually either white or black). There are sometimes extra points colored black or white due to numerical roundoff errors.

In Figure **??** the coloring is only black and white, and $\mathcal{J}(G_\beta)$ is approximated by black points. Due to the existence of a parabolic fixed point, the numerical errors accumulate and show black butterflies at the fixed point and its preimages.

Theorem 4.1. *Let G_β be as in (**??**) for a square lattice with invariants $(g_2, 0)$, with $4 > g_2 > 3$. Then setting*

$$\beta = \sqrt{\frac{g_2}{12}} - \wp_\Lambda\left(\sqrt{\frac{g_2}{12}} + c_2\right) + c_1,$$

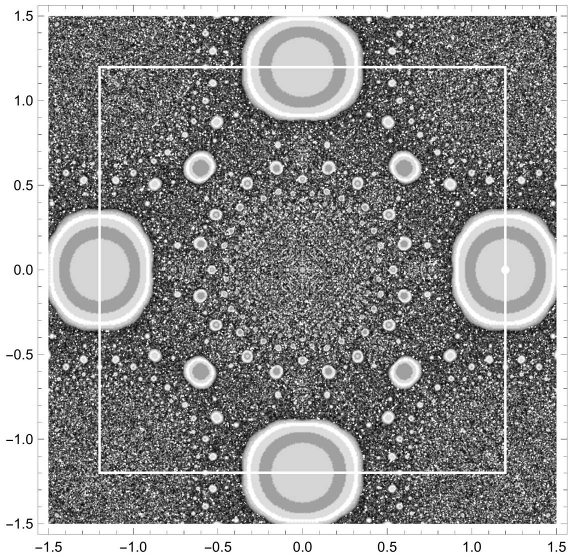

FIGURE 2. The Julia set for $\wp_\Lambda$ with the square lattice shown, is contained in the darkest set of points. There is a superattracting fixed point at $z_0 \approx 1.19$, and all Fatou components, colored shades of gray, are bounded. (See also Rk **??**.) There is also rotational symmetry since $\pm i\mathcal{J}(G_\beta) = \mathcal{J}(G_\beta)$.

(1) there exists a repelling fixed point $t_\eta = \sqrt{g_2/12} + c_1$, with $G'_\beta(t_\eta) > 1$.
(2) There exists an attracting fixed point $p_1 < t_\eta$, whose real immediate attracting basin is the interval $B_1 = (\sqrt{\frac{g_2}{12}} - c_1, \sqrt{\frac{g_2}{12}} + c_1)$.
(3) There exists an attracting fixed point p_2 with $p_2 > t_\eta$.
(4) The critical point at 0 lies in the real attracting basin of p_2, call it B_2, and the critical point $c_1 \in B_1$.
(5) We have that $G_\beta(V_0) \cap B_2 \neq \emptyset$; if $G_\beta(V_0) \subset B_2$, then $\mathcal{F}(G_\beta)$ has a vertical toral band.
(6) $G_\beta(c_2) = \beta + e_1$, so if $\beta + e_1 - 2c_1 \notin B_1$, then $\mathcal{F}(G_\beta)$ has a vertical toral band. This includes a range of values for g_2, and for a fixed g_2, a range of values for β.

PROOF. The proof has been outlined above. The existence of the two attracting fixed points follows from the concavity of the graph of G_β on $\mathbb{R}$ changing exactly at t_η, and the Mean Value Theorem, which shows that $G'_\beta(p_j) < 1$ for $j = 1, 2$. □

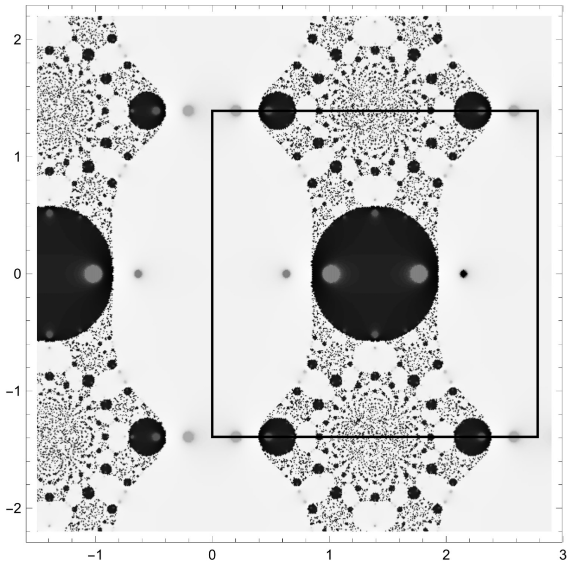

FIGURE 3. For G_β, with $g_2 = 3.15$ and $\beta \approx 2.458$ from Example **??**, there is a vertical toral band containing the attracting fixed point $p_2 \approx 2.143$, and there is another attracting fixed point at $p_1 \approx 1.765$. The repelling fixed point separating them is $t_\eta \approx 1.946$; a fundamental region is shown. (See Rk **??** for details.)

Remark 4.3. While Theorem **??** holds for larger values of g_2, this method does not yield a vertical toral band once g_2 approaches 4, since $G_\beta(c_2)$ lands in the interval B_1, the attracting basin of the smaller of the 2 fixed points, so $G_\beta(V_0) \not\subset B_2$. Nevertheless slightly different methods continue to yield vertical toral bands for other values of β as shown in Figure **??**.

5. The value $g_2 = 3$

We now have the tools to show that if $g_2 = 3$, even though we have no real repelling fixed points, a certain choice of β gives rise to a vertical toral band in $\mathcal{F}(G_\beta)$. This is shown in Figure **??** and proved in Theorem **??**.

Lemma 5.1. *If* $G_\beta = -\dfrac{3}{4\wp_\Lambda(z)} + \beta$, *with* $\beta \in [0, \lambda_1)$*, the following hold:*

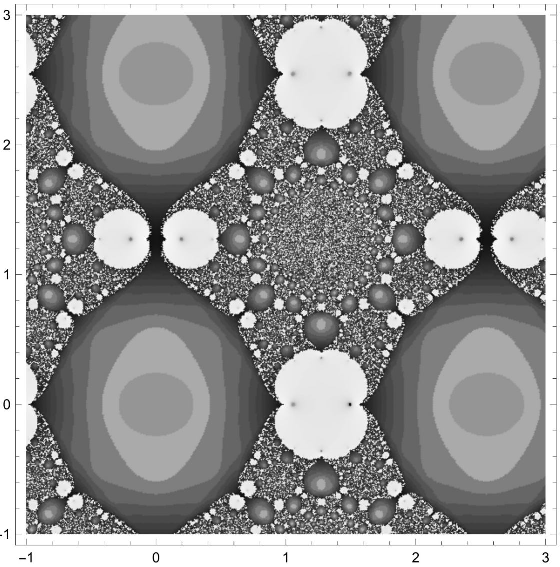

FIGURE 4. For G_β, with $g_2 = 4.5$ and $\beta \approx 2.450$ there is a gray vertical toral band containing the attracting fixed point $p_2 \approx 2.437$, and there is another attracting fixed point at $p_1 \approx 1.492$, even through the technique in Thm **??** for choosing β does not work. (See Rk **??** for details.)

(1) $|G'_\beta(z)| \leqslant 1$*, and there are two points in each periodic interval of* $\mathbb{R}$ *such that* $|G'_\beta(z)| = 1$*; one point is* t_η *where* $G'_\beta(t_\eta) = 1$*, and the other is* s_η *where* $G'_\beta(s_\eta) = -1$*.*

(2) We can choose $t_\eta \in (c_1, \lambda_1)$*, and the point* s_η *satisfies* $s_\eta = 2c_1 - t_\eta \in (0, c_1)$*.*

(3) If $G_\beta(t_\eta) = t_\eta$ *for some* β*, and* $G'_\beta(t_\eta) = 1$*, then* t_η *is attracting along* $\mathbb{R}$ *in the sense that there is a real basin of attraction containing a critical point and an open interval* $B \subset \mathbb{R}$ *with* $t_\eta \in B$*, with the property that for all* $t \in B$*,* $\lim_{n\to\infty} G^n_\beta(t) = t_\eta$*.*

PROOF. To show (1), our starting point is that on $L = \{t + c_2 : t \in \mathbb{R}\}$, in each periodic interval, there are two critical points for $\wp'_\Lambda$. The critical values for $\wp'_\Lambda$ are ± 1, and they occur at points u where $\wp_\Lambda(u + c_2) = -1/2$. Therefore we write $\wp_\Lambda^{-1}(-1/2) = u + c_2$, choosing the smallest possible $u > 0$ such that $\wp_\Lambda(u+c_2) = -1/2$. At the point $u+c_1$, for every β, $G'_\beta(u+c_1) = \wp'_\Lambda(u+c_1-c_3) =$

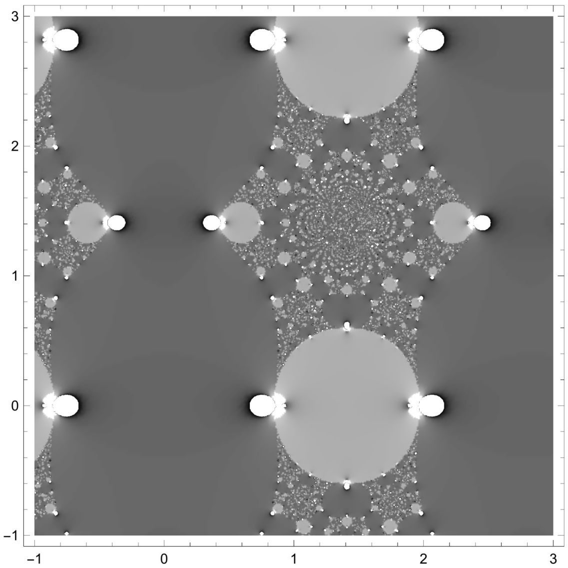

FIGURE 5. Using $g_2 = 3, g_3 = 0, and\beta \approx 2.47$, there is one parabolic fixed point, $t_\eta \approx 1.92$, with 2 Leau petals. $\mathcal{J}(G_\beta)$, separates the preimages of Leau petals shown in shades of gray. See also Rk **??**.

$\wp'_\Lambda(u + c_2) = 1$. To have $\wp'_\Lambda(u + c_2) = 1$, we choose the larger of the 2 possible values of u in the interval $[0, \lambda_1)$.

Since G_β is decreasing on $[0, c_1]$, to obtain $G'_\beta(z) = 1$, we have $u + c_1 > c_1$ (where G_β is increasing and therefore the derivative is positive). It follows that $t_\eta \in (c_1, \lambda_1)$.

We have that $\wp_\Lambda$ and G_β are even functions, but $\wp'_\Lambda$ and G'_β are odd functions. Therefore $G_\beta(-t_\eta) = G_\beta(t_\eta)$, and $G'_\beta(t_\eta) = 1$ while $G'_\beta(-t_\eta) = -1$. Therefore setting $s_\eta = \lambda_1 - t_\eta$ gives the result. Statement (3) holds because the Schwarzian derivative of $G_\beta < 0$ on $\mathbb{R}$ [**?**], and therefore a parabolic fixed point must be topologically attracting [**?**]. $\square$

We now find a parameter β for which the point t_η is fixed under G_β, and then show G_β has a vertical toral band. We first discuss how to choose β.

Algorithm 5.2. Choosing β for which G_β has a vertical toral band.

(i) By Proposition **??** (5), the maximum value of G'_β on $\mathbb{R}$ is 1. We choose the smallest value of $u > 0$ such that $\wp'_\Lambda(u + c_2) = 1$.

(ii) To make this choice, applying Proposition **??**, the derivative condition occurs at points z_0 where $\wp_\Lambda(z_0) = -1/2$. Since we know that there is such a point on the half lattice line L in each fundamental region, we choose $z_0 = u + c_2$, with $u > 0$. Since the derivative is positive, it follows that u must lie in the interval along L where $\wp_\Lambda$ is increasing, so $0 < u < c_1$ and in fact $u \approx 0.5604$ (see also Lemma **??** below).

(iii) Using (**??**) it follows that $G_0(u+c_1) = -1/2$, and $G'_\beta(u+c_1) = 1$. Setting $t_\eta = u + c_1$, we now choose a value of β for which t_η is a fixed point.

(iv) For this we use a technique established in [**?**]. If we choose $\beta = u+c_1+1/2$, then $G_\beta(t_\eta) = G_0(u + c_1) + \beta = -1/2 + 1/2 + t_\eta = t_\eta$.

(v) Since $t_\eta > c_1$, by symmetry of $\wp_\Lambda$, hence G_β, about critical points, there is a symmetric point $s_\eta = c_1 - u$ on $\mathbb{R}$ such that $G_\beta(s_\eta) = t_\eta$.

(vi) The fixed point t_η is a parabolic fixed point, so $t_\eta \in \mathcal{J}(G_\beta)$. We look at the Taylor series expansion for G_β expanded about t_η to determine the number of Leau petals (in the Fatou set) occurring at t_η.

(vii) We have that $G'_\beta(t_\eta) = 1$, and since it is the maximum value of the derivative, $G''_\beta(t_\eta) = 0$. Since $\wp'''_\Lambda = 12\wp_\Lambda\wp'_\Lambda$, it follows that $G'''_\beta(t_\eta) = 12(-1/2)1 = -6$. Using the product rule we also have that $G^{(iv)}_\beta(t_\eta) = 12$. Conjugating the fixed point to the origin, we write

$$\begin{aligned} G_\beta(z + t_\eta) - t_\eta &= z - z^3 + z^4/2 + O(z^5) \\ &= z\left(1 - z^2 + z^3/2 + O(z^4)\right), \end{aligned}$$

so there are two forward invariant petals at the fixed point t_η.

Theorem 5.1. *For the value of β above, $\mathcal{F}(G_\beta)$ contains a vertical toral band.*

Proof. Since $u \in (0, c_1)$, the point $s_\eta = c_1 - u$ satisfies $G_\beta(s_\eta) = t_\eta$ and the interval $A = [s_\eta, t_\eta]$ satisfies $G_\beta(A) = [\beta - \sqrt{3}/2, t_\eta] = [t_\eta + (1 - \sqrt{3})/2, t_\eta]$. In particular, A is forward invariant under G_β, and all points iterate to t_η. Also, $s_\eta, t_\eta \in \mathcal{J}(G_\beta)$. This follows from Lemma **??** (2) and the fact that $G_\beta(c_1) = \beta + e_2 = (u + c_1 + 1/2) - \sqrt{3}/2$.

We now turn to an analysis of the image of the imaginary axis under G_β. We have that $G_\beta(0) = \beta$, and $G_\beta(c_2) = \beta + e_1$, so by periodicity, continuity, and symmetry around fixed points, $G_\beta(\mathbb{I}) = [\beta, \beta + e_1] = K$. It remains to show that this interval lies completely in the Fatou set.

We use the following: (a) $0 < u < c_1/2$. This follows since $\wp_\Lambda(u + c_2) = -1/2$, and using Corollary **??**, $0 > \wp_\Lambda(c_1/2 + c_2) = (\sqrt{3} - \sqrt{6})/2 > -1/2$; and $\wp_\Lambda$ is increasing on $(0, c_1)$. (b) We have $\beta + e_1 - \lambda_1 = u + 1/2 + \sqrt{3}/2 - c_1$.

We claim that $u + 1/2 + \sqrt{3}/2 - c_1 < c_1/2 < s_\eta$. To prove this we use (a) and reduce it to showing:

$$1 + \sqrt{3} < 2c_1. \tag{13}$$

This can be shown with standard estimation techniques. For example, using $\kappa > 1.85$, and $3^{-1/4} > .75$, we have $2c_1 > 2.775$. Similarly we can show that $1 + \sqrt{3} < 2.733$ and the inequality follows.

We have shown that the imaginary axis and all parallel lines along the vertical boundary curves of square regions, are mapped under G_β to a line segment along $\mathbb{R}$ that does not intersect $A + \Lambda$. Then intervals $\{G_\beta^n(K)\}_{n\in\mathbb{N}} \subset (t_\eta, \beta]$ and the family of maps is normal there. Therefore $\mathbb{I} \subset \mathcal{F}(G_\beta)$ and is contained in a Fatou component. $\square$

Since there are two disjoint immediate basins of attraction for each of the two Leau petals for G_β, the Julia set cannot be totally disconnected; therefore $\mathcal{J}(G_\beta)$ is not a Cantor set, as shown in Figure **??**.

5.1. **The case when s_η is a fixed point.** Using $s_\eta = -u + c_1$, with u as in Algorithm **??** (i), suppose $\beta = s_\eta + 1/2$. (Note that all other choices for β yield a Cantor Julia set except for the value in Theorem **??**.) Then $G_\beta(s_\eta) = \wp_\Lambda(-u + c_2) + \beta = -1/2 + s_\eta + 1/2 = s_\eta$, and by our choice above, $G'_\beta(s_\eta) = -1$. For this value of β, we label the critical values of G_β as follows:

- $b_0 = G_\beta(0) = \beta = s_\eta + 1/2$;
- $b_1 = G_\beta(c_1) = \beta - e_1 = s_\eta + \dfrac{1-\sqrt{3}}{2}$;
- $b_2 = G_\beta(c_2) = \beta + e_1 = s_\eta + \dfrac{1+\sqrt{3}}{2}$;

In this case there are also two petals, P_0 and P_1, at the neutral fixed point s_η, but they alternate under G_β; P_j is mapped into itself under G_β^2 for $j = 0, 1$. We show that $\mathcal{F}(G_\beta)$ does not contain a vertical toral band. This will follow from the next several results. In what follows, let Q denote the fundamental square region for $\mathbb{C}/\Lambda$ with vertices $\{0, \lambda_1, \lambda_1 + i\lambda_1, i\lambda_1\}$.

Lemma 5.3. $u \in (.5, .65)$.

PROOF. We give the heuristic argument first. Recall that u is defined so that the derivative $\wp'_\Lambda$ achieves its maximum value, which is 1, at its image $\wp_\Lambda(u + c_2) = 1/2$. Therefore we have that $\wp_\Lambda(t + c_2)$ is approximately the identity along L at $t = u$. Hence u is approximately 1/2; in fact as mentioned above, we have $u \approx .56$. To prove it is in the interval given involves looking at the series expansion of the

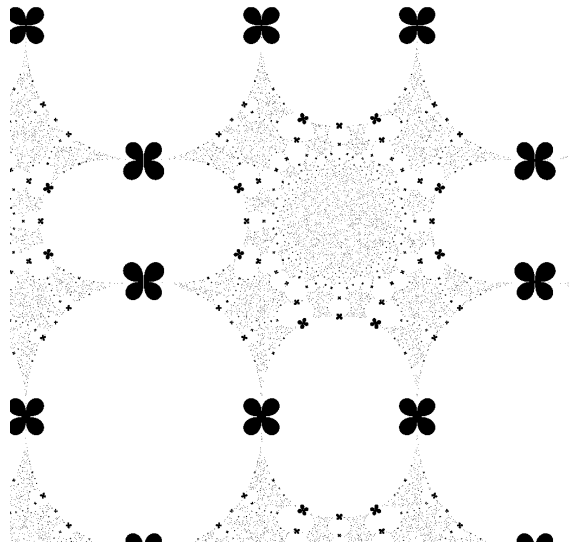

FIGURE 6. Connected Julia set for G_β, with $g_2 = 3$ and $\beta = s_\eta + 1/2$, with no vertical toral band. There is exactly one real fixed point, $z_0 \approx .875$, and $G'(z_0) = -1$.

analytic function $\wp_\Lambda$ near c_2 (with radius of convergence c_1). The concavity of the graph changes at u from positive to negative.

There are precise (but cumbersome) formulas for the derivatives $\{\wp_\Lambda^{(k)}(c_2)\}_{k\geqslant 1}$ for $g_2 = 3$, so we write the Taylor series expansion for $\wp_\Lambda$. Recall that $\wp_\Lambda$ is an even function so only even powers appear, and the series alternates:

$$\wp_\Lambda(t + c_2) = G_0(t + c_1) = -\frac{\sqrt{3}}{2} + \frac{3}{2}t^2 - \frac{3\sqrt{3}}{4}t^4 + \frac{648}{6!}t^6 - \dots.$$

A few more terms are needed before we find polynomial approximations to $\wp_\Lambda(t + c_2)$ that establish that $\wp_\Lambda''(.5 + c_2) > 0$ and $\wp_\Lambda''(.65 + c_2) < 0$, so that $\wp_\Lambda''(u + c_2) = 0$ for a value of u between these values. This proves the result. $\square$

Proposition 5.2. *Assume* $(g_2, g_3) = (3, 0)$, $\beta = -u + c_1 + 1/2$ *with* G_β *and* u *as above, and* $s_\eta = -u + c_1$, $t_\eta = u + c_1$. *Then the following hold:*

(1) $G_\beta(s_\eta) = s_\eta$ *and* $G_\beta'(s_\eta) = -1$.

(2) For the critical values $b_0, b_1,$ *and* b_2,

$$0 < b_1 < s_\eta < b_0 < c_1 < t_\eta < b_2 < \lambda_1. \tag{14}$$

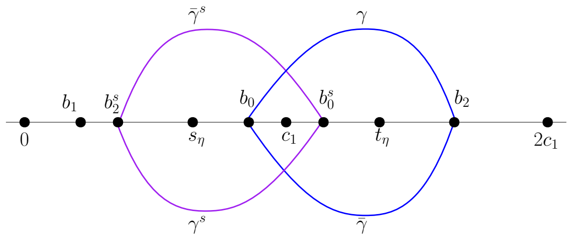

FIGURE 7. Illustration of the proof of Proposition **??**.

PROOF. By our choice of β and s_η, it suffices to prove (**??**). Using the estimates in the proof of Theorem **??**, Lemma **??**, and the choice of β, it is straightforward to see that $0 < b_1 < c_1$. The inequality $b_2 < \lambda_1$ follows from (**??**), and $b_0 < c_1$ by Lemma **??**. Therefore $G_\beta(\mathbb{R}) = [b_1, b_0]$ contains no critical point of G_β, so $G_\beta^2 = G_\beta \circ G_\beta$ has the same critical points in $[0, \lambda_1]$ as G_β. (A critical point ζ for G_β^2, apart from $[0]$ and $[c_1]$, necessarily satisfies $G_\beta(\zeta) \in [0] \cup [c_1]$.)

By the theory of Leau petals, G_β^2 has two forward invariant petals, each containing a (real) critical point (of G_β^2) on its boundary. However G_β^2 has the same critical points as G_β on $\mathbb{R}$, so by the periodicity of G_β and symmetry around critical points, the petals have attracting directions along $\mathbb{R}$. Since one petal contains a subinterval of $[0, s_\eta]$, it must contain the smallest critical value, $b_1 < s_\eta$ so we denote it by P_1. We denote the other petal by P_0. Since $(1-\sqrt{3})/2 < 0$ and all critical values are positive, $G_\beta(t_\eta) = s_\eta$, and if $J = [s_\eta, t_\eta]$, $G_\beta(J) = [b_1, s_\eta] = [s_\eta + (1-\sqrt{3})/2, s_\eta]$. Therefore if J contains the critical value $b_0 = s_\eta + 1/2 < t_\eta$, then $J \subset \overline{P_0}$.

This follows since $t_\eta - s_\eta = 2u > 1$ by Lemma **??**; therefore $s_\eta < b_0 < t_\eta$. Showing $b_2 > t_\eta$ is equivalent to proving

$$-u + \frac{1+\sqrt{3}}{2} > u, \text{ or } \quad 2u < \frac{1+\sqrt{3}}{2}.$$

This follows from Lemma **??**, and the inequalities are established. □

Proposition 5.3. *Using the notation as above, the three critical values* $0 < b_1 < b_0 < b_2 < \lambda_1$ *lie in distinct components of* $\mathcal{F}(G_\beta)$.

PROOF. As in the proof of Proposition **??**, there are two disjoint forward invariant Fatou components F_0 and F_1, for G_β, each containing a petal P_0 and P_1 respectively, with $b_0 \in F_0$, and $b_1 \in F_1$. It remains to check whether or not b_2 lies in F_0 or F_1. By (**??**) in Proposition **??**, $b_2 \in (t_\eta, \lambda_1)$. By symmetry with respect to the critical points 0 (and λ_1), $G_\beta(t) = G_\beta(\lambda_1 - t)$ for all t, so $G_\beta([0, s_\eta]) = G_\beta([t_\eta, \lambda_1]) = [s_\eta, b_1] \subset F_1$.

Therefore if b_2 is in one of F_0 or F_1, it must be F_0 (since F_1 does not map into itself under G_β); therefore we assume that b_0 and b_2 lie in F_0. From (**??**) we have that $c_1 \in (b_0, b_2)$ and if we write $\delta = c_1 - b_0$, we consider the point symmetric to b_0 about c_1 and call it $b_0^s = c_1 + \delta$. Since $2\delta < e_1$ by Lemma **??**, we have that $b_0 < c_1 < b_0^s < b_2$; we note that $[b_0, b_2] \cap \mathcal{J}(G_\beta) \neq \emptyset$. Nevertheless from our assumption it follows that there is a smooth path $\gamma : [0,1] \to F_0$ such that $\gamma(0) = b_0$ and $\gamma(1) = b_2$, since F_0 is path connected. Referring to both the map and its image as γ, we can choose γ to lie completely in Q except at its endpoints, since F_0 is open, by reflecting about any boundary lines of Q.

Defining $\overline{\gamma}$ by $\overline{\gamma}(t) = \overline{\gamma(1-t)}$, let Ω be the inside of the region in $\mathbb{C}$ bounded by the simple closed curve $\gamma^* = \gamma \cup \overline{\gamma}$, where the component (of $\mathbb{C}$) containing c_1 is called the inside of γ^*. We have that $c_1, b_0^s \in \Omega$. We next set $b_2^s = c_1 - e_1 + \delta$ to be the point symmetric to b_2 with respect to c_1. Now $b_1 = \beta - e_1 = c_1 - e_1 - \delta < b_2^s < s_\eta$, so $b_2^s \in F_1$; b_2^s lies outside of γ^* by (**??**).

We define the symmetric path γ^s starting at b_2^s and ending at b_0^s to be the reflection of γ about c_1; this is given by $\gamma^s(t) = \lambda_1 - \gamma(1-t)$. By the symmetry shown in Lemma **??**, it follows that $\gamma^s \subset \mathcal{F}(G_\beta)$ (see Figure **??**).

Since b_0^s is inside γ^*, and b_2^s is outside γ^*, it follows that $\overline{\gamma^s} \cap \gamma \neq \emptyset$ by the Jordan Curve Theorem. Choose a point $z_0 \in \overline{\gamma^s} \cap \gamma$. This gives a (piecewise smooth) path ρ given by $\overline{\gamma}^s$ from $b_2^s \in F_1$ to z_0, and γ from z_0 to $b_2 \in F_0$, and lying completely in $\mathcal{F}(G_\beta)$; this is impossible. Therefore b_2 must lie in a component of the Fatou set disjoint from both F_0 and F_1. ☐

Corollary 5.4. *For $(g_2, g_3) = (3, 0)$, and $\beta = -u + c_1 + 1/2$ as above, $\mathcal{J}(G_\beta)$ is connected and $\mathcal{F}(G_\beta)$ does not have a vertical toral band component.*

Proof. By Proposition **??**, each Fatou component contains at most one critical value. Therefore we apply a result from ([**?**], Theorem 3.1) to conclude that $\mathcal{J}(G_\beta)$ is connected.

To show G_β does not have a vertical toral band, it is enough to show that $\mathbb{I} \cap \mathcal{J}(G_\beta) \neq \emptyset$. The point $t_\eta \in \mathcal{J}(G_\beta)$ since $G_\beta(t_\eta) = s_\eta$, and by (**??**) in Proposition **??**, there is a preimage of t_η on the interval in $\mathbb{I}$ from 0 to c_2, as well as its reflected image about c_2 along $\mathbb{I}$ on $(c_2, 2c_2)$, using Remark **??** (3) and (4). By invariance of $\mathcal{J}(G_\beta)$, the result follows. ☐

The Julia set for G_β from Corollary **??** is shown in Figure **??** and Figure **??** shows a sketch supporting the proof of Proposition **??**

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