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Ergodic Properties and Julia Sets of Weierstrass Elliptic Functions

By

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Abstract. We discuss properties of the Julia and Fatou sets of Weierstrass elliptic $\wp$ functions arising from real lattices. We give sufficient conditions for the Julia sets to be the whole sphere and for the maps to be ergodic, exact, and conservative. We also give examples for which the Julia set is not the whole sphere.

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1. Introduction

The importance of the Weierstrass elliptic $\wp$ function in complex dynamics goes back to the inception of the subject [18]. Indeed, if Λ denotes any lattice in the plane of the form $\{m + n\lambda\}$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$, $m, n \in \mathbb{Z}$, then for each integer $d \geqslant 2$, we obtain a rational map $R_{d,\Lambda}$ of the sphere of degree d^2. This construction, which yields a rational map with Julia set the whole sphere, dates back to Lattès [18] and Böettcher [8] and proceeds as follows. We consider the toral endomorphism $z \mapsto dz$ on $\mathbb{C}/\Lambda$, then look at the quotient endomorphism induced on the sphere via the branched 2-fold covering of the sphere by $\wp_\Lambda$; i.e., $R_{d,\Lambda} \circ \wp_\Lambda(z) = \wp_\Lambda(dz)$ for all $z \in \mathbb{C}$. In addition, the resulting map $R_{d,\Lambda}$ is ergodic, exact, and conservative with respect to m, the normalized Lebesgue measure on the sphere (i.e., the Riemannian volume form). Furthermore it is known that for a fixed degree $d \geqslant 2$, all rational maps arising this way are measure theoretically isomorphic to the one-sided Bernoulli shift on d^2 states with equal probability [5], hence to each other. Studies on the exactness and isomorphism to Bernoulli shifts have been extended to non-integer toral endomorphisms using other elliptic functions as well [16].

Moreover, elliptic functions are meromorphic functions on $\mathbb{C}$ which can be iterated (except at poles) and exhibit some dynamical behavior distinct from that of rational maps (see [7], [10], and [15] and the references contained therein).

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The dynamics of elliptic functions are studied in [17], and it is the results obtained in [17] that motivated this paper.

Here we consider the Weierstrass elliptic $\wp$ function in an attempt to understand how the dynamics of $\wp_\Lambda$ depend on the lattice Λ. In contrast to the Lattès examples, for square lattices generated by λ and λi for some $\lambda > 0$, the side length λ affects the dynamics of $\wp_\Lambda$ as does the placement of the square in the plane (e.g., $e^{\pi i/4}\lambda$ and $e^{-\pi i/4}\lambda$ can generate an elliptic function with different dynamics).

We discuss some properties of the Julia and Fatou sets of Weierstrass elliptic $\wp$ functions arising from general lattices. We give sufficient conditions for the Julia sets to be the whole sphere and for the maps to be ergodic, exact, and conservative. We also give examples of Weierstrass elliptic maps for which the Julia set is not the whole sphere. Most of the results in this paper are restricted to the study of real lattices – that is, lattices for which $\overline{\Lambda} = \Lambda$.

We begin with some preliminaries on the dynamics of meromorphic functions and elliptic functions, and then discuss some properties of real lattices and their associated Weierstrass elliptic $\wp$ functions. In Sections 4–6 we discuss the dynamics and the structure of the Fatou and Julia sets for $\wp_\Lambda$, where Λ is a real lattice. In Section 7 we give sufficient conditions for an elliptic function to be conservative and exact with respect to Lebesgue measure, and using the results of [17] we show that many Weierstrass elliptic functions can be shown to satisfy these conditions. Results about ergodicity of meromorphic functions have also appeared in [15], and we use some of the basic ideas for showing ergodicity of meromorphic functions given there.

We conclude with examples of Weierstrass elliptic $\wp$ functions exhibiting both exact and nonergodic behavior with respect to the measure m.

2. Iteration of Meromorphic Functions

Let $f : \mathbb{C} \to \mathbb{C}_\infty$ be a meromorphic function where $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. The *Fatou set* $F(f)$ is the set of points $z \in \mathbb{C}_\infty$ such that $\{f^n\colon n \in \mathbb{N}\}$ is defined and normal in some neighborhood of z. The *Julia set* is the complement of the Fatou set on the sphere, $J(f) = \mathbb{C}_\infty \backslash F(f)$. Notice that $\mathbb{C}_\infty \backslash \overline{\bigcup_{n \geqslant 0} f^{-n}(\infty)}$ is the largest open set where all iterates are defined. If f has at least one pole that is not an omitted value, then $\bigcup_{n \geqslant 0} f^{-n}(\infty)$ has more than two elements. Since $f(\mathbb{C}_\infty \backslash \overline{\bigcup_{n \geqslant 0} f^{-n}(\infty)}) \subset \mathbb{C}_\infty \backslash \overline{\bigcup_{n \geqslant 0} f^{-n}(\infty)}$, Montel's theorem implies that

$$J(f) = \overline{\bigcup_{n \geqslant 0} f^{-n}(\infty)}.$$

Let $\mathrm{Crit}(f)$ denote the set of critical points of f, *i.e.*,

$$\mathrm{Crit}(f) = \{z : f'(z) = 0\}.$$

If z_0 is a critical point then $f(z_0)$ is a *critical value*. We say that w is an *asymptotic value* of f if there exists a path α: $[0, \infty) \to \mathbb{C}$ such that $\lim_{t\to\infty} |\alpha(t)| = \infty$ and $\lim_{t\to\infty} f(\alpha(t)) = w$. The *singular set* $\mathrm{Sing}(f)$ of f is the set of critical and finite asymptotic values of f and their limit points. A function is called Class S if f has only finitely many critical and asymptotic values. The *postcritical set* of f is $P(f) = \overline{\bigcup_{n \geqslant 0} f^n(\mathrm{Sing}(f))}$.

A point z_0 is *periodic* if there exists a $p \geqslant 1$ such that $f^p(z_0) = z_0$. A periodic point z_0 is called *attracting*, *repelling*, or *neutral* if $|(f^p)'(z_0)|$ is less than, greater than, or equal to 1 respectively. If $|(f^p)'(z_0)| = 0$ then z_0 is called a *superattracting* periodic point. As in the case of rational maps, the Julia set is the closure of the repelling periodic points [1].

Suppose U is a connected component of the Fatou set. We say that U is *preperiodic* if there exists $n > m \geqslant 0$ such that $f^n(U) = f^m(U)$, and the minimum of $n - m = p$ for all such n, m is the *period* of the cycle. A component of F which is not preperiodic is called a *wandering domain*.

The classification of periodic components of the Fatou set of a meromorphic function is slightly more complicated than that of rational maps of the sphere. Periodic components of $F(f)$ may be attracting domains, Leau domains, Siegel disks, Herman rings, or Baker domains [1]. The definitions of the first four types of behavior are the same as for rational maps of the sphere. A periodic component U of period p is called a *Baker domain* if there exists a $z_0 \in \partial U$ such that $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$, but $f^p(z_0)$ is not defined.

If f is Class S then f does not have wandering domains [3] or Baker domains [12]. Thus the classification of periodic components of the Fatou set of meromorphic functions which are Class S is no more complicated than that of rational maps of the sphere.

Let $C = \{U_0, U_1, \ldots, U_{p-1}\}$ be a periodic cycle of components of $F(f)$. If C is a cycle of immediate attractive basins or Leau domains, then $U_j \cap \mathrm{Sing}(f) \neq \emptyset$ for some $0 \leqslant j \leqslant p-1$. If C is a cycle of Siegel Disks or Herman rings, then $\partial U_j \subset \overline{\bigcup_{n \geqslant 0} f^n(\mathrm{Sing}(f))}$ for all $0 \leqslant j \leqslant p-1$. In particular, singular points are required for any type of preperiodic Fatou component.

3. Lattices in the Plane

3.1. Lattices and fundamental regions. Let $\lambda_1, \lambda_2 \in \mathbb{C} \backslash \{0\}$ such that $\lambda_2/\lambda_1 \notin \mathbb{R}$. We define a lattice of points in the complex plane by $\Lambda = [\lambda_1, \lambda_2] := \{m\lambda_1 + n\lambda_2 : m, n \in \mathbb{Z}\}$. It is well-known that two different sets of vectors can generate the same lattice Λ. If $\Lambda = [\lambda_1, \lambda_2]$, then any other generators λ_3, λ_4 of Λ are obtained by multiplying the vector (λ_1, λ_2) by the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$.

We can view Λ as a group acting on $\mathbb{C}$ by translation, each $\omega \in \Lambda$ inducing the transformation of $\mathbb{C}$:

$$T_\omega : z \mapsto z + \omega.$$

Definition 3.1. A closed, connected subset Q of $\mathbb{C}$ is defined to be a *fundamental region* for Λ if

1. for each $z \in \mathbb{C}$, Q contains at least one point in the same Λ-orbit as z;
2. no two points in the interior of Q are in the same Λ-orbit.

If Q is any fundamental region for Λ, then for any $s \in \mathbb{C}$, the set

$$Q + s = \{z + s : z \in Q\}$$

is also a fundamental region. Usually (but not always) we choose Q to be a polygon with a finite number of parallel sides, in which case we call Q a period parallelogram for Λ.

3.2. Real lattices. Frequently we refer to types of lattices by the shapes of the corresponding period parallelograms. In this paper we will focus on real lattices and specific examples of these.

Definition 3.2. A meromorphic function $f : \mathbb{C} \to \mathbb{C}_\infty$ is *real* if $f(\overline{z}) = \overline{f(z)}$ for all $z \in \mathbb{C}$ (where $\overline{\infty} = \infty$). A lattice Λ is *real* if $\overline{\Lambda} = \Lambda$.

The real lattices have distinctive "shapes" for their period parallelograms.

Definition 3.3. We say that $\Lambda = [\lambda_1, \lambda_2]$ is *real rectangular* if there exist generators such that λ_1 is real and λ_2 is purely imaginary. We say that $\Lambda = [\lambda_1, \lambda_2]$ is *real rhombic* if there exist generators such that $\lambda_2 = \overline{\lambda_1}$. In each case the period parallelogram with vertices $0, \lambda_1, \lambda_2$, and $\lambda_1 + \lambda_2$ is rectangular or rhombic respectively.

Theorem 3.4. [14] *A lattice Λ is real if and only if it is real rectangular or real rhombic.*

Among real lattices, those having the most regular period parallelograms are distinguished.

Definition 3.5. A lattice is *square* if $\Lambda = i\Lambda$. A lattice is *triangular* if $\Lambda = e^{2\pi i/3}\Lambda$ (in which case a period parallelogram can be made from two equilateral triangles).

Lemma 3.6. *A lattice Λ is square if and only if it has generators of equal length and forming a right angle with each other. More precisely, every real square lattice is either rectangular and $\Lambda = [\lambda, \lambda i]$ for some positive λ, or is rhombic and $\Lambda = [e^{\pi i/4}\lambda, e^{-\pi i/4}\lambda]$ for some positive λ.*

Lemma 3.7. *A lattice is triangular if and only if it has generators of equal length and forming an angle of $2\pi/3$ with each other. More precisely, every real triangular lattice is given by $\Lambda = [\lambda e^{\pi i/3}, \lambda e^{-\pi i/3}]$ for some positive λ.*

Sometimes the shape of a lattice is not evident from its presentation.

Example. $\Lambda = [1, 1 + i]$ is a rectangular square lattice since $\Lambda = [1, i]$ as well.

In the following, if Λ is rectangular we assume that generators have been selected so that λ_1 is positive real and λ_2 is positive imaginary. If Λ is rhombic we assume that λ_1 is in quadrant I and $\lambda_2 = \overline{\lambda_1}$.

4. The Weierstrass Elliptic Function

For any $z \in \mathbb{C}$ and any lattice Λ, the Weierstrass elliptic function is defined by

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right).$$

Replacing every z by $-z$ in the definition we see that $\wp_\Lambda$ is an even function. It is well-known that $\wp_\Lambda$ is meromorphic and is periodic with respect to Λ.

Lemma 4.1. [11] *If Λ is any lattice and $u \in \mathbb{C}$, let $\Omega_u = \{u + s\lambda_1 + t\lambda_2 : 0 \leqslant s, t < 1\}$. Then $\wp_\Lambda : \Omega_u \to \mathbb{C}_\infty$ is onto and 2-to-1 except at the points that lie in the Λ-orbit of $0, \lambda_1/2, \lambda_2/2$, and $(\lambda_1 + \lambda_2)/2$.*

The derivative of the Weierstrass elliptic function is also an elliptic function which is periodic with respect to Λ defined by

$$\wp'_\Lambda(z) = -2 \sum_{w \in \Lambda \setminus \{0\}} \frac{1}{(z-w)^3}.$$

The Weierstrass elliptic function and its derivative are related by the differential equation

$$\wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 - g_2\wp_\Lambda(z) - g_3, \tag{1}$$

where $g_2(\Lambda) = 60 \sum_{w \in \Lambda \setminus \{0\}} \frac{1}{w^4}$ and $g_3(\Lambda) = 140 \sum_{w \in \Lambda \setminus \{0\}} \frac{1}{w^6}$. The numbers $g_2(\Lambda)$ and $g_3(\Lambda)$ are invariants of the lattice Λ. Given any g_2 and g_3 such that $g_2^3 - 27g_3^2 \neq 0$ there exists a lattice having g_2 and g_3 as its invariants.

Theorem 4.2. [14] *The following are equivalent*:

1. $\wp_\Lambda$ *is a real function*;
2. Λ *is a real lattice*;
3. $g_2, g_3 \in \mathbb{R}$.

For any lattice Λ, the Weierstrass elliptic function and its derivative satisfy the following properties: for $u \neq v + \Lambda$ and $k \in \mathbb{C} \setminus \{0\}$,

$$\wp_{k\Lambda}(ku) = \frac{1}{k^2}\wp_\Lambda(u), \quad \text{(homogeneity of } \wp_\Lambda\text{)}, \tag{2}$$

$$\wp'_{k\Lambda}(ku) = \frac{1}{k^3}\wp'_\Lambda(u), \quad \text{(homogeneity of } \wp'_\Lambda\text{)},$$

$$\wp_\Lambda(u+v) = -\wp_\Lambda(u) - \wp_\Lambda(v) + \frac{1}{4}\left(\frac{\wp'_\Lambda(u) - \wp'_\Lambda(v)}{\wp_\Lambda(u) - \wp_\Lambda(v)} \right)^2 \text{ (addition property)}.$$

Verification of the first two properties can be seen by substitution into the series definitions. For a proof of the addition property see [11].

Lemma 4.3. *If Λ is real then $\wp_\Lambda$ maps the imaginary axis to the real axis.*

Proof. If Λ is rectangular then $i\Lambda$ is also rectangular. Theorem 4.2 implies that $\wp_{i\Lambda}$ maps the reals to the reals, and Equation 2 gives, for $b \in \mathbb{R}$

$$\wp_{i\Lambda}(-b) = \frac{1}{i^2}\wp_\Lambda(ib) = -\wp_\Lambda(ib),$$

so $\wp_\Lambda(ib)$ is real as well.

If $\Lambda = [\lambda, \bar{\lambda}]$ where $\lambda = a + ib$ is rhombic then $i\Lambda = [i\bar{\lambda}, \overline{i\bar{\lambda}}]$ is also rhombic. An application of Theorem 4.2 and Equation 2 gives the result. □

Lemma 4.4. *If Λ is rectangular or rhombic square then $\wp_\Lambda(a+ai)$ and $\wp_\Lambda(a-ai)$ are pure imaginary.*

Proof. By the addition property of $\wp_\Lambda$ and the homogeneity properties of $\wp_\Lambda$ and $\wp'_\Lambda$ (see Equation 2),

$$\begin{aligned}\wp_\Lambda(a+ai) &= -\wp_\Lambda(a) - \wp_\Lambda(ai) + \frac{1}{4}\left(\frac{\wp'_\Lambda(a) - \wp'_\Lambda(ai)}{\wp_\Lambda(a) - \wp_\Lambda(ai)}\right)^2 \\ &= -\wp_\Lambda(a) + \wp_\Lambda(a) + \frac{1}{4}\left(\frac{\wp'_\Lambda(a) - i\wp'_\Lambda(a)}{\wp_\Lambda(a) + \wp_\Lambda(a)}\right)^2 \\ &= \frac{-i}{8}\left(\frac{\wp'_\Lambda(a)}{\wp_\Lambda(a)}\right)^2\end{aligned}$$

But $\wp_\Lambda(a)$ and $\wp'_\Lambda(a)$ are both real by the previous lemma. A similar argument works for $a - ai$. □

We can easily determine the critical values of the Weierstrass elliptic function on an arbitrary lattice $\Lambda = [\lambda_1, \lambda_2]$. For $j = 1, 2$, notice that $\wp_\Lambda(\lambda_j - z) = \wp_\Lambda(z)$ for all z. Taking derivatives of both sides we obtain $-\wp'_\Lambda(\lambda_j - z) = \wp'_\Lambda(z)$. Substituting $z = \frac{\lambda_1}{2}, \frac{\lambda_2}{2}$, or $\frac{\lambda_1+\lambda_2}{2}$, we see that $\wp'_\Lambda(z) = 0$ at these values. We use the notation

$$e_1 = \wp_\Lambda\left(\frac{\lambda_1}{2}\right), \quad e_2 = \wp_\Lambda\left(\frac{\lambda_2}{2}\right), \quad e_3 = \wp_\Lambda\left(\frac{\lambda_1+\lambda_2}{2}\right)$$

to denote the critical values. Since e_1, e_2, e_3 are the distinct zeros of Equation 1, we also write

$$\wp'_\Lambda(z)^2 = 4(\wp_\Lambda(z) - e_1)(\wp_\Lambda(z) - e_2)(\wp_\Lambda(z) - e_3). \tag{3}$$

Equating like terms in Equations 1 and 3, we obtain

$$e_1 + e_2 + e_3 = 0, \quad e_1e_3 + e_2e_3 + e_1e_2 = \frac{-g_2}{4}, \quad e_1e_2e_3 = \frac{g_3}{4}. \tag{4}$$

Naturally, the lattice shape relates to the properties and dynamics of the corresponding Weierstrass elliptic function. Denote $p(x) = 4x^3 - g_2x - g_3$, the polynomial associated with Λ. Let $\triangle = g_2^3 - 27g_3^2 \neq 0$ denote its discriminant.

Proposition 4.5. [11] 1. *If Λ is real rectangular, then the discriminant is positive and $g_2 > 0$; in this case the roots of p are three distinct real roots. If $g_3 > 0$ then the vertical legs of the rectangles are longer than the horizontal legs,* *and if* $g_3 < 0$ *then the horizontal legs of the rectangles are longer than the vertical legs.*

2. *If* Λ *is rectangular square, then* $g_2 > 0$ *and* $g_3 = 0$; *in this case the roots of p are* $0, \pm\sqrt{g_2}/2$.

3. *If* Λ *is real rhombic, then the discriminant is negative*; *in this case the roots of p are one real and a complex conjugate pair. If* $g_3 > 0$ *then the vertical diagonal of the rhombus is longer than the horizontal diagonal, and if* $g_3 < 0$ *then the horizontal diagonal of the rhombus is longer than the vertical diagonal.*

4. *If* Λ *is rhombic square, then* $g_2 < 0$ *and* $g_3 = 0$; *in this case the roots of p are* $0, \pm\sqrt{g_2}/2$.

5. *If* Λ *is triangular, then* $g_2 = 0$. *In this case the roots of p are the cube roots of* $g_3/4$.

The following corollary, which can be obtained using Equations 1 and 4, provides useful information for the study of the Fatou set of the Weierstrass elliptic function.

Corollary 4.6. 1. *If* Λ *is real rectangular then the critical values are all real. If* $g_3 > 0$ *then* $e_2 < e_3 < 0 < e_1$, *and if* $g_3 < 0$ *then* $e_2 < 0 < e_3 < e_1$.

2. *If* Λ *is rectangular square then* $e_1 = \sqrt{g_2}/2 > 0$, $e_2 = -e_1$, *and* $e_3 = 0$.

3. *If* Λ *is rhombic then* $e_2 = \overline{e_1}$ *are the complex roots, and* e_3 *is real. If* $g_3 > 0$ *then* $e_3 > 0$, *and if* $g_3 < 0$ *then* $e_3 < 0$.

4. *If* Λ *is rhombic square then* $e_3 = 0$, *and* $e_1 = \sqrt{g_2}/2 = -e_2$ *are pure imaginary.*

5. *If* Λ *is triangular then* $e_3 \neq 0$ *is real,* $e_1 = e^{4\pi i/3}e_3$, *and* $e_2 = e^{2\pi i/3}e_3$.

Figure 1 shows the Weierstrass elliptic function restricted to the real line for the lattice with invariants $g_2 = 4$, $g_3 = 0$. Using Equation 4, $e_1 = 1$ for this example. In fact, the graph of the Weierstrass elliptic function restricted to the real line for any real lattice has a similar shape. Lemma 4.7 discusses these properties, which are then used in the proof of Proposition 4.8.

Lemma 4.7. *If* $\wp_\Lambda$ *is real, then it is periodic as a map on* $\mathbb{R}$ *and has infinitely many real critical points and at least one real critical value. If* Λ *is rhombic, then there is exactly one real critical value* e_3 *which is negative if and only if* $g_3 < 0$.

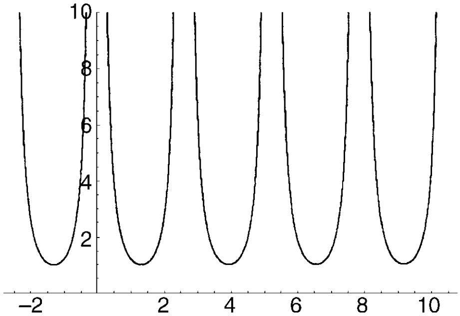

Figure 1. $\wp_\Lambda$ on $\mathbb{R}$ when $g_2 = 4$, $g_3 = 0$

Otherwise there is at least one nonnegative critical value, and in any case the image of the real critical point is the minimum of $\wp_\Lambda$ *on* $\mathbb{R}$. *In particular, if* e_r *denotes the real critical value, then* $\wp_\Lambda|_{\mathbb{R}} : \mathbb{R} \to [e_r, \infty]$ *is piecewise monotonic and onto.*

Proof. By Theorem 3.4, Λ is either real rectangular or real rhombic. If Λ is rectangular then one of the generators is real so the periodicity is clear. If Λ is rhombic then it has generators which are complex conjugate. If $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$, then $\wp_\Lambda(2a) = \wp_\Lambda(\lambda_1 + \lambda_2) = \wp_\Lambda(m(\lambda_1 + \lambda_2)) = \wp_\Lambda(m(2a))$ for all $m \in \mathbb{Z}$, and so the period of $\wp_\Lambda$ on the real line is $2a$.

Since the critical points are the half lattice points, and the sums of the half lattice points, then for a rhombic lattice, all points of the form ma, where m is odd, are critical points. In the rectangular case, infinitely many half lattice points lie along the real axis.

Obviously, since there are infinitely many critical points, one in each periodic interval, then their common value is a real critical value. The last statements follow from Equation (1)–(3) and Proposition 4.5. Since $\wp'_\Lambda$ is strictly increasing, and is negative to the left of a critical point and positive to the right (in each periodic interval), then e_r is the minimum on $\mathbb{R}$. In particular, if a is the real period of $\wp_\Lambda$ and m is any integer, then $\wp_\Lambda : (ma, ma + a/2] \to [e_r, \infty)$ is monotonic and onto, as is $\wp_\Lambda : [ma + a/2, (m+1)a) \to [e_r, \infty)$. □

The symmetry of the lattices affects the dynamics of $\wp_\Lambda$, particularly the postcritical sets.

Proposition 4.8. 1. *If* Λ *is real rectangular, then the postcritical set is real and, except for at most finitely many points (possibly including* ∞*),* $P(\wp_\Lambda)$ *is contained in the nonnegative real axis.*

2. *If* Λ *is real rhombic, then* $P(\wp_\Lambda)$ *consists of a real part (possibly including* ∞*) and two parts that are complex conjugate.*

3. *If* Λ *is rhombic or rectangular square, then* $P(\wp_\Lambda)$ *is real and positive except for finitely many points, and includes* ∞. *Further* $e_3 \in J(\wp_\Lambda)$.

4. *If* Λ *is triangular, then* $P(\wp_\Lambda)$ *is contained in three invariant rays: one ray* $\alpha = [e_3, \infty]$ *along the real axis, and the rays* $e^{2\pi/3}\alpha$ *and* $e^{4\pi/3}\alpha$. *Note that* e_3 *can be positive or negative; if* $e_3 < 0$ *then the three rays intersect at the origin.*

Proof. In each case, we use the results of Proposition 4.5, Corollary 4.6, and Lemma 4.7.

In the rectangular case, since the roots of $\wp_\Lambda$ are real they stay real under iteration by $\wp_\Lambda$, and Lemma 4.7 implies that $P(\wp_\Lambda) \subset [e_1, \infty) \cup \{e_2, e_3, \infty\}$.

If Λ is real rhombic then the forward orbit of e_3 lies along $[e_3, \infty]$. Since $e_2 = \overline{e_1}$ and $\wp_\Lambda$ is real, the forward orbit of e_2 consists of points which are the complex conjugates of the points in the orbit of e_1.

If Λ is rhombic square then these complex conjugate pieces eventually coincide. We know that $e_3 = 0$, which is a prepole. Both e_1 and e_2 lie on the imaginary axis by Proposition 4.5 (4) and also $e_2 = -e_1$. Lemma 4.3 implies that $\wp_\Lambda(e_2) = \wp_\Lambda(e_1)$ is real and negative, and so $\wp^n_\Lambda(e_2)$ lies on the nonnegative real axis for all $n \geqslant 2$. So for rhombic square we have $P(\wp_\Lambda) \subset [0, \infty) \cup \{e_1, e_2, \wp(e_1), \infty\}$.

If Λ is rectangular square then $e_3 = 0$ is a prepole and must lie in $J(\wp_\Lambda)$. Thus $P(\wp_\Lambda) \subset [e_1, \infty) \cup \{0, e_2, \infty\}$.

If Λ is triangular then the forward orbit of the real critical value lies along the ray $\alpha = [e_3, \infty]$ on the real axis by Lemma 4.7. Let $\epsilon = e^{2\pi i/3}$. Since $\epsilon\Lambda = \Lambda$, the homogeneity property (Equation 2) implies that

$$\wp_\Lambda(\epsilon u) = \wp_{\epsilon\Lambda}(\epsilon u) = \frac{1}{\epsilon^2}\wp_\Lambda(u) = \epsilon\wp_\Lambda(u).$$

Similarly, $\wp_\Lambda(\epsilon^2 u) = \epsilon^2\wp_\Lambda(u)$. Thus the forward orbit of e_1 remains on the ray $\epsilon^2\alpha$ and the forward orbit of e_2 remains on the ray $\epsilon\alpha$. Note that if $g_3 > 0$ then these three rays only intersect at ∞. □

4.1. Fixed points. Since $\wp_\Lambda$ is meromorphic, it is known that for any lattice Λ, $\wp_\Lambda$ has infinitely many periodic points of period n for every $n \geqslant 2$. In general, a transcendental meromorphic function need not have any fixed points [7]. However, for real lattices it can easily be shown that $\wp_\Lambda$ has infinitely many fixed points.

Proposition 4.9. *Let Λ be a real lattice and $\wp_\Lambda$ the corresponding Weierstrass elliptic function. Then $\wp_\Lambda$ has infinitely many fixed points in $\mathbb{R}$.*

Proof. For a real lattice, Proposition 4.5 implies that $\wp_\Lambda$ has a critical point on the real axis. Since $\wp_\Lambda$ and $\wp'_\Lambda$ are periodic on $\mathbb{R}$, there are infinitely many critical points c along the real axis. Let e_r denote the critical value of the real critical points. (If Λ is rectangular then $e_r = e_1$ and if Λ is rhombic then $e_r = e_3$). We find a critical point c_m such that $c_m > e_r$ and $c_m > 0$. If the period along the real axis is p, then we consider $\wp_\Lambda : (c_m - p/2, c_m] \to [e_r, \infty)$ which is continuous and monotonically decreasing. We consider $\wp_\Lambda(x) - x$, which is positive near $c_m - p/2$ and negative at c_m. By the intermediate value theorem, there is a point in $(c_m - p/2, c_m]$ where $\wp_\Lambda(x) = x$. This is true for each segment of the form $(c - p/2, c]$ or $[c, c + p/2]$, where c is a critical point and $c > e_r$, and so there are infinitely many fixed points. □

The proof of the preceding result implies that in each periodic interval on $\mathbb{R}$, there will be 0, 1, or 2 fixed points. After we have encountered the first periodic interval $\mathscr{I}$ containing a fixed point, all periodic intervals to the right of $\mathscr{I}$ will contain two fixed points.

More can be said about periodic points of higher periods. In particular, $\wp_\Lambda$ maps each periodic interval in the nonnegative real axis onto $[e_r, \infty)$; if we choose a periodic interval $\mathscr{I}$ so that there are no poles in the interior of $\mathscr{I}$, then $\wp_\Lambda \circ \wp_\Lambda$ is defined everywhere inside $\mathscr{I}$, and the following holds.

Proposition 4.10. *For every real lattice Λ and for any periodic interval $\mathscr{I}$ to the right of e_r that contains no poles of $\wp_\Lambda$ in its interior, there are infinitely many real cycles of period 2 for $\wp_\Lambda$. The endpoints of $\mathscr{I}$ are accumulation points of the period 2 cycles.*

Proof. Points ω of the form $\omega = \wp_\Lambda(z)$ for $z \in \mathscr{I}$ include all real critical points, so as a map from $\mathscr{I}$ onto $[e_r, \infty)$, $\wp_\Lambda^2$ is piecewise monontone with infinitely many monotone pieces, and each interval of monotonicity has a fixed point. This corresponds to a periodic point of period less than or equal to 2. □

The same result holds for periodic points of higher periods, but there exists no periodic interval $\mathscr{I}$ for $\wp_\Lambda$ such that $\wp_\Lambda^3$ is defined everywhere in $\mathscr{I}$.

5. Symmetry of Fatou and Julia Sets

We begin with a symmetry result for an arbitrary lattice.

Theorem 5.1. *If Λ is any lattice then*

1. $J(\wp_\Lambda) + \Lambda = J(\wp_\Lambda)$ *and* $F(\wp_\Lambda) + \Lambda = F(\wp_\Lambda)$.
2. $(-1)J(\wp_\Lambda) = J(\wp_\Lambda)$ *and* $(-1)F(\wp_\Lambda) = F(\wp_\Lambda)$.

Proof. Suppose that $z \in F(\wp_\Lambda)$ and let $w \in \Lambda$. Then $\wp_\Lambda^n(z)$ is defined for all $n \in \mathbb{Z}$ and there exists a neighborhood U of z where $\wp_\Lambda^n(U)$ forms a normal family. But $\wp_\Lambda^n(z+w) = \wp_\Lambda^n(z)$ and $\wp_\Lambda^n(U+\Lambda) = \wp_\Lambda^n(U)$, so $z + w \in F(\wp_\Lambda)$.

For the second statement, let $z \in F(\wp_\Lambda)$. By definition, $\wp_\Lambda^n(z)$ exists and is normal for all n. Let U be a neighborhood of z such that $\{\wp_\Lambda^n(U)\}$ forms a normal family. Let $V = -U$. Since $\wp_\Lambda$ is even, we have that $\wp_\Lambda^n(V) = \wp_\Lambda^n(U)$ for all $n \geqslant 1$ and thus $\{\wp_\Lambda^n(V)\}$ forms a normal family. The proof of the converse is identical. So $z \in F(\wp_\Lambda)$ if and only if $-z \in F(\wp_\Lambda)$ and the Fatou set is symmetric with respect to the origin. This of course forces the Julia set to be symmetric with respect to the origin as well. $\square$

Theorem 5.2. *If Λ is real, then $\overline{J(\wp_\Lambda)} = J(\wp_\Lambda)$, and hence $\overline{F(\wp_\Lambda)} = F(\wp_\Lambda)$.*

Proof. Since $\overline{\Lambda} = \Lambda$, we have that $\wp_{\overline{\Lambda}} = \wp_\Lambda$. Defining $\phi(z) = \overline{z}$, we see that $\phi \circ \wp_\Lambda = \wp_{\overline{\Lambda}} \circ \phi$ for all lattices Λ, so for a general lattice the map $\wp_\Lambda$ is conjugate to $\wp_{\overline{\Lambda}}$, and the Julia sets are conjugate under ϕ. But $\wp_\Lambda = \wp_{\overline{\Lambda}}$, so $\phi(J(\wp_\Lambda)) = J(\wp_\Lambda)$ as claimed. $\square$

Corollary 5.3. *For a general lattice $\Lambda = [\lambda_1, \lambda_2]$, $\overline{J(\wp_\Lambda)} = J(\wp_{\overline{\Lambda}})$.*

Corollary 5.4. *Let Λ be a real lattice.*

1. *If F_0 is a Fatou component that intersects the real axis then F_0 is symmetric with respect to the real axis.*
2. *If F_0 is a Fatou component that intersects the imaginary axis then F_0 is symmetric with respect to the imaginary axis.*
3. *If F_0 is a Fatou component that intersects a critical point c then F_0 is symmetric with respect to c.*

If the lattice has a distinguished shape then the Fatou set and the Julia set exhibits additional symmetry.

Theorem 5.5. 1. *If Λ is rectangular square or rhombic square, then $e^{\pi i/2}J(\wp_\Lambda) = J(\wp_\Lambda)$ and $e^{\pi i/2}F(\wp_\Lambda) = F(\wp_\Lambda)$.*
2. *If Λ is triangular, then $e^{2\pi i/3}J(\wp_\Lambda) = J(\wp_\Lambda)$ and $e^{2\pi i/3}F(\wp_\Lambda) = F(\wp_\Lambda)$.*

Proof. Let $z \in F(\wp_\Lambda)$. By definition, $\wp_\Lambda^n(z)$ exists and is normal for all n. By Equation 2, $\wp_\Lambda(iz) = -\wp_\Lambda(z)$ and since $\wp_\Lambda$ is even we know that $\wp_\Lambda^n(iz) = \wp_\Lambda^n(z)$ for all $n \geqslant 2$. So $\wp_\Lambda^n(iz)$ exists for all n. Let U be a neighborhood of z such that $\{\wp_\Lambda^n(U)\}$ forms a normal family. Let $V = iU$. Repeating our argument, we have that $\wp_\Lambda(V) = \wp_\Lambda(iU) = -\wp_\Lambda(U)$ and $\wp_\Lambda^n(V) = \wp_\Lambda^n(U)$ for all $n \geqslant 2$ and thus $\{\wp_\Lambda^n(V)\}$ forms a normal family. The proof of the converse is identical.

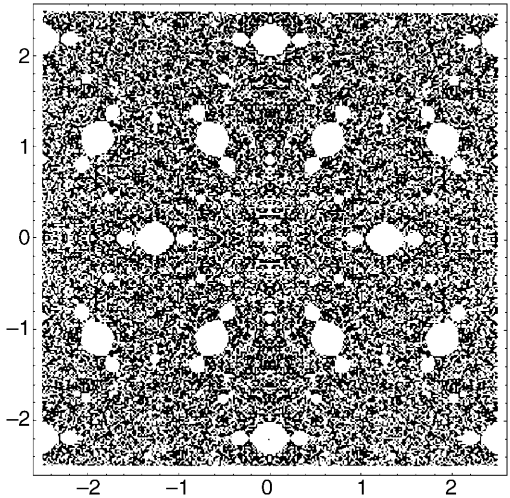

Figure 2. $J(\wp_\Lambda)$ when $g_2 = 0$, $g_3 = 3.082$

So $z \in F(\wp_\Lambda)$ if and only if $iz \in F(\wp_\Lambda)$. By Theorem 5.1, $-z, -iz \in F(\wp_\Lambda)$ and the Fatou set is symmetric with respect to rotation by $\pi/2$.

The second part of the theorem follows similarly using that the homogeneity property on the triangular lattice forces $\wp_\Lambda^n(\epsilon z) = \epsilon \wp_\Lambda^n(z)$ for all n. □

To illustrate these symmetry results, Figure 2 shows the Julia set of the Weierstrass elliptic function with invariants $g_2 = 0$ and $g_3 = 3.082$. The lattice for these invariants is triangular and can be generated by $\lambda_1 \approx 1.2682 + 2.1966i$, $\lambda_2 \approx 1.2682 - 2.1966i$. This particular Weierstrass elliptic function has three superattracting cycles of period two located at $\mathscr{A} = \{1.26825, .961764\}$, $\epsilon\mathscr{A}$, and $\epsilon^2\mathscr{A}$ (see Proposition 4.8 (4)). We discuss more examples in Section 8.

There are two types of square lattices; we will see that given a side length the Julia sets of the rectangular square and rhombic square lattice Weierstrass $\wp$ functions are quite different, but there is some symmetry between the maps.

Proposition 5.6. *Let Λ_1 be a rectangular square lattice with side length γ_o, and let Λ_2 be the rhombic square lattice with side length γ_o. Then for each $z \in \mathbb{C}$,*

$$\wp_{\Lambda_2}(e^{\pi i/4} z) = -i\wp_{\Lambda_1}(z).$$

Furthermore,

$$\wp'_{\Lambda_2}(e^{\pi i/4} z) = e^{-3\pi i/4}\wp'_{\Lambda_1}(z),$$

so $|\wp'_{\Lambda_2}(e^{\pi i/4} z)| = |\wp'_{\Lambda_1}(z)|$. If the invariants for $\wp_{\Lambda_1}$ are $g_2 > 0$ and $g_3 = 0$, then for $\wp_{\Lambda_2}$ they are $-g_2$ and 0.

Proof. We have that $e^{\pi i/4}\Lambda_1 = \Lambda_2$. By Equation 2, for each z, $\wp_{\Lambda_2}(e^{\pi i/4} z) = -i\wp_{\Lambda_1}(z)$. The properties of the derivative follows from Equation 2 as well. Furthermore, from Equation 1 and Proposition 4.5 we have that $g_2 > 0$ and $g_3 = 0$ are the invariants for Λ_1, and then the invariants for Λ_2 are $-g_2$ and 0. □

6. The Structure of Periodic Components of the Fatou Set

We turn to a discussion about the possible periodic Fatou components for $\wp_\Lambda$. The first few results hold for every lattice.

Lemma 6.1. *For any lattice* Λ, *the Fatou set of* $\wp_\Lambda$ *has no wandering domains and no Baker domains.*

Proof. $\wp_\Lambda$ has only three critical values. Since $\wp_\Lambda$ is doubly periodic, it cannot have any asymptotic values. Therefore it is a Class S function and has no wandering domains [3] or Baker domains [12]. □

We mention some well-known results which give criteria for determining if $J(\wp_\Lambda)$ is connected.

Proposition 6.2. *Let* Λ *be any lattice.*

1. $J(\wp_\Lambda)$ *is connected if and only if each component of* $F(\wp_\Lambda)$ *is simply connected.*
2. *If* $F(\wp_\Lambda)$ *is connected, then either it is simply connected or it is infinitely connected.*
3. *If* $\wp_\Lambda$ *has no Siegel disks or Herman rings, then a forward invariant component of* $F(\wp_\Lambda)$ *is either simply connected or it is infinitely connected; in the second case,* $J(\wp_\Lambda)$ *has uncountably many components.*

Proof. If D is an open subset of the sphere, then $\mathbb{C}_\infty \backslash D$ is connected if and only if each component of D is simply connected (Proposition 5.1.5 [6]). This gives 1. The second statement can be found in [6] (Chapter 5) for rational maps, and it holds here as well.

By Lemma 6.1 and the hypotheses, if U is a forward invariant component of $F(\wp_\Lambda)$, then U is an attracting, superattracting or a parabolic invariant component of $F(\wp)$. It follows from [2] that U is either infinitely or simply connected. □

Lemma 6.3. *For rectangular lattices, rhombic square lattices, and triangular lattices when* $g_3 > 0$, $\wp_\Lambda$ *has no Siegel disks or Herman rings.*

Proof. If $C = \{U_0, U_1, \ldots, U_{p-1}\}$ is a cycle of Siegel disks or Herman rings, then $\partial U_j \subset \overline{\bigcup_{n \geqslant 0} f^n(\mathrm{Sing}(f))}$ for all $0 \leqslant j \leqslant p-1$. (cf. [7], Theorem 7). By Proposition 4.8, the postcritical set for a rectangular lattice or a rhombic square lattice is contained in the nonnegative real axis, except for finitely many points, and thus $\partial U_j \subset [0, \infty)$ for all $0 \leqslant j \leqslant p-1$. Since the closure of the postcritical set is a ray, there can be at most one Siegal disk or Herman ring U_0 and $(-\infty, b) \subset U_0$ for some finite $b < 0$. This contradicts Theorem 5.1, and thus $P(\wp_\Lambda)$ cannot bound a Herman ring or Siegel disk.

If Λ is triangular and $g_3 > 0$, then Corollary 4.6 implies that $e_3 > 0$, and Proposition 4.8 implies that the postcritical set lies on three rays $\alpha = [e_3, \infty]$, $e^{2\pi/3}\alpha$ and $e^{4\pi/3}\alpha$ whose only possible point of intersection is ∞. If these rays were to bound a Siegal disk or Herman ring U_0, then U_0 must be contained in the complement of the union of the three rays, contradicting Theorem 5.1. Thus $\wp_\Lambda$ has no Herman rings or Siegel disks. □

Corollary 6.4. *For rectangular lattices, rhombic square lattices, and triangular lattices when $g_3 > 0$, every forward invariant component of $F(\wp_\Lambda)$ is either simply connected or infinitely connected.*

We now discuss in detail the possible Fatou components that can occur for various types of real lattices. We need the following result.

Lemma 6.5. *Suppose that Λ is a rhombic square, rectangular square, or triangular lattice so that $\wp_\Lambda|_\mathbb{R} : \mathbb{R} \to [e_r, \infty)$. Let λ be the period of $\wp_\Lambda$ on $\mathbb{R}$. Suppose $n\lambda \leqslant e_r < (n+1)\lambda$; define $I_{e_r} = \{x \in \mathbb{R} : n\lambda \leqslant x < (n+1)\lambda\}$ to be the periodic interval containing e_r. Assume that there is exactly one real nonrepelling cycle for $\wp_\Lambda$. If $A = \{p_1, \dots, p_n\}$ denotes the nonrepelling cycle, then $A \cap I_{e_r} \neq \emptyset$.*

Proof. First we assume that p is a fixed point so that $A = \{p\}$. We claim that if there is a nonrepelling fixed point on $\mathbb{R}$, it must be the smallest fixed point, the one lying furthest to the left.

To prove the claim, it follows from Lemma 4.7 that there are only finitely many fixed points lying to the left of 0, so there is a smallest fixed point. Let x_1 denote the smallest, so $x_1 = \wp_\Lambda(x_1) \geqslant e_r$; if x_2 is a fixed point such that $x_1 < x_2$, then since $\wp_\Lambda(x_1) < \wp_\Lambda(x_2)$, it is necessarily the case that x_2 lies closer to a pole (a lattice point on $\mathbb{R}$) than x_1.

Equation 1 implies that $\wp''_\Lambda(z) = 6\wp^2_\Lambda(z) - \frac{g_2}{2}$. Lemma 4.6 implies that the image of the real critical point is the minimum of $\wp_\Lambda$ on $\mathbb{R}$ and $\wp_\Lambda|_\mathbb{R} : \mathbb{R} \to [e_r, \infty]$ is piecewise monotonic and onto. If the lattice is rectangular square, then $e_1 = \frac{\sqrt{g_2}}{2} > 0$ by Lemma 4.7. Since $\wp_\Lambda \geqslant \frac{\sqrt{g_2}}{2}$, we have $\wp''_\Lambda > 6\wp^2_\Lambda(z) - \frac{3g_2}{2} \geqslant 0$ and thus $\wp'_\Lambda$ is strictly monotonic. If the lattice is rhombic square then $g_2 < 0$ and thus $\wp''_\Lambda(z) = 6\wp^2_\Lambda(z) - \frac{g_2}{2} > 0$. If the lattice is triangular then $g_2 = 0$ and thus $\wp''_\Lambda(z) = 6\wp^2_\Lambda(z) \geqslant 0$ and $\wp''_\Lambda(z) = 0$ only at the isolated roots of $\wp_\Lambda$. Thus in all of these cases, $\wp'_\Lambda$ is strictly monotonic.

Since x_2 lies closer to a pole (a lattice point on $\mathbb{R}$) than x_1 and $\wp'_\Lambda$ is strictly monotonic, $|\wp'_\Lambda(x_1)| < |\wp'_\Lambda(x_2)|$. This proves the claim.

We consider the periodic interval containing p. If $e_r = p$ we are done, so we assume that $p > e_r$. We denote the periodic interval containing p by $I_p = \{x \in \mathbb{R} : k\lambda \leqslant x < (k+1)\lambda\}$; i.e., $k\lambda \leqslant p < (k+1)\lambda$. There is a critical point contained in I_r, which is $c_p = (n + 1/2)\lambda$. We note that since $p \neq e_p$, it follows that $p \neq c_p$. Then there are two cases to consider. If $p < c_p$, the interval $[p, c_p)$ maps injectively onto $(e_r, p]$ without any poles so $e_r \in I_p$. Otherwise, $p > c_p$, so the interval $(c_p, p]$ maps injectively onto $(e_r, p]$ without any poles so again $e_r \in I_p$.

If p is periodic of a higher period $n \geqslant 1$, the same argument is applied to $\wp^n_\Lambda$ with the necessary adaptations; the higher iterates of $\wp_\Lambda$ are also piecewise monotonic with minimum real value e_r. □

We now can classify the types of behavior of periodic Fatou components for the Weierstrass elliptic function on a triangular lattice.

Proposition 6.6. *For any triangular lattice* $\Lambda = [\lambda_1, \lambda_2]$ *with* $g_3 > 0$, *one of the following must occur*:

1. $J(\wp_\Lambda) = \mathbb{C}_\infty$;
2. *There exist exactly three* (*super*)*attracting cycles for* $\wp_\Lambda$, *and these are the only periodic cycles for* $F(\wp_\Lambda)$;
3. *There exist exactly three rationally neutral cycles for* $\wp_\Lambda$, *and these are the only periodic cycles for* $F(\wp_\Lambda)$;
4. *In the case of* 2 *or* 3, *the following also hold*:
   (*a*) *One of the nonrepelling periodic orbits lies along the ray* $\alpha = [e_3, \infty)$ *on the real axis*, *one lies on the ray* $e^{2\pi i/3}\alpha$, *and one lies on* $e^{4\pi i/3}\alpha$.
   (*b*) *Let* $a = \lambda_1 + \lambda_2$ *be the period of* $\wp_\Lambda$ *on* $\mathbb{R}$. *Suppose* $na \leqslant e_3 < (n+1)a$; *let* $I_{e_3} = \{x \in \mathbb{R} : na \leqslant x < (n+1)a\}$ *denote the periodic interval containing* e_3. *If* $A = \{p_1, \ldots, p_n\}$ *denotes the nonrepelling cycle*, *then* $A \cap I_{e_3} \neq \emptyset$.

Proof. If $J(\wp_\Lambda) \neq \mathbb{C}_\infty$ then there must exist a periodic Fatou cycle which cannot contain any Siegel disks, Herman rings, or Baker domains by Lemmas 6.1 and 6.3. Proposition 4.8 implies that the postcritical set is contained in three invariant disjoint rays: $\alpha = [e_3, \infty)$, $e^{2\pi i/3}\alpha$, and $e^{4\pi i/3}\alpha$. Since the forward orbits of each critical value remain on disjoint rays, no immediate basin can contain more than one critical value. If $\{p_0, \ldots, p_{k-1}\}$ is a periodic cycle, then the homogeneity property of $\wp_\Lambda$ implies that $\{e^{2\pi i/3}p_0, \ldots, e^{2\pi i/3}p_{k-1}\}$ and $\{e^{4\pi i/3}p_0, \ldots, e^{4\pi i/3}p_{k-1}\}$ are periodic cycles. The homogeneity property of $\wp'_\Lambda$ implies that $(\wp^n_\Lambda)'(p_i) = (\wp^n_\Lambda)'(e^{2\pi i/3}p_i) = (\wp^n_\Lambda)'(e^{4\pi i/3}p_i)$ and so the three periodic orbits must have the same classification. Since the nonrepelling orbit must lie in the postcritical set, the last two statements follow from Proposition 4.8 and Lemma 6.5. □

We turn to the case of square lattices in which the situation is somewhat different. In this setting, at most one periodic Fatou cycle can occur. Recall that for a square lattice Λ with invariants $\{g_2, 0\}$ we always have that $e_1 = \frac{\sqrt{g_2}}{2}$, where $g_2 > 0$ for rectangular square and $g_2 < 0$ for rhombic square.

Proposition 6.7. *For any rectangular square lattice* $\Lambda = [\lambda, \lambda i]$, *one of the following must occur*:

1. $J(\wp_\Lambda) = \mathbb{C}_\infty$;
2. *There exists exactly one* (*super*)*attracting cycle for* $\wp_\Lambda$, *and the immediate basin contains* e_1; *this is the only periodic cycle for* $F(\wp_\Lambda)$;
3. *There exists exactly one rationally neutral cycle for* $\wp_\Lambda$, *and the immediate basin contains* e_1; *this is the only periodic cycle for* $F(\wp_\Lambda)$;
4. *In the case of* 2 *or* 3, *the following also hold*:
   (*a*) *The nonrepelling periodic orbit lies completely in* $[e_1, \infty)$.
   (*b*) *Let* λ *be the period of* $\wp_\Lambda$ *on* $\mathbb{R}$. *Suppose* $n\lambda \leqslant e_1 < (n+1)\lambda$; *let* $I_{e_1} = \{x \in \mathbb{R} : n\lambda \leqslant x < (n+1)\lambda\}$ *denote the periodic interval containing* e_1. *If* $A = \{p_1, \ldots, p_n\}$ *denotes the nonrepelling cycle*, *then* $A \cap I_{e_1} \neq \emptyset$.

Proof. If $J(\wp_\Lambda) \neq \mathbb{C}_\infty$ then there must exist a periodic Fatou cycle which cannot contain any Siegel disks, Herman rings, or Baker domains by Lemmas 6.1 and 6.3.

By Corollary 4.6 and Proposition 4.8 $e_3 \in J(\wp_\Lambda)$, and $e_2 = -e_1$. Since $\wp_\Lambda(e_2) = \wp_\Lambda(e_1)$ there is only one postcritical orbit which lies in $[e_1, \infty)$, and e_2 must land in the same Fatou cycle as e_1. This exhausts the possible Fatou components so we are done. The last two statements follow from Proposition 4.8 and Lemma 6.5. □

The theorem for rhombic square lattices is similar to the rectangular square case, although the critical values e_1 and e_2 must both lie in the same Fatou component. Further, no superattracting cycles can occur.

Proposition 6.8. *For any rhombic square lattice* Λ, *one of the following must occur*:

1. $J(\wp_\Lambda) = \mathbb{C}_\infty$;
2. *There exists exactly one attracting cycle for* $\wp_\Lambda$, *and* e_1 *and* e_2 *lie in the same component of the Fatou set*; *this is the only periodic cycle for* $F(\wp_\Lambda)$;
3. *There exists exactly one rationally neutral cycle for* $\wp_\Lambda$, *and* e_1 *and* e_2 *lie in the same component of the Fatou set*; *this is the only periodic cycle for* $F(\wp_\Lambda)$;
4. *In the case of* 2 *or* 3, *the nonrepelling periodic orbit lies completely in* $[0, \infty)$.

Proof. The proof that only one forward invariant cycle can exist is the same as the rectangular square case. Since the minimum of $\wp_\Lambda$ on $\mathbb{R}$ is $e_3 = 0$, no superattracting cycle can occur.

Since any nonrepelling orbit lies on the positive real axis, we know that all components of the Fatou cycle must intersect the real axis. One of these components, say U_0 must contain e_1. By Corollary 5.4, U_0 is symmetric with respect the real axis and $e_2 \in U_0$. □

Corollary 6.9. *If* Λ *is rhombic square then* $\wp_\Lambda$ *has no superattracting cycles.*

For a general real rectangular lattice, we have the following result.

Proposition 6.10. *For any real rectanglar lattice* $\Lambda = [\lambda_1, \lambda_2 i]$, *with* $\lambda_1, \lambda_2 > 0$, *one of the following must occur*:

1. $J(\wp_\Lambda) = \mathbb{C}_\infty$;
2. *There exist at most three* (*super*)*attracting or rationally neutral cycles for* $\wp_\Lambda$;
3. *If* 2 *holds*, *then the nonrepelling periodic orbits are real and lie completely in* $[e_1, \infty)$.

Proof. For any real lattice, $\wp_\Lambda(\mathbb{R})$ is nonnegative by Lemma 3.6. We assume that $g_3 \neq 0$ since this is taken care of by the rectangular square lattice results in Proposition 6.7. Therefore none of e_1, e_2, or e_3 are zero. The minimum value obtained by $\wp_\Lambda$ on $\mathbb{R}$ is $e_1 = \wp_\Lambda(\frac{\lambda_1}{2})$; Corollary 4.6 implies that at least one of e_2 or e_3 must be negative. Since every nonrepelling orbit is contained in the postcritical set of $\wp_\Lambda$, by Proposition 3.7, we have that all nonrepelling orbits are contained in $[e_1, \infty)$. □

Next, we discuss completely invariant components of the Fatou set. It is shown in [2] that Class S functions have a maximum of two completely invariant components, and if two completely invariant components exist then each is simply connected. We begin with a result that holds for any lattice.

Proposition 6.11. *If Λ is any lattice and F_o is a completely invariant component of $F(\wp_\Lambda)$, then F_o contains infinitely many critical points and F_o is unbounded.*

Proof. If F_o is a completely invariant component of $F(\wp_\Lambda)$, then F_o contains a critical value e_i. But then there are infinitely many critical points of the form $c+\Lambda$ satisfying $\wp_\Lambda(c+\Lambda)=e_i$. Since F_o is completely invariant, then all points $c+\Lambda$ are in F_o, and hence it is unbounded. □

Theorem 6.12. *If Λ is any real lattice and F_o is a completely invariant component of $F(\wp_\Lambda)$, then F_o must be infinitely connected. Therefore, there exists at most one completely invariant component.*

Proof. It is enough to eliminate the possibility that F_o is simply connected. Every open simply connected subset of the plane is path connected. We know that F_o contains at least one critical value e_i and infinitely many critical points $c+\Lambda$. For $i=1,2$, let c_i be a critical point lying in the quadrant i. We take a continuous path α_1 in F_o connecting the critical point c_1 to c_2 and lying completely in the upper half plane. This is possible by the symmetry of F_o given in Corollary 5.4 (1) when we construct α_1; i.e., for each point on the first attempt that lies below the x axis, there is a corresponding conjugate point lying above which we choose each time the path hits the real axis. Similarly, we can find a continuous path α_2 in F_o connecting c_2 to $-c_1$ that lies entirely in the second and third quadrants, using Corollary 5.4 (2). We then take the paths $-\alpha_1$ and $-\alpha_2$. This forms a loop which encloses a pole, 0, so does not shrink to a point. Therefore F_o is not simply connected.

Since $\wp_\Lambda$ is Class S (see proof of Lemma 6.1), if F has two completely invariant components then each must be simply connected. Thus there can be at most one completely invariant component of the Fatou set. □

Proposition 6.13. 1. *If Λ is triangular and $g_3>0$ then there are no completely invariant Fatou components.*

2. *If Λ is rectangular square or rhombic square and F_o is a completely invariant component of the Fatou set, then $F_o=F(\wp_\Lambda)$.*

Proof. Suppose that Λ is triangular and F_o is a completely invariant component. By Proposition 6.6 F_o must contain a (super)attracting or neutral fixed point and a critical value, say e_3. Clearly, $e^{2\pi i/3}e_3$, $e^{4\pi i/3}e_3\notin F_o$. But then $e^{2\pi i/3}F_o$ and $e^{4\pi i/3}F_o$ are completely invariant and then $\wp_\Lambda$ would have three completely invariant Fatou components. This is a contradiction, so there are no completely invariant Fatou components.

The second statement follows from Propositions 6.7 and 6.8. □

We do not yet know of any examples where the Fatou set contains a completely invariant component. When the lattice is rhombic square, we cannot have a connected Julia set unless the Fatou set is empty.

Proposition 6.14. *If Λ is rhombic square, then the Julia set is connected if and only if $J(\wp_\Lambda) = \mathbb{C}_\infty$.*

Proof. Suppose $J(\wp_\Lambda) \neq \mathbb{C}_\infty$. As in the proof of Proposition 6.8 we have a periodic component of the Fatou set U_1 that contains points on the positive real axis, e_1 on the positive imaginary axis, and e_2 on the negative imaginary axis. Corollary 5.4 implies that U_1 is also symmetric with respect the imaginary axis. This allows us to construct a loop in U_1 around the origin, which is a prepole, and therefore lies in the Julia set. Therefore, U_1 is not simply connected and by Proposition 6.2 the Julia set is not connected. □

For rectangular square or triangular lattices with a superattracting fixed point, the Julia set must be connected.

Proposition 6.15. *Let $\wp_\Lambda$ be the Weierstrass elliptic function for a rectangular square lattice with a superattracting fixed point. Then $J(\wp_\Lambda)$ is connected.*

Proof. If p_1 is a superattracting fixed point then $p_1 = e_1 = m\lambda/2$ for some positive integer m. By Corollary 4.6, we know that $e_2 = -e_1$ and $\wp(e_2) = \wp(e_1)$. Let U_o denote the forward invariant component of the Fatou set containing the fixed point e_1. By Corollary 5.4, U_o is symmetric with respect to e_1.

We claim that $e_2 \notin U_o$. Since U_o is a superattracting invariant Fatou component, we have that there is a conformal change of coordinates from U_o onto the open unit disk $\mathbb{D}$ which conjugates $\wp|_{U_o}$ to the $z \mapsto z^2$ power map on $\mathbb{D}$ ([21], Section 68, Theorem 4). If the claim were false, then the local coordinate change of coordinates about $e_1 \in U_o$ extends throughout a region $V \subset U_o$ and $\partial V \subset U_o$ contains the critical value (point) e_2. However, e_2 and V cannot both lie in the same period parallelogram (there are points $z \in V \cap \mathbb{R}$ such that $|z - e_2| > \lambda$), and the local coordinate change cannot extend over into the adjoining period parallelogram since $\wp$ is periodic but the power map is not. Therefore the claim is true, and U_o is simply connected ([21], Section 68, Theorem 4).

Since U_o is contained inside one period parallelogram and is symmetric with respect to $e_1 = m\lambda/2$, Lemma 4.1 implies that $\wp_\Lambda : U_o \to U_o$ is two-to-one except at e_1. Further, $\wp_\Lambda^{-1}(U_o) = \{U_o + w : w \in \Lambda\}$, a pairwise disjoint collection of translates of U_o, and therefore each component is simply connected. Let W denote the component of $\wp_\Lambda^{-1}(U_o)$ that contains e_2. By Corollary 5.4, $W = -U_o$.

Let $E = iU_o$, so E is simply connected and contained entirely within one period parallelogram. By Equation 2,

$$\wp_\Lambda(E) = \wp_\Lambda(iU_o) = -\wp_\Lambda(U_o) = -U_o = W.$$

Since E contains the critical point $mi\lambda/2$, Corollary 5.4 implies that E is symmetric with respect to the critical point $mi\lambda/2$. Again, we have that $\wp_\Lambda : E \to W$ is a two-to-one map except at $mi\lambda/2$ by Lemma 4.1. Thus $\wp_\Lambda^{-1} W = \{E + w : w \in \Lambda\}$, each of which is just a disjoint translate of E and so is simply connected.

No other components of the Fatou set contain critical values since $e_3 \in J(\wp_\Lambda)$ by Proposition 4.8. We can therefore find a local univalent inverse for any Fatou component except U_o and W, and thus all Fatou components are simply connected. Using Proposition 6.2 it follows that $J(\wp)$ is connected. □

Theorem 6.16. *Let $\wp_\Lambda$ be the Weierstrass elliptic function for a triangular lattice with $g_3 > 0$ that has a superattracting fixed point. Then $J(\wp_\Lambda)$ is connected.*

Proof. If p_1 is a superattracting fixed point then $p_1 = e_3 = m(\lambda_1 + \lambda_2)/2$ for some positive integer m. By Corollary 4.6, we know that $e_1 = e^{4\pi i/3}e_3$, and $e_2 = e^{2\pi i/3}e_3$. Let U_o denote the forward invariant component of the Fatou set containing the fixed point e_3. By Corollary 5.4, U_o is symmetric with respect to e_3. By an argument similar to that in the proof of Theorem 6.15, U_o cannot contain any other critical value, must be contained inside one period parallelogram, and must be simply connected. Further, $\wp_\Lambda : U_o \to U_o$ is two-to-one except at e_3. Again, the components of $\wp_\Lambda^{-1}(U_o) = \{U_o + w : w \in \Lambda\}$ are just pairwise disjoint translates of U_o, and therefore each is simply connected. None of these preimages can contain another critical value by the homogeneity property (Equation 2), so each component of $\wp_\Lambda^{-n}(U_o)$ is simply connected. By Proposition 6.6 and Equation 2, the behavior at the other two critical values is simply a rotation of what occurs at the fixed point e_3, and thus all Fatou components are simply connected. Proposition 6.2 implies that $J(\wp)$ is connected. $\square$

7. Ergodicity, Exactness and Conservativity

7.1. Some measure theoretic results. We begin with some preliminary definitions from nonsingular measure theory. Let $f : (X, \mathscr{B}, m) \to (X, \mathscr{B}, m)$ denote a nonsingular transformation of a Lebesgue probability space X onto itself; that is, $m(f^{-1}(A)) = 0$ if and only if $m(A) = 0$ for every $A \in \mathscr{B}$.

Definition 7.1. We say f is

1. *ergodic* with respect to m if $f^{-1}(B) = B$ for some $B \in \mathscr{B}$ implies that $m(B) = 0$ or 1;
2. *conservative* if for every $B \in \mathscr{B}$, $m(B) > 0$ there exists $n \in \mathbb{N}$ such that
$$m(B \cap f^{-n}B) > 0;$$
3. *exact* if $B = f^{-n} \circ f^n(B)$ for all $n \in \mathbb{N}$, $B \in \mathscr{B}$, implies that $m(B) = 0$ or 1; (we call B a *tail set* in this case).
4. *lim sup full* if for every $B \in \mathscr{B}$, $m(B) > 0$,
$$\lim \sup_{n \to \infty} m(f^n(B)) = 1.$$

Every exact transformation is ergodic, but the converse does not hold. Conservativity is independent of the notions of ergodicity and exactness when the map is not invertible, even in the case of smooth maps on manifolds [9].

We use several results from [4] on exactness and conservativity of rational maps that can be modified to the meromorphic setting. The proofs in [4] are for a rational map, which is d-to-1 on a set of full m measure, and here we have maps which are countable-to-1. We give modifications of the proofs given in [4] to extend to this setting. If A is any set, let $A^c = X \backslash A$.

Definition 7.2. We say that a nonsingular countable-to-1 map $f : (X, \mathscr{B}, m) \to (X, \mathscr{B}, m)$ is *uniformly countable-to-1* if there exists a partition of X into countably many pieces $Q_1, Q_2, \ldots, Q_i, \ldots$, with $m(Q_i) > 0$ for all i, $m(Q_i \cap Q_j) = 0$ if $i \neq j$, and such that $f|_{Q_i}$ is nonsingular, one-to-one, and onto X (up to sets of measure 0).

Lemma 7.3. *If $f : (X, \mathscr{B}, m) \to (X, \mathscr{B}, m)$ is nonsingular and uniformly countable-to-one transformation of a probability space, then for every $\epsilon > 0$, there exists a $\delta > 0$ such that for any $A \in \mathscr{B}$, $m(A^c) < \delta$ implies that $m(f(A)) > 1 - \epsilon$.*

Proof. We denote $f|_{Q_i}$ by f_i. Given $\epsilon > 0$, we have that on each Q_i, the measures m and $m \circ f_i$ are equivalent. We choose any $N \in \mathbb{N}$; we find $\delta > 0$ such that if B is any measurable set in Q_i, $i \leqslant N$, such that $m(B) < \delta$, then $m(f_i(B)) < \epsilon$. Therefore given any set $A \in \mathscr{B}$ such that $m(A) > 1 - \delta$, let $B = A^c$; so $m(B \cap Q_i) \leqslant m(B) < \delta$ for each $i \leqslant N$. Denote $B_1 = B \cap Q_1 = A^c \cap Q_1$. Then $m(f_1(B)) < \epsilon$ and

$$m(f(A)) \geqslant m(f(A \cap Q_1)) = 1 - m(f_1(B)) > 1 - \epsilon. \qquad \square$$

Using Lemma 7.3 above and the proof given in [4], we have the following.

Lemma 7.4. *If $f : (X, \mathscr{B}, m) \to (X, \mathscr{B}, m)$ is nonsingular and uniformly countable-to-one transformation of a probability space which is* lim sup *full, and $A \in \mathscr{B}$ is any set satisfying $0 < m(A) < 1$, then for infinitely many j, $m(f^j(A) \cap f^j(A^c)) > 0$.*

Theorem 7.5. [4] *If $f : (X, \mathscr{B}, m) \to (X, \mathscr{B}, m)$ is nonsingular and uniformly countable-to-one, and f is* lim sup *full, then f is conservative and exact.*

Proof. Let $A = f^{-n} f^n(A)$ $m - a.e.$; then

$$0 = m(A \cap A^c) = m(f^{-n} f^n(A) \cap f^{-n} f^n(A^c)) = m(f^{-n}(f^n(A) \cap f^n(A^c))).$$

By the nonsingularity of f, we have therefore that $m(f^n(A) \cap f^n(A^c)) = 0$ for all $n \in \mathbb{N}$. However, since f is lim sup full, Lemma 7.4 implies that $m(A) = 0$ or 1. Therefore f is exact.

To show f is conservative, we consider any set of positive measure, say $A \in \mathscr{B}$ and $m(A) = \alpha > 0$. Since f is lim sup full, there are infinitely many positive integers n_k such that $m(f^{n_k}(A)) > 1 - \alpha$. Therefore a set of points in A of positive m measure returns to A under f^{n_k}, so f is conservative. $\square$

The next result gives a simpler condition for determining if a nonsingular transformation is lim sup full. A set $U \in \mathscr{B}$ is *finite full* if there exists $k \in \mathbb{N}$ such that $m(f^k(U)) = 1$.

Proposition 7.6. *If $f : (X, \mathscr{B}, m) \to (X, \mathscr{B}, m)$ is nonsingular and uniformly countable-to-one transformation of a probability space, and for every positive measure set A there is a finite full set U intersecting at most finitely many Q_j's such that* $\limsup_{n\to\infty} m(f^n(A) \cap U) = m(U)$, *then f is* lim sup *full.*

Proof. Let A be any set of positive measure, and U be finite full. We can assume without loss of generality that $U \subset \bigcup_{i=1}^N Q_i$. Given $\epsilon > 0$, we assume $m(f^k(U)) = 1$, and we find an increasing sequence $\{n_j\}$ such that

$$\lim_{j\to\infty} |m(f^{n_j}(A) \cap U) - m(U)| = 0.$$

Denote $U_j = U \backslash f^{n_j} A$; finding δ for Lemma 7.3 applied to f^k we can choose $\delta > 0$ small enough and M large enough so that for all $j \geqslant M$, $m(U_j) < \delta$ and

$m(f^k(U_j)) < \epsilon$. Hence $m(f^k(f^{n_j}(A))) \geqslant m(f^k(U \backslash U_j)) > 1 - \epsilon$ and therefore f is lim sup full. $\square$

7.2. Lebesgue measure properties of elliptic Weierstrass functions on real lattices. For any lattice Λ, $\wp_\Lambda$ is nonsingular with respect to Lebesgue measure (either normalized on $\mathbb{C}_\infty$ or the equivalent planar measure). We will use m to denote the normalized Lebesgue measure (the Riemannian volume form) on $\mathbb{C}_\infty$. We note that every period parallelogram is finite full using $k = 1$; in fact the upper half of any period parallelogram is finite full using $k = 1$ (and each period parallelogram could be cut in two finite full pieces in many different ways).

Recall that an *exceptional point* of a meromorphic function f is a point a such that $\bigcup_{n>0} f^{-n}(a)$ is finite.

Lemma 7.7. *For any lattice* Λ, $\wp_\Lambda$ *has no exceptional points.*

Proof. Let $a \in \mathbb{C}$ and suppose $\wp_\Lambda(z) = a$. Since $\wp_\Lambda$ is doubly periodic, $\wp_\Lambda(z + w) = a$ for any $w \in \Lambda$ so $\wp^{-1}(a)$ is infinite. $\square$

Let $O^+(z) = \bigcup_{n \geqslant 0} \wp_\Lambda^n(z)$ denote the forward orbit of z, where the union is only taken over those $n \geqslant 0$ for which $\wp_\Lambda^n$ is defined. For a subset W of $\mathbb{C}_\infty$, we define $O^+(W) = \bigcup_{z \in W} O^+(z)$.

Theorem 7.8. *If W is an open set that intersects $J(\wp_\Lambda)$, then $O^+(W) = \mathbb{C}_\infty$.*

Proof. Let $W_0 = O^+(W)$ and let $K = \mathbb{C}_\infty \backslash W_0$. If K contains three distinct points, then Montel's Theorem implies that $\{\wp_\Lambda^n\}$ is normal in W, contradicting the assumption that W contains points in $J(\wp_\Lambda)$. Therefore K contains 2 or fewer points.

Suppose $z_0 \in K$. By Lemma 7.7, z_0 has an infinite backwards orbit and thus the backwards orbit must intersect W_0. Thus for some point w and some nonnegative integers p and q, $\wp_\Lambda^p(w) = z_0$ and $w \in \wp_\Lambda^q(W)$. Thus $z_0 \in \wp_\Lambda^{p+q}(W)$. This contradiction establishes the theorem. $\square$

Corollary 7.9. *For any lattice Λ, any open set W such that $W \cap J(\wp_\Lambda) \neq \emptyset$ is finite full.*

Proof. Since $J(\wp_\Lambda)$ is the closure of the repelling periodic points, we can choose $z_0 \in W \cap J(\wp_\Lambda)$ such that $\wp_\Lambda^k(z_0) = z_0$ and $|(\wp_\Lambda')^k(z_0)| > 1$. Choose a disk V around z_0 such that $V \subset W$ and $V \subset \wp_\Lambda(V)$. Then $V \subset \wp_\Lambda(V) \subset \wp_\Lambda^2(V) \subset \ldots$ Theorem 7.8 implies that $O^+(V)$ covers the sphere, which is compact. We can choose a finite subcover, which implies that W is finite full. $\square$

We establish some conditions under which we obtain a lim sup full Weierstrass elliptic function. We need the following two results. The first was extended from the rational to meromorphic case in [13].

Proposition 7.10. [13] *Let f be a meromorphic function and let U be an open disk such that $U \cap J(f) \neq \emptyset$. Suppose there exist holomorphic univalent branches $g_{n_k} \in f^{-n_k}$ on U with $n_k \to \infty$. Then $g'_{n_k} \to 0$ uniformly on every compact subset of U.*

We also recall the following classical theorem.

Theorem 7.11 (Koebe Distortion). (*cf.* [20]) *If* $g : B_1(0) \to \mathbb{C}$ *is a holomorphic univalent map such that* $g(0) = 0$, *then for every* $0 < k < 1$, *and* $z, w \in B_k(0)$, *there is a constant* T_k *such that*

$$\frac{1}{T_k} < \frac{|g'(z)|}{|g'(w)|} < T_k.$$

We give examples for which the following theorem holds in Section 8. We strengthen a result on ergodicity that was shown in [15]. Let d denote the spherical metric on $\mathbb{C}_\infty$.

Theorem 7.12. *Suppose* Λ *is a lattice such that the following holds*: *there exists an* $\epsilon > 0$ *and a measurable set* $B \subset J(\wp_\Lambda)$, $m(B) > 0$, *such that for all* $z \in B$, $\limsup_n d(\wp_\Lambda^n(z), P(\wp_\Lambda)) > \epsilon$. *Then* $J(\wp_\Lambda) = \mathbb{C}_\infty$, *and* $\wp_\Lambda$ *is exact*, *ergodic and conservative*, *with respect to m*.

Proof. We show that $\wp_\Lambda$ is lim sup full. In order to establish this, we show that every set A of positive measure satisfies the hypotheses of Proposition 7.6 and we then apply Theorem 7.5. We denote the postcritical set of $\wp_\Lambda$ by P. Assume the set B is maximal; i.e., $B = \{z : \limsup_n d(\wp_\Lambda^n z, P(\wp_\Lambda)) > \epsilon\}$; then B is a completely invariant subset of $J(\wp_\Lambda)$. Given $A \in \mathscr{B}$, $m(A) > 0$, we assume that $m(A \cap B) > 0$, (otherwise we work with A^c). We remove from $A \cap B$ all prepoles and denote the resulting set by C. Then for every point in $z \in C$, $\limsup_{n\to\infty} d(\wp_\Lambda^n(z), P) > \epsilon$.

Given a point of Lebesgue density of C, say z_o, we can find a sequence $\{n_k\}$ such that $\lim_{k\to\infty} \wp_\Lambda^{n_k} z_o$ stays bounded away from the postcritical set of $\wp_\Lambda$. Therefore, the forward orbit of z_o has a finite accumulation point in $\mathbb{C}$, call it y; that is, $y = \lim_{k\to\infty} \wp_\Lambda^{n_k}(z_o)$ for some sequence $\{n_k\}$; denote $z_k = \wp_\Lambda^{n_k}(z_o)$. Furthermore, $d(y, P) > \epsilon/2 > 0$, so we consider the sequence of balls $U_k = B_{\epsilon/4}(z_k)$. Each set U_k is finite full since it is open, and each clearly lies only in finitely many fundamental periods of $\wp_\Lambda$. Furthermore, $m(U_k) = M$ for all k. Therefore there exists a univalent branch g_k of $\wp_\Lambda^{-n_k}$ defined on $B_{\epsilon/2}(z_k)$ such that $g_k(z_k) = z_o$.

Using Koebe Distortion Theorem 7.11 (in its spherical metric version) and the expansion result Proposition 7.10, plus the definition of a density point, we have that

$$\frac{m(g_k(U_k) \cap C)}{m(g_k(U_k))} \to 1$$

and

$$\frac{m(U_k \cap \wp_\Lambda^{n_k}(C))}{m(U_k)} \to 1.$$

From this we have that $\lim_k m(\wp_\Lambda^{n_k}(C) \cap U) = m(U)$, where U is the finite full set $U = B_{\epsilon/4}(y)$.

Since B is completely invariant and fills C (by definition of C), then $\lim_k m(\wp_\Lambda^{n_k}(B) \cap U) = m(U)$ as well, and by finite fullness there is a positive integer j such that

$$\lim_k m(\wp_\Lambda^{n_k+j}(B) \cap \wp_\Lambda^j(U)) = m(\wp_\Lambda^j(U)) = m(\mathbb{C}_\infty) = 1$$

using Corollary 7.9. Therefore by invariance of B, $m(B) = 1$, and the result follows. $\square$

As a corollary, we have the following result. An example is given in the next section.

Corollary 7.13. *If Λ is a lattice such that all critical values of $\wp_\Lambda$ are pre-periodic but not periodic, then $\wp_\Lambda$ is conservative, exact, and ergodic.*

Proof. Under the hypotheses, we have that $P(\wp_\Lambda)$ is a finite compact subset of $\mathbb{C}$, which implies that $J(\wp_\Lambda) = \mathbb{C}_\infty$. Then every measurable set B will necessarily have to satisfy the hypotheses of Theorem 7.12, since the postcritical set acts as a repellor. $\square$

8. Examples

We give examples of Weierstrass elliptic functions with properties discussed in Sections 6 and 7.

8.1. Rectangular square lattices. In the case where $g_2 = 4$ and $g_3 = 0$, Proposition 4.5 enables us to solve for the critical values explicity. This particular example is used to construct lattices with prescribed properties, so we denote the rectangular square lattice associated with $g_2 = 4$ and $g_3 = 0$ by $\Delta = [\delta, \delta i]$. Corollary 4.6 implies that $e_3 = 0$, $e_1 = 1$ and $e_2 = -1$.

8.1.1. Examples with a superattracting Fatou component. In this section we give an infinite family of rectangular square lattices $\Gamma = [\gamma, \gamma i]$ such that $\wp_\Gamma$ has a fixed critical point and therefore one superattracting Fatou component.

Theorem 8.1. *Let $\Delta = [\delta, \delta i]$ be the lattice associated with the invariants $g_2 = 4$ and $g_3 = 0$. If $\gamma = \sqrt[3]{\frac{2\delta^2}{m}}$ where m is an odd, positive integer and $\Gamma = [\gamma, \gamma i]$, then $F(\wp_\Gamma) \neq \emptyset$ and $J(\wp_\Gamma)$ is connected.*

Proof. Recall that with $\Delta = [\delta, \delta i]$ associated with the invariants $g_2 = 4$ and $g_3 = 0$, $\wp_\Delta(\frac{\delta}{2}) = 1$.

Let $\gamma = \sqrt[3]{\frac{2\delta^2}{m}}$ and $\Gamma = [\gamma, \gamma i]$. Let $k = \frac{\sqrt[3]{4m\delta}}{2}$, and note that $k\gamma = \delta$. If $u = \frac{\gamma}{2}$, then Equation 2 gives

$$\frac{1}{\left(\frac{\sqrt[3]{4m\delta}}{2}\right)^2}\wp_\Gamma\left(\frac{\gamma}{2}\right) = \wp_{\left(\frac{\sqrt[3]{4m\delta}}{2}\right)\Gamma}\left(\frac{\sqrt[3]{4m\delta}}{2}\cdot\frac{\gamma}{2}\right) = \wp_{\left(\frac{\sqrt[3]{4m\delta}}{2}\right)\Gamma}\left(\frac{\sqrt[3]{4m\delta}}{2}\cdot\sqrt[3]{\frac{2\delta^2}{m}}\right)$$

$$= \wp_\Delta\left(\frac{\delta}{2}\right) = 1$$

so

$$\wp_\Gamma\left(\frac{\gamma}{2}\right) = \left(\frac{\sqrt[3]{4m\delta}}{2}\right)^2 = \frac{m\gamma}{2}$$

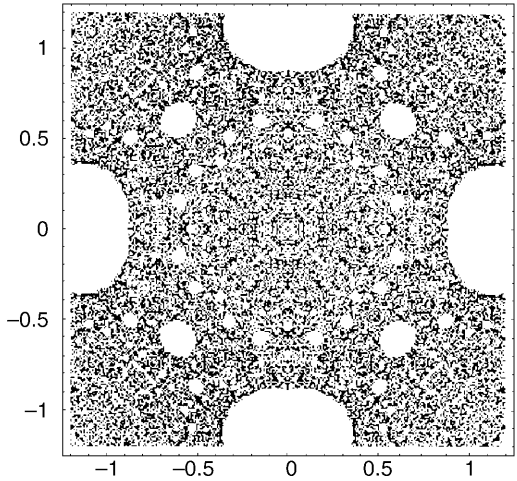

Figure 3. $J(\wp_\Gamma)$ when $m = 1$ in Theorem 8.1

But $\frac{m\gamma}{2}$ mod $\Gamma = \frac{\gamma}{2}$, so by periodicity we have $\wp_\Gamma\left(\frac{m\gamma}{2}\right) = \wp_\Gamma\left(\frac{\gamma}{2}\right) = \frac{m\gamma}{2}$, and $\frac{m\gamma}{2}$ is a fixed point. Also, $\wp'_\Gamma\left(\frac{m\gamma}{2}\right) = \wp'_\Gamma\left(\frac{\gamma}{2}\right) = 0$, so the critical point is superattracting. Therefore there must exist a superattracting component of the Fatou set. By Proposition 6.7 this is the only forward invariant cycle for $F(\wp_\Gamma)$, and by Proposition 6.15 the Julia set is connected. □

Figure 3 shows the Julia set for the case when $m = 1$ in Theorem 8.1, restricted to one period parallelogram centered at the origin. The lattice can be generated by $\gamma \approx 2.395$, and the superattracting fixed point occurs at $e_1 = \gamma/2 \approx 1.197$.

8.1.2. Examples with $J(\wp_\Gamma) = \mathbb{C}_\infty$. We find an infinite family of square lattices $\Gamma = [\gamma, \gamma i]$ such that all critical points of $\wp_\Gamma$ are prepoles and hence lie in the Julia set.

Theorem 8.2. *Let $\Delta = [\delta, \delta i]$ be the lattice associated with the invariants $g_2 = 4$ and $g_3 = 0$; let $\gamma = \sqrt[3]{\frac{\delta^2}{m}}$ where $m \in \mathbb{N}$ and $\Gamma = [\gamma, \gamma i]$. Then $J(\wp_\Gamma) = \mathbb{C}_\infty$.*

Proof. Again we will use the special lattice $\Delta = [\delta, \delta i]$ associated with the invariants $g_2 = 4$ and $g_3 = 0$.

Let $\gamma = \sqrt[3]{\frac{\delta^2}{m}}$ and $\Gamma = [\gamma, \gamma i]$. Let $k = \sqrt[3]{m\delta}$, and note that $k\gamma = \delta$. If $u = \frac{\gamma}{2}$, then Equation 2 gives

$$\frac{1}{(\sqrt[3]{m\delta})^2}\wp_\Gamma\left(\frac{\gamma}{2}\right) = \wp_{(\sqrt[3]{m\delta})\Gamma}\left(\sqrt[3]{m\delta}\cdot\frac{\gamma}{2}\right) = \wp_\Delta\left(\frac{\delta}{2}\right) = 1$$

so

$$\wp_\Gamma\left(\frac{\gamma}{2}\right) = \sqrt[3]{m^2\delta^2} = m\gamma.$$

But $m\gamma$ is a lattice point and a pole of $\wp_\Gamma$. Hence $\frac{\gamma}{2}$, $\frac{\gamma i}{2} \in \wp_\Gamma^{-2}(\infty)$. Since $\frac{\gamma+\gamma i}{2} \in \wp_\Gamma^{-2}(\infty)$, by periodicity all critical points are prepoles and hence lie in the Julia set. By Proposition 6.7, the Julia set is the entire sphere. $\square$

8.2. Rhombic square lattices. We give an infinite family of rhombic square lattices $\Gamma = [\gamma_1, \gamma_2]$ such that all critical points of $\wp_\Gamma$ are prepoles.

Theorem 8.3. *Let $\Delta = [\delta_1, \delta_2]$ be the lattice associated with the invariants $g_2 = -4$ and $g_3 = 0$ where δ_1 lies in quadrant I and δ_2 lies in quadrant IV. For any nonzero integer m, let $\gamma_1 = \delta_1/\sqrt[3]{m(\delta_1+\delta_2)}$ and $\gamma_2 = \delta_2/\sqrt[3]{m(\delta_1+\delta_2)}$ and $\Gamma = [\gamma_1, \gamma_2]$. Then $J(\wp_\Gamma) = \mathbb{C}_\infty$.*

Proof. Corollary 4.6 gives $e_1 = \wp_\Delta(\frac{\delta_1}{2}) = i$. Let $k = 1/\sqrt[3]{m(\delta_1+\delta_2)}$, and note that k is real and $k\delta_i = \gamma_i$. Then Equation 2 gives

$$\wp_\Gamma\left(\frac{\gamma_1}{2}\right) = \wp_{k\Delta}\left(\frac{k\delta_1}{2}\right) = \frac{1}{k^2}\wp_\Delta\left(\frac{\delta_1}{2}\right) = \frac{i}{k^2} = \frac{im(\delta_1+\delta_2)}{\sqrt[3]{m(\delta_1+\delta_2)}} = m(\gamma_1+\gamma_2).$$

But $m(\gamma_1+\gamma_2)$ is a lattice point and a pole of $\wp_\Gamma$. The result follows from Proposition 6.8. $\square$

8.3. Triangular lattices. Let $\epsilon = e^{2\pi i/3}$. If $g_2 = 0$ and $g_3 = 4$, we can find the critical values explicity. We denote the lattice associated with the invariants $g_2 = 0$ and $g_3 = 4$ by $\Omega = [\omega_1, \omega_2]$ where $\omega_2 = \epsilon\omega_1$. Using Proposition 4.5 and Corollary 4.6, we have $e_1 = \epsilon^2, e_2 = \epsilon, e_3 = 1$.

8.3.1. Examples with a superattracting Fatou component. We construct an infinite family of triangular lattices $\Gamma = [\gamma_1, \gamma_2]$ such that $\wp_\Gamma$ has three superattracting fixed points.

Theorem 8.4. *Let $\Omega = [\omega_1, \omega_2]$ be the lattice associated with the invariants $g_2 = 0$ and $g_3 = 4$. If $\gamma_1 = \sqrt[3]{\frac{2\omega_1^2\epsilon^2}{m}}$ where m is negative and odd, $\gamma_2 = \frac{\gamma_1\omega_2}{\omega_1}$ and $\Gamma = [\gamma_1, \gamma_2]$, then $F(\wp_\Gamma)$ has exactly three super-attracting Fatou components, and the Julia set is connected.*

Proof. First, we verify that Γ is a triangular lattice. Recall that $\Omega = [\omega_1, \omega_2]$ is triangular, where we have chosen generators so that $\omega_1 = \epsilon\omega_2$. Then $\epsilon\gamma_2 = \epsilon\frac{\gamma_1\omega_2}{\omega_1} = \gamma_1$, so $\epsilon\Gamma = \Gamma$ and Γ is triangular.

Let $\gamma_1 = \sqrt[3]{\frac{2\omega_1^2\epsilon}{m}}$ where m is odd. Let $k = \frac{\gamma_1}{\omega_1}$. If $u = \frac{\gamma_1}{2}$, then Equation 2 and the property that $\wp_\Omega\left(\frac{\omega_1}{2}\right) = \epsilon^2$ gives

$$\wp_\Gamma\left(\frac{\gamma_1}{2}\right) = \wp_{(\frac{\gamma_1}{\omega_1})\Omega}\left(\frac{\gamma_1}{\omega_1}\cdot\frac{\omega_1}{2}\right) = \frac{1}{(\frac{\gamma_1}{\omega_1})^2}\wp_\Omega\left(\frac{\omega_1}{2}\right) = \frac{1}{(\frac{\gamma_1}{\omega_1})^2}\epsilon^2$$
$$= \frac{\omega_1^2\epsilon^2}{\left(\sqrt[3]{\frac{2\omega_1^2\epsilon^2}{m}}\right)^2} = \frac{m\gamma_1}{2}.$$

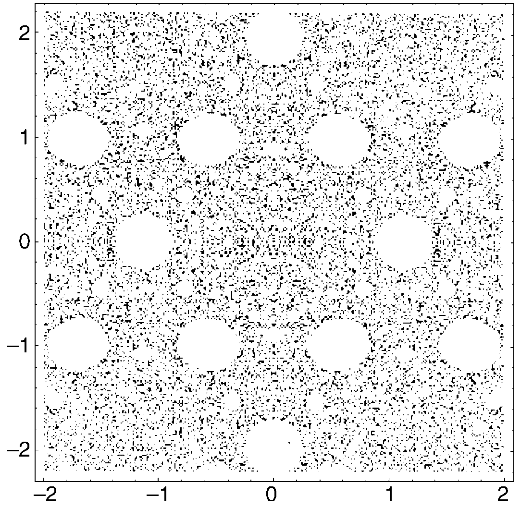

Figure 4. $J(\wp_\Gamma)$ when $m = -1$ in Theorem 8.4

Therefore $\wp_\Gamma\left(\frac{m\gamma_1}{2}\right) = \wp_\Gamma\left(\frac{\gamma_1}{2}\right) = \frac{m\gamma_1}{2}$, and $\frac{m\gamma_1}{2}$ is a fixed point. Also, $\wp'_\Gamma\left(\frac{m\gamma_1}{2}\right) = 0$, so the critical point is superattracting. Therefore there must exist a superattracting component of the Fatou set. By Proposition 6.6, each critical point is contained in a unique superattracting Fatou component. Proposition 6.16 implies that $J(\wp_\Gamma)$ is connected. □

Figure 4 shows the Julia set for the case when $m = -1$ in Theorem 8.4. The lattice can be generated by $\lambda_1 \approx 1.1382 + 1.9714i$, $\lambda_2 \approx 1.1382 - 1.9714i$ with three superattracting fixed points at 1.1382, 1.1382ϵ, and $1.1382\epsilon^2$.

8.3.2. Examples where $J(\wp_\Gamma) = \mathbb{C}_\infty$. We exhibit an infinite family of triangular lattices $\Gamma = [\gamma_1, \gamma_2]$ such that all critical points of $\wp_\Gamma$ are prepoles.

Theorem 8.5. *Let $\Omega = [\omega_1, \omega_2]$ be the lattice associated with the invariants $g_2 = 0$ and $g_3 = 4$. Let $\gamma_1 = \sqrt[3]{\frac{\omega_1^2\epsilon^2}{m}}$ where m is odd and negative, $\gamma_2 = \frac{\gamma_1\omega_2}{\omega_1}$ and $\Gamma = [\gamma_1, \gamma_2]$. Then $J(\wp_\Gamma) = \mathbb{C}_\infty$.*

Proof. The verification that Γ is a triangular lattice is the same as in the proof of Theorem 8.4.

Let $\gamma_1 = \sqrt[3]{\frac{\omega_1^2\epsilon^2}{m}}$. Let $k = \frac{\gamma_1}{\omega_1}$. If $u = \frac{\gamma_1}{2}$, then Equation 2 and the property that $\wp_\Omega\left(\frac{\omega_1}{2}\right) = \epsilon^2$ gives

$$\wp_\Gamma\left(\frac{\gamma_1}{2}\right) = \wp_{\left(\frac{\gamma_1}{\omega_1}\right)\Omega}\left(\frac{\gamma_1}{\omega_1}\cdot\frac{\omega_1}{2}\right) = \frac{1}{\left(\frac{\gamma_1}{\omega_1}\right)^2}\wp_\Omega\left(\frac{\omega_1}{2}\right) = \frac{1}{\left(\frac{\gamma_1}{\omega_1}\right)^2}\epsilon^2 = \frac{\omega_1^2\epsilon^2}{\left(\sqrt[3]{\frac{\omega_1^2\epsilon^2}{m}}\right)^2} = m\gamma_1.$$

The result follows from Proposition 6.6. □

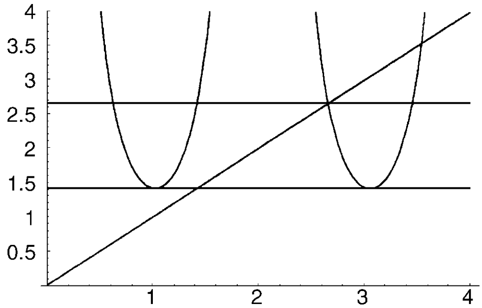

Figure 5. A preperiodic example

Next, we prove the existence of a Weierstrass elliptic function where the critical points are preperiodic but not prepoles. In particular, we prove that we can find a triangular lattice Λ such that $\wp_\Lambda(e_3) = a \neq e_3$ and $\wp_\Lambda(a) = a$, as in Figure 5. Since Λ is triangular, $\wp(e_1)$ and $\wp(e_2)$ are also fixed points.

Theorem 8.6. *There exists a triangular lattice Λ such that e_1, e_2, and e_3 are preperiodic but are not prepoles, and therefore $J(\wp_\Lambda) = \mathbb{C}_\infty$ and $\wp_\Lambda$ is exact, conservative, and ergodic with respect to normalized Lebesgue measure.*

Proof. We denote the lattice associated with the invariants $g_2 = 0$ and $g_3 = 4$ by $\Omega = [\omega_1, \omega_2]$ where we choose ω_1 in Quadrant I and $\omega_2 = \overline{\omega_1}$ (note that $\omega_1 = \epsilon\omega_2$). If $\omega_3 = \omega_1 + \omega_2$ denotes the real period, then using the tables in [19], $\omega_3 \approx 2.42\ldots$, and so $2 < \omega_3 < 3$.

If $\Lambda = k\Omega = [k\omega_1, k\omega_2] = [\lambda_1, \lambda_2]$ then $\frac{\lambda_3}{2} = \frac{\lambda_1+\lambda_2}{2}$ is a critical point. Using Equation 2, we have

$$\wp_\Lambda\left(\frac{\lambda_3}{2}\right) = \wp_{k\Omega}\left(\frac{k(\omega_1+\omega_2)}{2}\right) = \frac{1}{k^2}\wp_\Omega\left(\frac{\omega_1+\omega_2}{2}\right) = \frac{1}{k^2}.$$

Thus $e_{3,\Lambda} = 1/k^2$.

We are looking for a value of $k > 0$ such that $\wp_\Lambda(e_{3,\Lambda}) = \wp_\Lambda(1/k^2)$ is fixed. Using the periodicity and symmetry of $\wp_\Lambda$, it suffices to find a k such that

$$\wp_\Lambda\left(\frac{-1}{k^2} + 2\lambda_3\right) = \frac{-1}{k^2} + 2\lambda_3$$

(see Figure 5). Equation 2 implies that

$$\wp_\Lambda\left(\frac{-1}{k^2} + 2\lambda_3\right) = \wp_{k\Omega}\left(k\left(\frac{-1}{k^3} + 2\omega_3\right)\right) = \frac{1}{k^2}\wp_\Omega\left(\frac{-1}{k^3}\right).$$

Thus we need to find k such that

$$\frac{1}{k^2}\wp_\Omega\left(\frac{-1}{k^3}\right) = \frac{-1}{k^2} + 2\lambda_3 = \frac{-1}{k^2} + 2k\omega_3,$$

or

$$\wp_\Omega\left(\frac{-1}{k^3}\right) = -1 + 2k^3\omega_3.$$

Let $g(k) = \wp_\Omega(-1/k^3)$ and $f(k) = -1 + 2k^3\omega_3$. We will be looking as k ranges over the interval $[r,s] = \left[\sqrt[3]{\frac{1}{\omega_3}}, \sqrt[3]{\frac{2}{\omega_3}}\right]$. Since $2 < \omega_3 < 3$, we have that $r > .5$ and $s < 1$. Note that r is a pole of g and s is a critical point of g, and g is monotone decreasing on $[r,s]$. Since r is a pole of g, we can find a small number δ such that $g(r+\delta) > 10$. Clearly, $r+\delta < 1$ since $s < 1$. Thus $f(r+\delta) < -1 + 2\omega_3 < -1 + 6 = 5$. Thus $g - f > 0$ at $k = r + \delta$.

We know that $g(s) = \wp_\Omega(-\omega_3/2) = 1$. Since $\omega_3 > 2$ and $s > .5$, we have $f(s) > -1 + 2(.5)2 = 1$. Thus $g - f < 0$ at $k = s$. Since both g and f are continuous on the interval $[r+\delta, s]$, the Intermediate Value Theorem implies that there is a $k \in (r+\delta, s)$ such that $g(k) = f(k)$. This implies the existence of a triangular lattice Λ such that $e_{3,\Lambda}$ is preperiodic (but not a prepole). Since Λ is triangular, all critical values are preperiodic (but are not prepoles) by Equation 2 and $J(\wp_\Lambda) = \mathbb{C}_\infty$. By Corollary 7.13, $\wp_\Lambda$ is conservative, exact, and ergodic. □

Numerically we can approximate $k = .8396$ in Theorem 8.6, and obtain the example shown in Figure 5 using $g_2 = 0$ and $g_3 = 11.4217$.

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